SINGULARITIES IN BAKER DOMAINS

WALTER BERGWEILER

Dedicated to the memory of Professor I. N. Baker

ABSTRACT. Let U be a Baker domain of a transcendental entire function f. Denote by λ_U the hyperbolic metric in U and, for $w \in U$ and $n \in \mathbb{N}$, define $\rho_n(w) = \lambda_U(f^{n+1}(w), f^n(w))$ and $\rho(w) = \lim_{n\to\infty} \rho_n(w)$. Here f^n denotes the n–th iterate of f. It is shown that if the set of singularities of f^{-1} that are contained in U is bounded, then

$$\rho_n(w) = \frac{1}{2n} + a\frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right)$$

for some $a \in \mathbb{R}$ if $\rho(w) = 0$ and

$$\rho_n(w) = \rho(w) + \frac{b}{n^3} + O\left(\frac{1}{n^4}\right)$$

for some $b \geq 0$ if $\rho(w) > 0$, but $\inf_{w\in U} \rho(w) = 0$. The result is applied to certain entire functions of finite order.

1. Introduction and main results

Let f be an entire transcendental function and denote by f^n the n–th iterate of f. An invariant component U of the Fatou set of f is called a *Baker domain* if $f^n|_U \to \infty$ as $n \to \infty$.

Eremenko and Lyubich [11, Theorem 1] have shown that if $\text{sing}(f^{-1})$, the set of finite singularities of f^{-1}, is bounded, then f has no Baker domains. However, Baker domains need not contain singularities of f^{-1}, see [8, Theorem 1], [10, Example 3] or [12, p. 609]. We shall establish some criteria that imply that there are even infinitely many singularities of f^{-1} in a Baker domain.

First we note that Baker domains of entire functions are simply-connected [3, Theorem 1]. Thus, if U is a Baker domain of f, then there exists a conformal map ϕ from the unit disk $\mathbb{D}$ onto U. Then $g := \phi^{-1} \circ f \circ \phi$ is an inner function; cf. [4, 5]. Since f has no fixed point in U and thus g has no fixed point in $\mathbb{D}$, the Denjoy-Wolff theorem (see, e. g., [15, p. 55] or [18, §2.7]) implies that $g^n \to z_0$ for some $z_0 \in \partial\mathbb{D}$. The point z_0 is called the *Denjoy-Wolff point.* Our results are based on the following observation.

Theorem 1. *Let U be a Baker domain of a transcendental entire function f. Suppose that $U \cap \text{sing}(f^{-1})$ is bounded. Let $\phi : \mathbb{D} \to U$ be a conformal homeomorphism and define $g := \phi^{-1} \circ f \circ \phi$. Then g extends analytically to a neighborhood of its Denjoy-Wolff point.*

It is not difficult to see that if a holomorphic function $g : \mathbb{D} \to \mathbb{D}$ extends analytically to a neighborhood of its Denjoy-Wolff point, then the Denjoy-Wolff point is a fixed point of (the analytic continuation of) g.

Recall that a fixed point z_0 of a holomorphic function h is called *attracting* if $|h'(z_0)| < 1$. If $h'(z_0) = 1$, then z_0 is a *multiple fixed point*; that is, z_0 is a multiple zero of $z - h(z)$. The smallest integer $n \geq 2$ such that $h^{(n)}(z_0) \neq 0$ is then called the *multiplicity* of the fixed point z_0. Following [15, §12] we call the residue of $1/(z - h(z))$ at a fixed point z_0 the *residue fixed point index* and denote it by $\iota(h, z_0)$. Of course, if z_0 is not multiple,

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then $\iota(h, z_0) = 1/(1 - h'(z_0))$. The residue fixed point index is invariant under holomorphic changes of variables [15, Lemma 12.3]. This is used to define the residue fixed point index in the case that $z_0 = \infty$.

We denote the hyperbolic metric in a hyperbolic domain $U \subset \mathbb{C}$ by λ_U. We also use the notation λ_U for the density of the hyperbolic metric. This should lead to no confusion, as the density is defined on U while the hyperbolic metric is defined on $U \times U$. Of the two common normalizations we use the one where the density has constant curvature equal to -1 so that in the unit disk it is given by $\lambda_{\mathbb{D}}(z) = 2/(1 - |z|^2)$, while in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ it is given by $\lambda_{\mathbb{H}}(z) = 1/\operatorname{Im} z$.

For a map $f : U \to U$, $w \in U$ and $n \in \mathbb{N}$ we put

$$\rho_n(w, U, f) := \lambda_U(f^{n+1}(w), f^n(w)).$$

Note that if f, U, ϕ and g are as above, and $z \in \mathbb{D}$, then

$$\rho_n(z) := \rho_n(z, \mathbb{D}, g) = \rho_n(\phi(z), U, f).$$

By Schwarz's lemma, $\rho_n(z)$ is non-increasing and thus $\rho(z) := \lim_{n\to\infty} \rho_n(z)$ exists. The sequences $(\rho_n(z))$ have been studied by Bargmann [6], Bonfert [9] and König [14].

We show that under the hypotheses of Theorem 1 the asymptotic behavior of these sequences is of a very special form.

Theorem 2. *Let $g : \mathbb{D} \to \mathbb{D}$ be an inner function with Denjoy-Wolff point $z_0 \in \partial\mathbb{D}$. Suppose that g extends analytically to a neighborhood of z_0 (and denote the analytic continuation again by g). Then we have one of the following three cases:*

(i) *z_0 is an attracting fixed point and $\inf_{z\in\mathbb{D}} \rho(z) = \log \frac{1}{|g'(z_0)|} > 0$.*

(ii) *z_0 is a multiple fixed point of multiplicity 2, $\rho(z) > 0$ for all $z \in \mathbb{D}$ but $\inf_{z\in\mathbb{D}} \rho(z) = 0$, and*

$$\rho_n(z) = \rho(z) + \frac{\iota(g, z_0) - 1}{3\tanh\left(\frac{\rho(z)}{2}\right)} \cdot \frac{1}{n^3} + O\left(\frac{1}{n^4}\right)$$

as $n \to \infty$ for all $z \in \mathbb{D}$. Moreover, $\iota(g, z_0) \geq 1$, with equality if and only if g is univalent.

(iii) *z_0 is a multiple fixed point of multiplicity 3, $\rho(z) = 0$ for all $z \in \mathbb{D}$ and*

$$\rho_n(z) = \frac{1}{2n} + \left(\frac{\iota(g, z_0)}{4} - \frac{3}{8}\right)\frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right)$$

as $n \to \infty$ for all $z \in \mathbb{D}$.

2. Proof of Theorems 1 and 2

Proof of Theorem 1. Let $w \in U$ and let $\gamma : [0, 1] \to U$ be a curve connecting w with $f(w)$; that is, $\gamma(0) = w$ and $\gamma(1) = f(w)$. We extend γ to a curve $\gamma : [0, \infty) \to U$ by putting $\gamma(n + s) = f^n(\gamma(s))$ for $s \in [0, 1)$ and $n \in \mathbb{N}$. Then $\gamma(t + 1) = f(\gamma(t))$ for $t \in [0, \infty)$ and $\gamma(t) \to \infty$ as $t \to \infty$. Thus there exists a crosscut $\sigma \subset U$ that separates $\operatorname{sing}(f^{-1})$ and $\gamma([r, \infty))$ for some $r \in [0, \infty)$. Let V be the component of $U\backslash\sigma$ that contains $\gamma([r, \infty))$. Then there exists a branch α of f^{-1} in V satisfying $\alpha(\gamma(t + 1)) = \gamma(t)$ for $t \geq r$. Let $W = \phi^{-1}(V)$. Then $\beta = \phi^{-1} \circ \alpha \circ \phi$ is an inverse branch of g defined in W. Thus g is univalent in $\beta(W)$. Now $\phi^{-1}(\sigma)$ is a crosscut in $\mathbb{D}$ and $\phi^{-1} \circ \gamma$ is a curve ending at the Denjoy-Wolff point z_0 of g. It follows (cf. [16, Proposition 2.14]) that $W \supset \mathbb{D} \cap D(z_0, \varepsilon)$ for some $\varepsilon > 0$. Here $D(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - t_0| < \varepsilon\}$. Thus $\beta(W) \supset \mathbb{D} \cap D(z_0, \delta)$ for some $\delta > 0$. It follows that g extends continously to $\mathbb{D} \cup (\partial\mathbb{D} \cap D(z_0, \delta))$ and thus g extends analytically to $\mathbb{D} \cup (\partial\mathbb{D} \cap D(z_0, \delta)) \cup \widehat{\mathbb{C}}\backslash\mathbb{D}$ by the Schwarz reflection principle, where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. $\square$

Proof of Theorem 2. It will be convenient to consider inner functions of the upper half plane instead of inner functions of the unit disk, with the Denjoy-Wolff point placed at ∞. Instead of $g : \mathbb{D} \to \mathbb{D}$ we thus consider $G := L \circ g \circ L^{-1} : \mathbb{H} \to \mathbb{H}$, with $L : \mathbb{D} \to \mathbb{H}$, $L(z) = -i\frac{z+z_0}{z-z_0}$. Then $G^n \to \infty$ as $n \to \infty$ and there exists $R > 0$ such that G extends analytically to $\widehat{\mathbb{C}}\backslash[-R, R]$. As already remarked after Theorem 1, it follows easily that ∞ is a fixed point of (the analytic continuation of) G. Moreover, it follows directly that ∞ is either an attracting or a multiple fixed point of G. The dynamics near multiple fixed points are well understood. In particular, if m is the multiplicity, then there are $m - 1$ invariant components of the Fatou set where the iterates tend to the fixed point. (Here the Fatou set of a function analytic in some domain on the sphere is the maximal open set where the iterates are defined and normal.) In the present case, the upper and lower half plane are contained in such a component. This implies that $m - 1 \le 2$ so that $m \le 3$. If the upper and lower half plane are in the same component of the Fatou set, then $m = 2$. If they are in different components, then $m = 3$.

We now consider the different cases separately.

(i) *The point ∞ is an attracting fixed point of G.* Then $G(z) \sim az$ as $z \to \infty$, with $a = 1/|g'(z_0)| > 1$. From the explicit form of the hyperbolic metric (see [7, p. 130]) given by

$$\lambda_{\mathbb{H}}(w, aw) = \log \frac{|a - \frac{\overline{w}}{w}| + a - 1}{|a - \frac{\overline{w}}{w}| - a + 1} \ge \log a$$

we conclude that

$$\rho(z) \ge \log a$$

for all $z \in \mathbb{H}$. Moreover, we can deduce that $\rho(it) \to \log a$ as $t \to \infty$, $t \in \mathbb{R}$.

(ii) *The point ∞ is a multiple fixed point of G of multiplicity* 2. Then $G(z) = z + a + o(1)$ as $z \to \infty$ for some $a \in \mathbb{R}\backslash\{0\}$. We may assume that $a = \pm 1$ since otherwise we conjugate G by $z \mapsto |a|z$. It suffices to consider the case $a = +1$. Thus

$$G(z) = z + 1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

as $z \to \infty$, with $b_1, b_2, \dots \in \mathbb{R}$, $b_1 = 1 - \iota(G, \infty) = 1 - \iota(g, z_0)$. Also, G has a representation (see [1, p. 216]) of the form

$$G(z) = z + 1 + \int_{\mathbb{R}} \frac{d\nu(t)}{t - z}$$

with a singular, compactly supported measure ν on $\mathbb{R}$. Thus $b_1 < 0$ except when $G(z) \equiv z+1$.

By classical results in complex dynamics (see, e. g., [15, §10] or [18, §3.5]), there exists a function ψ holomorphic and univalent in a halfplane $H_\sigma := \{z \in \mathbb{C} : \operatorname{Re} z > \sigma\}$ such that

$$\psi(z + 1) = G(\psi(z))$$

and

$$\psi(z) \sim z$$

as $z \to \infty$. This function ψ is unique up to precomposition by a translation. We may assume that $\psi(x) \in \mathbb{R}$ for $x \in \mathbb{R}$.

The asymptotics of ψ are known [13]. There exist $c \in \mathbb{R}$ and $c_{jk} \in \mathbb{R}$, $k \ge 1$, $0 \le j \le k$, such that for $N \in \mathbb{N}$

$$\psi(z) = z + b_1 \log z + c + \sum_{k=1}^{N} \sum_{j=0}^{k} c_{jk} \frac{(\log z)^j}{z^k} + O\left(\left(\frac{\log z}{z}\right)^{N+1}\right)$$

as $z \to \infty$. Since ψ is unique only up to precomposition by a translation, we may normalize ψ by choosing $c = 0$. The c_{jk} can then be computed from the b_m. For example, we have $c_{01} = b_1^2 - \frac{1}{2}b_1 - b_2$, $c_{11} = b_1^2$, $c_{02} = \frac{1}{2}b_1^3 - \frac{1}{4}b_1^2 - \frac{1}{12}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3$, $c_{12} = \frac{1}{2}b_1^2 + b_1 b_2$ and $c_{22} = -\frac{1}{2}b_1^3$.

Let now $z \in \mathbb{H}$. Then there exists $M \in \mathbb{N}$ with $G^M(z) \in \psi(H_\sigma)$. Thus $G^M(z) = \psi(u)$ for some $u \in H_\sigma \cap \mathbb{H}$. Let $v := u - M \in \mathbb{H}$. For $n \geq M$ we then have

$$G^n(z) = G^{n-M}(G^M(z)) = G^{n-M}(\psi(u)) = \psi(u + n - M) = \psi(v + n).$$

Thus

$$\rho_n(z) = \lambda_{\mathbb{H}}(G^{n+1}(z), G^n(z)) = \lambda_{\mathbb{H}}(\psi(v + n + 1), \psi(v + n)).$$

Combining this with the asymptotics of ψ and the explicit form (see again [7, p. 130]) of $\lambda_{\mathbb{H}}$ given by

$$\lambda_{\mathbb{H}}(a, b) = 2 \operatorname{arctanh} \left| \frac{a - b}{a - \overline{b}} \right|$$

we obtain by a lengthy computation (which I did using MAPLE) that

$$\rho_n(z) = 2 \operatorname{arctanh} \frac{1}{\sqrt{1 + 4s^2}} - \frac{b_1\sqrt{1 + 4s^2}}{3n^3} + O\left(\frac{1}{n^4}\right)$$

with $s = \operatorname{Im} v$.

Thus $\rho(z) = 2 \operatorname{arctanh} \frac{1}{\sqrt{1+4s^2}}$ and hence

$$\rho_n(z) = \rho(z) + \frac{b_1}{3 \tanh\left(\frac{\rho(z)}{2}\right)} \cdot \frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

(iii) *The point ∞ is a multiple fixed point of G of multiplicity* 3. Then

$$G(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \ldots$$

with $b_1, b_2, \ldots \in \mathbb{R}$, $b_1 \neq 0$. As in case (ii) we have $b_1 < 0$. Conjugating with a suitable affine function we may assume that $b_1 = -1$ and $b_2 = 0$ so that

$$G(z) = z - \frac{1}{z} + \frac{b_3}{z^3} + \ldots$$

as $z \to \infty$. Then $b_3 = 1 - \iota(G, \infty)$. We put $S := \mathbb{C} \backslash (-\infty, 0]$ and conjugate G by $z \mapsto -\frac{1}{2}z^2$ to a function $F : S \to S$; that is, we consider

$$F(z) = -\frac{G(i\sqrt{2z})^2}{2}.$$

Then

$$F(z) = z + 1 + \frac{a}{z} + \sum_{k=3}^{\infty} \frac{a_k}{z^{\frac{k}{2}}}$$

in a neighborhood of ∞, with $a = \frac{1}{2}b_3 + \frac{1}{4} = \frac{3}{4} - \frac{1}{2}\iota(G, \infty)$ and $a_3, a_4, \dots \in \mathbb{R}$. As in (ii) there exists a function ψ holomorphic and univalent in some right half plane with $\psi(z+1) = F(\psi(z))$ and $\psi(z) \sim z$. The asymptotics of ψ are again known [13]. With a suitable normalization we have

$$\psi(z) = z + a \log z + o(1)$$

as $z \to \infty$. Given $z \in S$, we again find $v \in \mathbb{C}$ with

$$F^n(z) = \psi(v + n)$$

for sufficiently large n. Combining this with the explicit form of the hyperbolic metric (and again with a lengthy computation, for which I used MAPLE) yields

$$\rho_n(z) = \frac{1}{2n} - \frac{a}{2} \cdot \frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right)$$

as $n \to \infty$. The conclusion follows. □

3. An application

An application of Theorem 1 and 2 is the following result.

Theorem 3. *Let f be an entire transcendental function of finite order of growth. Suppose there exists $a, \eta > 0$ such that $f(z) = z + a + o(1)$ as $z \to \infty$, $|\arg z| \le \eta$. Then there exists a Baker domain U containing $\{z \in \mathbb{C} : |\arg z| \le \eta$ and $\operatorname{Re} z > R\}$ for some $R > 0$, and $U \cap \operatorname{sing}(f^{-1})$ is unbounded.*

The proof requires the following consequence of Koebe's one quarter theorem.

Lemma 1. *Let $K \subset \mathbb{C}\backslash(-\infty, 0]$ be compact. Then there exists $\varepsilon > 0$ such that $\lambda_\Omega(1) \ge \frac{1}{2} + \varepsilon$ for all simply-connected domains $\Omega \subset \mathbb{C}$ that satisfy $0 \notin \Omega$, $1 \in \Omega$ and $K \cap \mathbb{C}\backslash\Omega \ne \emptyset$.*

Proof. Let $\Omega \subset \mathbb{C}$ be simply-connected with $0 \notin \Omega$ and $1 \in \Omega$. If $\psi : \mathbb{D} \to \Omega$ is a conformal homeomorphism with $\psi(0) = 1$, then

$$\lambda_\Omega(1) = \lambda_\Omega(\psi(0)) = \frac{\lambda_{\mathbb{D}}(0)}{|\psi'(0)|} = \frac{2}{|\psi'(0)|}.$$

Since the Koebe one quarter theorem says that Ω contains a disk of radius $\frac{1}{4}|\psi'(0)|$ around 1, but the largest disk around 1 contained in Ω has radius at most 1 because of $0 \notin \Omega$, we have $\frac{1}{4}|\psi'(0)| \le 1$ and thus $\lambda_\Omega(1) \ge \frac{1}{2}$. Moreover, the Koebe one quarter theorem shows that $\lambda_\Omega(1) = \frac{1}{2}$ implies $\Omega = S := \mathbb{C}\backslash(-\infty, 0]$.

Suppose now that the lemma is false. Then there exists a sequence of domains (Ω_n) with the properties given in the lemma such that $\lambda_{\Omega_n}(1) \to \frac{1}{2}$. Let $\psi_n : \mathbb{D} \to \Omega_n$ be conformal with $\psi_n(0) = 1$. Passing to a subsequence if necessary, we may assume that (ψ_n) converges, say $\psi_n \to \psi : \mathbb{D} \to \Omega$. Then Ω has the properties stated in lemma; that is, $0 \notin \Omega$, $1 \in \Omega$ and $K \cap \mathbb{C}\backslash\Omega \ne \emptyset$. On the other hand, we have $\lambda_\Omega(1) = \frac{1}{2}$ so that $\Omega = S$. This is a contradiction since $K \cap (-\infty, 0] = \emptyset$. □

Proof of Theorem 3. Without loss of generality we may assume that $a = 1$. It is easy to see that there exists $R > 0$ such that $V := \{z \in \mathbb{C} : |\arg z| \le \eta$ and $\operatorname{Re} z > R\}$ is invariant and that $f^n|_V \to \infty$ as $n \to \infty$. This implies that there exists a Baker domain U containing V. What we have to show is that $U \cap \operatorname{sing}(f^{-1})$ is unbounded.

As before we put

$$\rho_n(z) := \lambda_U(f^{n+1}(z), f^n(z))$$

for $z \in U$ and $n \in \mathbb{N}$. To estimate $\rho_n(z)$ we use a result of Baker ([2], see also [17]), which says that if μ is greater than the order of f, then the Julia set $J(f)$ of f cannot be contained in a sector of the form $\{z \in \mathbb{C} : |\arg(z - z_0) - \theta_0| \le \pi/2\mu\}$ where $\theta_0 \in \mathbb{R}$ and $z_0 \in \mathbb{C}$. It follows that there exist $\theta \in (-\pi, \pi)$ and a sequence (z_k) in $J(f)$ such that $r_k := |z_k| \to \infty$ and $\theta_k := \arg z_k \to \theta$. We choose $\rho > 0$ such that $K := \{z \in \mathbb{C} : |z - e^{i\theta}| \le \rho\}$ does not intersect $(-\infty, 0]$ and choose $\varepsilon > 0$ according to Lemma 1. Next we choose $\sigma \in (0, 1)$ such that $\sigma/(1 - \sigma) < \rho$ and $\frac{1}{2} + \varepsilon \ge \frac{1}{2}(1 + \varepsilon)(1 - \sigma)$. It follows that if $|z - r_k| \le \sigma r_k$, then

$$\left|\frac{z_k}{z} - e^{i\theta}\right| = \left|\frac{r_k - z}{z} e^{i\theta_k} + e^{i\theta_k} - e^{i\theta}\right| \le \frac{|r_k - z|}{|z|} + |e^{i\theta_k} - e^{i\theta}| \le \frac{\sigma}{1 - \sigma} + |\theta_k - \theta|$$

and thus $\frac{z_k}{z} \in K$ for sufficiently large k. Hence

$$\lambda_U(z) = \lambda_{\frac{1}{z}U}(1) \cdot \frac{1}{|z|} \geq \left(\frac{1}{2}+\varepsilon\right)\frac{1}{|z|} \geq \frac{\left(\frac{1}{2}+\varepsilon\right)}{(1-\sigma)r_k} \geq \frac{1+\varepsilon}{2r_k}$$

for $|z - r_k| \leq \sigma r_k$ and large k.

We now fix $z_0 \in U$ and note that $f^n(z_0) \sim n$ as $n \to \infty$. With $n_k := [r_k]$ we obtain $f^{n_k}(z_0) \sim r_k$ and $f^{n_k+1}(z_0) \sim r_k$. This implies that

$$\begin{aligned}\rho_{n_k}(z_0) &= \lambda_U(f^{n_k+1}(z_0, f^{n_k}(z_0)) \\ &\geq \frac{1+\varepsilon}{2r_k}|f^{n_k+1}(z_0) - f^{n_k}(z_0)| \\ &\geq (1-o(1))\frac{1+\varepsilon}{2r_k} \\ &= (1-o(1))\frac{1+\varepsilon}{2n_k}.\end{aligned}$$

On the other hand, we have $\lambda_U(z) \leq \frac{2}{\text{dist}(z,\partial U)}$, where $\text{dist}(\cdot,\cdot)$ denotes the Euclidean distance. Denoting by γ the straight line from $f^{n_k}(z_0)$ to $f^{n_k+1}(z_0)$ we obtain

$$\rho_{n_k}(z_0) \leq \int_\gamma \lambda_U(z)|dz| \leq (2+o(1))\frac{c}{r_k}|f^{n_k+1}(z_0) - f^{n_k}(z_0)| = (2+o(1))\frac{c}{n_k}$$

as $k \to \infty$, with $c = 1/\sin\eta$ if $\eta < \pi/2$ and $c = 1$ if $\eta \geq \pi/2$. Thus $\rho(z) = 0$.

From Theorem 2 we conclude that the inner function obtained by conjugation of $f|_U$ does not extend analytically to a neighborhood of its Denjoy-Wolff point. Thus $U \cap \text{sing}(f^{-1})$ is unbounded by Theorem 1. □

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References

[1] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Math. Surveys and Monographs 50, Amer. Math. Soc. Providence, RI, 1997.

[2] I. N. Baker, Sets of non-normality in iteration theory, *J. London Math. Soc.* 40 (1965), 499-502.

[3] ———, The domains of normality of an entire function, *Ann. Acad. Sci. Fenn. (Ser. A, I. Math.)* 1 (1975), 277-283.

[4] I. N. Baker and P. Domínguez, Boundaries of unbounded Fatou components of entire functions, *Ann. Acad. Sci. Fenn. (Ser. A, I. Math.)* 24 (1999), 437-464.

[5] I. N. Baker and J. Weinreich, Boundaries which arise in the iteration of entire functions, *Rev. Roumaine Math. Pures Appl.* 36 (1991), 413-420.

[6] D. Bargmann, Iteration of inner functions and boundaries of components of the Fatou set, preprint 1999.

[7] A. F. Beardon, *The Geometry of Discrete Groups*, Springer, New York, Berlin, Heidelberg 1983.

[8] W. Bergweiler, Invariant domains and singularities, *Math. Proc. Cambridge Philos. Soc.* 117 (1995), 525-532.

[9] P. Bonfert, On iteration in planar domains, *Mich. Math. J.* 44 (1997), 47-68.

[10] A. E. Eremenko and M. Yu. Lyubich, Examples of entire functions with pathological dynamics, *J. London Math. Soc.* (2) 36 (1987), 458-468.

[11] ———, Dynamical properties of some classes of entire functions, *Ann. Inst. Fourier* 42 (1992), 989-1020.

[12] M. Herman, Are there critical points on the boundary of singular domains?, *Comm. Math. Phys.* 99 (1985), 593-612.

[13] T. Kimura, On the iteration of analytic functions, *Funkc. Ekvacioj* 14 (1971), 197-238.

[14] H. König, Conformal conjugacies in Baker domains, *J. London Math. Soc.* (2) 59 (1999), 153-170.

[15] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, Braunschweig, Wiesbaden, 1999.

[16] Chr. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, New York, Berlin, Heidelberg 1992.

[17] J. Qiao, Stable sets for iteration of entire functions (Chinese), *Acta Math. Sin.* 37 (1994), 702-708.

[18] N. Steinmetz, *Rational Iteration*, Walter de Gruyter, Berlin, 1993.

Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany
E-mail address: bergweiler@math.uni-kiel.de