A QUESTION OF EREMENKO AND LYUBICH CONCERNING COMPLETELY INVARIANT DOMAINS AND INDIRECT SINGULARITIES

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Abstract. We give an example of an entire function with a completely invariant Fatou component which has an indirect singularity not contained in this Fatou component. The question whether such a function exists had been raised by Eremenko and Lyubich.

1. Introduction and result

Let f be an entire function and $a \in \mathbb{C}$. Then a is called a *critical value* of f if there exists $z_0 \in \mathbb{C}$ with $f(z_0) = a$ and $f'(z_0) = 0$, and a is called an *asymptotic value* if there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$. The critical and asymptotic values of f form the set $\text{sing}(f^{-1})$ of singularities of the inverse function of f. The critical values are also called algebraic singularities and the asymptotic values are also called transcendental singularities. The set $\text{sing}(f^{-1})$ plays an important role in complex dynamics.

Let a be an asymptotic value and γ as above. For $r > 0$ let $D(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$. By $\Omega(r)$ we denote the component of $f^{-1}(D(a, r))$ which contains the "tail" of γ; that is, $\Omega(r)$ is the (unique) component of $f^{-1}(D(a, r))$ for which there exists $t_r \geq 0$ such that $\gamma([t_r, \infty)) \subset \Omega(r)$. We say that a is a *direct singularity* if there exists $r > 0$ such that $f(z) \neq a$ for all $z \in \Omega(r)$. Otherwise a is called an *indirect singularity*; see [4, §XI.1] for this classification of singularities.

Note that the question whether a is direct or indirect actually depends not only on a but also on γ. A more precise terminology would be to say that a is (in)direct *with respect to* γ. For example, the function $\exp(-z^2)/\Gamma(z)$ tends to 0 along both the positive and negative real axis, and 0 is direct with respect to the positive real axis and indirect with respect to the negative real axis.

The *Fatou set* $F(f)$ of f is defined to be the set of all points where the family $\{f^n\}_{n \in \mathbb{N}}$ of iterates of f is normal. It is well-known that the Fatou set is completely invariant; that is, $f^{-1}(F(f)) = F(f)$. Baker [1] proved that the Fatou set of an entire transcendental function contains at most one completely invariant component. He [2, p. 178] raised the question whether for such a component U we necessarily have $U = F(f)$. This was proved by Eremenko and Lyubich [3, Theorem 6] in the case that $\text{sing}(f^{-1})$ is finite. They deduced this from a general result (see [3, Lemma 11 and Remark 1]) which says that a completely invariant component of $F(f)$ contains all critical values and all direct singularities of f. This sharpened an earlier result of Baker [2, Theorem 2]. Eremenko and Lyubich [3, Remark 1] asked

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whether a completely invariant domain must also contain all indirect singularities. We show that this is not the case.

Example. The function f defined by

$$f(z) = \frac{12\pi^2}{5\pi^2 - 48}\left(\frac{(\pi^2-8)z + 2\pi^2}{z(4z-\pi^2)}\cos\sqrt{z} + \frac{2}{z}\right)$$

is entire and has 0 as an indirect singularity, and $F(f)$ has a completely invariant component U with $0 \in \partial U$.

More specifically, we shall see that 0 is a fixed point of f of multiplier 1 and that U is the parabolic basin associated to 0. We have $(0,\infty) \subset U$ and $f(x) \to 0$ as $x \to \infty$, $x \in \mathbb{R}$.

By the same method of proof we see that there exists $\alpha_0 > 1$ such that if $1 < \alpha < \alpha_0$, then $f_\alpha(z) := \alpha f(z)$ has an attracting fixed point $x_\alpha > 0$ whose attracting basin U_α is completely invariant and satisfies $0 \in \partial U_\alpha$. Clearly, 0 is also an indirect singularity of f_α.

Computationally we find that $\alpha_0 = 6.348889\ldots$ and that for $\alpha = 2.594992\ldots$ the fixed point $x_\alpha = 20.599204\ldots$ is superattracting.

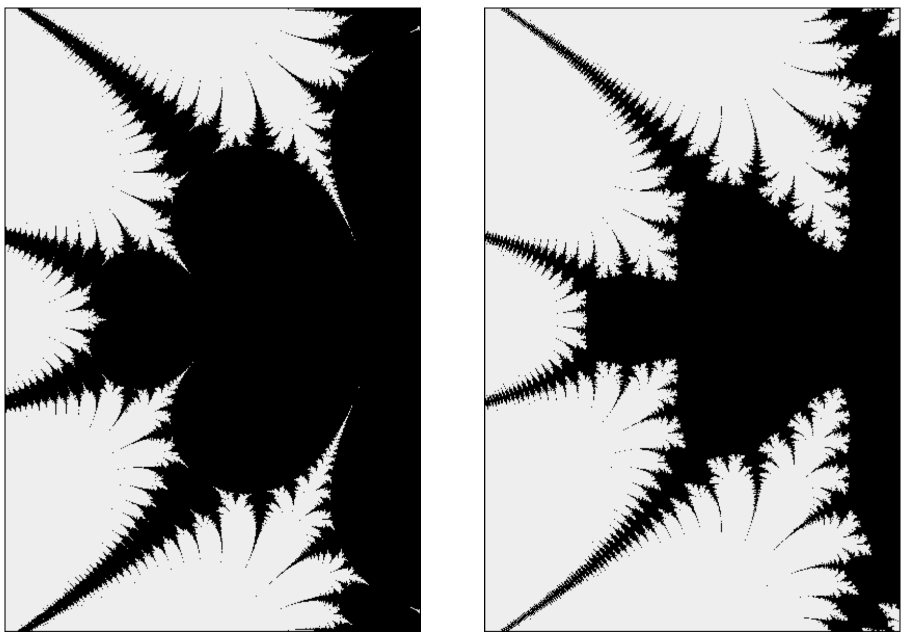

FIGURE 1. The dark sets are the completely invariant components of f (left) and f_α with $\alpha = 2.594992$ (right). The range shown is $-100 \le \operatorname{Re} z \le 300$, $|\operatorname{Im} z| \le 300$.

2. Verification of the properties of the example

Let

$$g(z) := \frac{z + \frac{2\pi^2}{\pi^2-8}}{z - \frac{\pi^2}{4}}\cos\sqrt{z} = \left(1 + \frac{\pi^4}{(\pi^2-8)(4z-\pi^2)}\right)\cos\sqrt{z}$$

and

$$h(z) := \frac{g(z) - g(0)}{z}.$$

Then g and h are entire and a computation shows that

$$f(z) = \frac{h(z)}{h'(0)}.$$

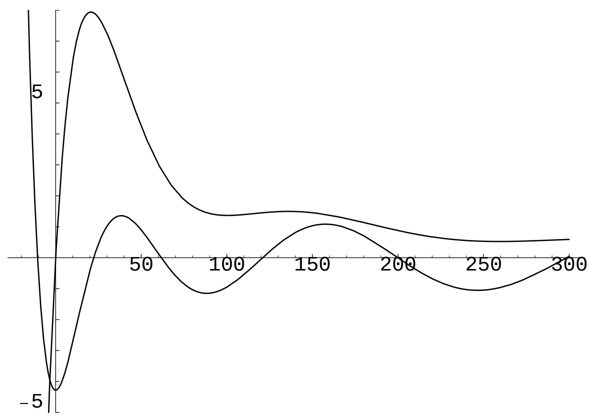

FIGURE 2. The graphs of f and g.

We shall first discuss some properties of the functions g and h. We note that g belongs to the Laguerre-Pólya class consisting of all entire functions which can be locally uniformly approximated by polynomials with real zeros and real coefficients. This implies that the only zeros of g' are simple, real zeros located in the intervals bounded by the zeros of g. Moreover, direct computation shows that $g'(0) = 0$. To summarize, if we denote the zeros of g' by ξ_k, with $\xi_1 < \xi_2 < \xi_3 < \ldots$, then

(i) $\xi_1 = 0$ and $\xi_k \in \left((k - \frac{1}{2})^2\pi^2, (k + \frac{1}{2})^2\pi^2\right)$ for $k \geq 2$.

We see that g has local maxima at ξ_{2k} and local minima at ξ_{2k-1} for $k \in \mathbb{N}$. For $x \geq \xi_2 \geq \frac{9}{4}\pi^2$ we have

$$|g(x)| \leq \left| 1 + \frac{\pi^4}{(\pi^2 - 8)(4x - \pi^2)} \right| \leq 1 + \frac{\pi^2}{8(\pi^2 - 8)} < \frac{4}{\pi^2 - 8} = -\frac{g(0)}{2}.$$

This implies that

(ii) $g(x) > g(0)$ for $x \in \mathbb{R} \setminus \{0\}$ and $g(x) < -\frac{1}{2}g(0)$ for $x \in [0, \infty)$.

Now we consider the function h. We have

$$h'(z) = M(z) - \frac{8}{(\pi^2 - 8)z^2} + R(z)\cos\sqrt{z}$$

where

$$M(z) := -\frac{z + \frac{2\pi^2}{\pi^2 - 8}}{z\left(z - \frac{\pi^2}{4}\right)} \cdot \frac{\sin\sqrt{z}}{2\sqrt{z}}$$

and

$$R(z) := -\frac{z^2 + \frac{4\pi^2}{\pi^2 - 8}z - \frac{\pi^4}{2(\pi^2 - 8)}}{z^2\left(z - \frac{\pi^2}{4}\right)^2} = O\left(\frac{1}{|z|^2}\right)$$

as $z \to \infty$. A computation shows that $h'(\pi^2) > 0$ and $(-1)^{k+1}h'\left(\left(k+\frac{1}{2}\right)^2\pi^2\right) > 0$ for $k \geq 2$. The intermediate value theorem implies that h' has a zero in the interval $\left(\pi^2, \left(\frac{5\pi}{2}\right)^2\right)$ and in each interval $\left((k+\frac{1}{2})^2\pi^2, (k+\frac{3}{2})^2\pi^2\right)$ for $k \geq 2$. To show that there is only one zero in each such interval and that there are no further zeros of h', we consider for $k \in \mathbb{N}$ and $T > 0$ the simple closed curve

$$\Gamma := \left\{\left(\left(k+\tfrac{1}{2}\right)\pi + iy\right)^2 : |y| \leq T\right\} \cup \left\{(x \pm iT)^2 : 0 \leq x \leq \left(k+\tfrac{1}{2}\right)\pi\right\}.$$

If k and T are large enough, then we have

$$|h'(z) - M(z)| = \left| -\frac{8}{(\pi^2-8)z^2} + R(z)\cos\sqrt{z}\right| < |M(z)|$$

for $z \in \Gamma$. Rouché's theorem implies that the number of zeros of h' in the interior of Γ equals the number of zeros minus the number of poles of M in the interior of Γ. Now M has $k+1$ zeros at $-\frac{2\pi^2}{\pi^2-8}, \pi^2, 4\pi^2, \ldots, k^2\pi^2$ and two poles at 0 and $\frac{\pi^2}{4}$ there. Hence h' has $k-1$ zeros in the interior of Γ, and we conclude that h' has no zeros besides the ones obtained from the intermediate value theorem. To summarize, if we denote the zeros of h' by η_k, with $\eta_1 < \eta_2 < \eta_3 < \ldots$, then

(iii) $\eta_1 \in \left(\pi^2, \left(\frac{5\pi}{2}\right)^2\right)$ and $\eta_k \in \left((k+\frac{1}{2})^2\pi^2, (k+\frac{3}{2})^2\pi^2\right)$ for $k \geq 2$.

We now find that h has local maxima at η_{2k-1} and local minima at η_{2k} for $k \in \mathbb{N}$. Using (ii) we find for $x \geq \eta_3 \geq \frac{49}{4}\pi^2$ that

$$h(x) \leq \frac{-\frac{3}{2}g(0)}{\frac{49}{4}\pi^2} = \frac{48}{49\pi^2(\pi^2-8)} < \frac{32}{25\pi^2(\pi^2-8)} = h\left(\tfrac{25}{4}\pi^2\right) \leq h(\eta_1).$$

It follows that h attains its maximum on $\mathbb{R}$ at η_1. Combining this with (ii) we see that

(iv) $0 < h(x) \leq h(\eta_1)$ for $x \in (0, \infty)$.

Next we observe that h' has order $\frac{1}{2}$. This implies that h' also belongs to the Laguerre-Pólya class. It follows that the only zeros of h'' are simple zeros in the intervals (η_k, η_{k+1}), $k \in \mathbb{N}$. In particular, $h''(x) \neq 0$ for $x < \eta_1$. Thus

(v) h' is decreasing on the interval $(-\infty, \eta_1]$.

Finally we consider the function f. Of course, the η_k are also the zeros of f', and since $h'(0) > 0$ we conclude that (iv) and (v) hold with h replaced by f. We have $f(0) = 0$, $f'(0) = 1$ and $f''(0) < 0$. Thus 0 is a fixed point of f of multiplier 1, with one immediate parabolic basin associated to it. We denote this parabolic basin by U. From (v) we obtain $0 < f'(x) < 1$ for $x \in (0, \eta_1)$ and $f'(x) > 1$ for $x < 0$. This implies that $f^n(x) \to -\infty$ as $n \to \infty$ if $x < 0$ and that $(0, \eta_1] \subset U$. Using (iv) we see that in fact $(0, \infty) \subset U$.

It is not difficult to see that $f(x) \to 0$ as $x \to \infty$, $x \in \mathbb{R}$. It follows from the result of Eremenko and Lyubich [3] mentioned in the introduction, or the Denjoy-Carleman-Ahlfors theorem [4, §XI.4], or direct computation that 0 is an indirect singularity. The Denjoy-Carleman-Ahlfors theorem also implies that f has no other (finite) asymptotic values. Thus $\text{sing}(f^{-1}) \subset [0, f(\eta_1)] \subset [0, \infty)$.

Next we show that the positive real axis is the only curve where f tends to zero and is real and positive. More precisely, we show that there does not exist a curve

$\gamma : [0,\infty) \to \mathbb{C} \setminus [0,\infty)$ with $\gamma(t) \to \infty$ and $f(\gamma(t)) \to 0$ as $t \to \infty$ such that $f(\gamma(t)) \in (0,\infty)$ for all t. In fact, suppose that such a curve γ exists and let

$$L := \left\{ (u+iv)^2 : |u| \geq 1, |v| \leq 4 \log |u| \right\}.$$

For $z = (u+iv)^2 \in \mathbb{C} \setminus L$ we have

$$|\cos \sqrt{z}| = |\cos(u+iv)| \geq \frac{1}{2}\left(e^{|v|} - 1\right) \geq \frac{1}{2}\left(u^4 - 1\right)$$

and

$$|\cos \sqrt{z}| \geq \frac{1}{2}\left(e^{|v|} - 1\right) \geq \frac{1}{48} v^4.$$

Because $|z|^2 = (u^2+v^2)^2 \leq 4 \max\left\{u^4, v^4\right\}$ we obtain $|\cos\sqrt{z}| \geq \frac{1}{192}|z|^2 - \frac{1}{2}$ for $z \in \mathbb{C} \setminus L$ and thus $|f(z)| \to \infty$ as $z \to \infty$, $z \in \mathbb{C} \setminus L$. Hence $\gamma(t) \in L$ for sufficiently large t.

For $k \in \mathbb{N}$ we define

$$\begin{aligned} L_k &:= L \cap \left\{ \left(\left(2k + \tfrac{1}{2}\right)\pi + iv\right)^2 : v \in \mathbb{R} \right\} \\ &= \left\{ \left(\left(2k + \tfrac{1}{2}\right)\pi + iv\right)^2 : |v| \leq 4 \log\left(\left(2k + \tfrac{1}{2}\right)\pi\right) \right\}. \end{aligned}$$

Then γ intersects L_k for sufficiently large k, say $\gamma(t_k) = \left(\left(2k + \frac{1}{2}\right)\pi + iv_k\right)^2 \in L_k$.

We note that for $z = (u+iv)^2 \in L$ we have $|v| = O(\log |u|) = o(|u|)$ and hence

$$|z| = u^2 + v^2 = u^2 \left(1 + O\left(\left(\frac{\log |u|}{|u|}\right)^2\right)\right)$$

as $z \to \infty$. We thus have

$$\Re\left(\frac{1}{z}\right) = \frac{u^2 - v^2}{|z|^2} = \frac{1}{u^2}\left(1 + O\left(\left(\frac{\log |u|}{|u|}\right)^2\right)\right)$$

and

$$\Im\left(\frac{1}{z}\right) = \frac{-2uv}{|z|^2} = -\frac{2v}{u^3}\left(1 + O\left(\left(\frac{\log |u|}{|u|}\right)^2\right)\right).$$

Now

$$h(z) = \left(\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right)\cos\sqrt{z} - \frac{g'(0)}{z}$$

as $z \to \infty$. With $z_k := \gamma(t_k) = \left(\left(2k + \frac{1}{2}\right)\pi + iv_k\right)^2 \in L_k$ we have $\cos\sqrt{z_k} = \cos\left(\frac{\pi}{2} + iv_k\right) = -i \sinh v_k$ and thus we obtain

$$\begin{aligned} \Im h(z_k) &= -\sinh v_k \frac{1}{\left(2k + \frac{1}{2}\right)^2 \pi^2}\left(1 + O\left(\left(\frac{\log k}{k}\right)^2\right)\right) \\ &\quad + v_k \frac{2g'(0)}{\left(2k + \frac{1}{2}\right)^3 \pi^3}\left(1 + O\left(\left(\frac{\log k}{k}\right)^2\right)\right). \end{aligned}$$

It follows that $\Im f(z_k) = (\Im h(z_k))/h'(0) \neq 0$ for sufficiently large k, a contradiction. Hence a curve γ as above does not exist.

We now show that U is completely invariant. We assume that this is not the case. Then $f^{-1}(U)$ has a component $V \neq U$. Since the zeros of f' are real and positive, and since 0 is the only asymptotic value, f maps V biholomorphically onto U. Let φ be the branch of f^{-1} which maps U to V. Then $\varphi(x) \not\to \infty$ as $x \to 0$, $x > 0$, since otherwise φ would define a curve tending to infinity where

f is real and positive and tends to 0. Thus φ accumulates at some finite point; that is, for $A := \cap_{\varepsilon>0}\overline{\varphi((0,\varepsilon])}$ we have $A \cap \mathbb{C} \neq \emptyset$. Since $A \cap \mathbb{C}$ can consist only of zeros of f and since A is connected, we in fact have $A = \{z_0\}$ for some zero z_0 of f. Thus $\varphi(t) \to z_0$ as $t \to z_0$. Since 0 is not a critical value of f we have $f'(z_0) \neq 0$ and thus φ has an analytic continuation to a neighborhood of z_0. Since $\text{sing}(f^{-1}) \subset [0,\infty) \subset U \cup \{0\}$ we see that φ can also be continued analytically to any point of $\mathbb{C} \setminus U$. This implies that φ extends to an entire function. Since φ is also univalent, we deduce that φ and hence f are linear, a contradiction.

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