On proper analytic maps with one critical point

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ABSTRACT. Let $U \subset \mathbb{C}$ be a domain containing $\{z \in \mathbb{C} : \operatorname{Re} z > \sigma, |\arg z| < \eta\}$ for some $\sigma, \eta > 0$ and let $f : U \to U$ be a proper holomorphic map satisfying $f(z) = z + 1 + a/z + o(1/|z|)$ as $z \to \infty$, $|\arg z| < \eta$, with $a \in \mathbb{C}$. We show that if U contains only one critical point of f, and this critical point is simple, then $\operatorname{Re} a \geq \frac{1}{4}$. This slightly generalizes a previous result concerning critical points in Leau domains. We also show that the condition $\operatorname{Re} a \geq \frac{1}{4}$ is sharp.

1. Introduction and results

Let f be a rational function with a fixed point $z_0 \in \mathbb{C}$. Then the function $h(z) := 1/(z - f(z))$ has a pole at z_0. The residue of h at z_0 is called the *residue fixed point index* and denoted by $\iota(f, z_0)$; see Milnor's book [**5**, §12] for a discussion of this concept. Clearly, if $f'(z_0) \neq 1$, then $\iota(f, z_0) = 1/(1 - f'(z_0))$. If $f'(z_0) = 1$, then z_0 is a multiple pole of h, say of multiplicity $m + 1$ with $m \in \mathbb{N}$. We also say that z_0 is a multiple fixed point of f of multiplicity $m + 1$. The residue fixed point index and, of course, the multiplicity of a fixed point are invariant under holomorphic changes of variables [**5**, Lemma 12.3]. This is used to define them for $z_0 = \infty$.

It is known classically (see, e. g., [**5**, §10] or [**7**, §3.5]) that if f has a multiple fixed point of multiplicity $m + 1$ in z_0, then there exist m invariant components $U_1, U_2, \ldots, U_m$ of the Fatou set of f such that $z_0 \in \partial U_k$ and $f^n|_{U_k} \to z_0$ as $n \to \infty$ for each $k \in \{1, 2, \ldots, m\}$. These domains U_k are called the *Leau domains* of f at z_0. Moreover, it is known that each Leau domain of f contains at least one critical point of f.

The following result was obtained (independently) in [**1**] and [**2**].

THEOREM A. *Let f be a rational function with a multiple fixed point z_0 of multiplicity $m + 1$. Suppose that each Leau domain of f at z_0 contains only one critical point of f, and that this critical point is simple. Then*

$$\operatorname{Re} \iota(f, z_0) \leq \frac{m}{4} + \frac{1}{2}. \tag{1.1}$$

The special case $m = 1$ of Theorem A had been stated (without proof) already earlier in [**6**].

2000 *Mathematics Subject Classification.* Primary 37F10; Secondary 30D05, 37F45.
Supported by G.I.F., G -643-117.6/1999, by INTAS-99-00089, and by DFG, BE 1508/3-1.

We remark that the methods of [**1**] and [**2**] also work if the critical points in the Leau domains are multiple. If we denote the multiplicity of the critical point in U_k by M_k, then the conclusion of Theorem A holds with (1.1) replaced by

$$\operatorname{Re}\iota(f,z_0) \leq \frac{7}{20}m + \frac{1}{2} - \frac{3}{10}\sum_{k=1}^{m}\frac{1}{M_k^2+2M_k}. \tag{1.2}$$

Note that (1.2) reduces to (1.1) if $M_k = 1$ for all k.

For simplicity we shall restrict ourselves in the following to the case of simple critical points, even though the results obtained below (Theorems 1.1 and 1.2) also extend to the case of multiple critical points.

Even though some of the underlying ideas in [**1**] and [**2**] are similar, there are also various differences in the proofs. One advantage of the method of [**2**] is that it leads to improved (though probably not sharp) bounds for polynomials. On the other hand, the method of [**1**] has the advantage that it uses only the asymptotics of f in the Leau domain and does not require that f is analytic in a neighborhood of z_0. For example, the method may be applicable to Baker domains of entire or meromorphic functions.

To explain this in more detail, we specialize Theorem A for simplicity to the case $m = 1$ and assume that $z_0 = \infty$; that is, we consider rational functions f with a double fixed point at ∞. After a normalization we may assume that

$$f(z) = z + 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

in a neighborhood of ∞. (This form can be achieved by conjugation with a linear map.) Then $a_1 = 1 - \iota(f,\infty)$. Theorem A says that if the Leau domain of f at ∞ contains only one critical point, and this critical point is simple, then $\operatorname{Re} a_1 \geq \frac{1}{4}$.

The method of [**1**] actually gives the following result.

THEOREM 1.1. *Let $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, be a simply-connected domain containing $\{z \in \mathbb{C} : \operatorname{Re} z > \sigma, |\arg z| < \eta\}$ for some $\sigma, \eta > 0$ and let $f : U \to U$ be a proper holomorphic map satisfying*

$$f(z) = z + 1 + \frac{a}{z} + o\left(\frac{1}{|z|}\right) \tag{1.3}$$

for some $a \in \mathbb{C}$ as $z \to \infty$, $|\arg z| < \eta$. Suppose that U contains exactly one critical point of f, and that this critical point is simple. Then $\operatorname{Re} a \geq \frac{1}{4}$.

We note that the usual arguments ([**5**, §10], [**7**, §3.5]) for the existence of critical points in Leau domains imply that f as in Theorem 1.1 has *at least* one critical point in U. The hypotheses thus means that f has *only* one critical point.

We also remark that it is not necessary to make the hypothesis that U be simply-connected. This follows also from the other hypotheses made; cf. [**1**, Lemma 4] and [**7**, Exercises 3.4.6 and 3.5.10].

For completeness, we include a proof of Theorem 1.1 in §3. The basic idea of this proof is the same as in [**1**], but the argument is arranged somewhat differently.

The following result shows that Theorem 1.1 is sharp.

THEOREM 1.2. *Let $a \in \mathbb{C}$ with $\operatorname{Re} a \geq \frac{1}{4}$. Then there exists $\sigma > 0$, a simply-connected domain $U \neq \mathbb{C}$ containing $\{z \in \mathbb{C} : \operatorname{Re} z > \sigma\}$ and a proper holomorphic map $f : U \to U$ satisfying (1.3) and having exactly one critical point, which is simple.*

The methods of [**2**] show that if, under the hypotheses of Theorem A, we have equality in (1.1), then $m = 1$ and $\iota(f, z_0) = \frac{3}{4}$ or $m = 2$ and $\iota(f, z_0) = 1$. Thus the function f in Theorem 1.2 cannot be rational if $\operatorname{Re} a = \frac{1}{4}$ and $\operatorname{Im} a \neq 0$. I do not know whether the function f in Theorem 1.2 can always be chosen to be rational $\operatorname{Re} a > \frac{1}{4}$.

2. Preliminary Lemmas

LEMMA 2.1. *Let $U \subset \mathbb{C}$ be a domain containing $\{z \in \mathbb{C} : \operatorname{Re} z > \sigma, |\arg z| < \eta\}$ for some $\sigma, \eta > 0$ and let $f : U \to U$ be a holomorphic map satisfying* (1.3) *as $z \to \infty$. Then $f^n(z) = n + a \log n + o(\log n)$ as $n \to \infty$ for fixed $z \in U$.*

This result is known; see, e. g., Fatou's memoir [**3**, §9] for a proof of this result under the slightly stronger assumption that $o(1/|z|)$ in (1.3) is replaced by $O(1/|z|^\gamma)$ with $\gamma > 1$. (Fatou actually shows that then $f^n(z) - n - a \log n$ tends to a limit as $n \to \infty$, and this limit is a solution of Abel's functional equation.) For completeness, we include a proof of the above result.

PROOF OF LEMMA 2.1. It follows easily from (1.3) that $f^k(z) \sim k$ as $k \to \infty$, say $f^k(z) = k + \tau(k)$ with $\tau(k) = o(k)$ as $k \to \infty$. Substituting this in (1.3) yields

$$\tau(k+1) = \tau(k) + \frac{a}{k + \tau(k)} + o\left(\frac{1}{k}\right) = \tau(k) + (a + o(1))\frac{1}{k}.$$

It follows that

$$\tau(n) = \tau(1) + \sum_{k=1}^{n-1} (\tau(k+1) - \tau(k)) = (a + o(1)) \sum_{k=1}^{n-1} \frac{1}{k} = a \log n + o(\log n)$$

as $n \to \infty$. □

LEMMA 2.2. *Let $S := \mathbb{C} \backslash (-\infty, 0]$ and let $U \subset \mathbb{C}$ be a simply-connected domain with $0 \notin U$. If $\psi : S \to U$ is biholomorphic, then $|\psi(x)|/x$ is decreasing for $x \in (0, \infty)$.*

PROOF. Fix $x \in (0, \infty)$ and define a biholomorphic map h from the unit disk onto S by $h(z) := x\left(\frac{1+z}{1-z}\right)^2$. Then $F := \psi \circ h$ is a biholomorphic map from the unit disk onto U. By Koebe's one quarter theorem, U contains the disk of radius $\frac{1}{4}|F'(0)|$ around $F(0)$. Since $0 \notin U$ this implies $\frac{1}{4}|F'(0)| \leq |F(0)|$. Because $F'(0) = \psi'(h(0))h'(0) = 4x\psi'(x)$ and $F(0) = \psi(x)$ we obtain

$$\left|\frac{\psi'(x)}{\psi(x)}\right| \leq \frac{1}{x}.$$

It follows that

$$\frac{d}{dx} \log \frac{|\psi(x)|}{x} = \operatorname{Re}\left(\frac{\psi'(x)}{\psi(x)}\right) - \frac{1}{x} \leq 0.$$

□

An immediate consequence is the following result.

LEMMA 2.3. *If ψ is as in the previous lemma and if $|\psi(x)|/x \to 1$ as $x \to \infty$, then $|\psi(x)| \geq x$.*

The following result is easy to prove. It can already be found in Fatou's memoir [**4**, p. 310].

Lemma 2.4. *The function* $g(z) := z + 1 + \frac{1}{4z}$ *has a fixed point of multiplicity* 2 *at* ∞ *and* $S := \mathbb{C}\backslash(-\infty, 0]$ *is the Leau domain of* g *at* ∞. *The only critical point of* g *contained in* S *is the simple critical point* $\frac{1}{2}$.

3. Proof of Theorem 1.1

Without loss of generality we may assume that $0 \notin U$. There exists a biholomorphic map $\psi : S \to U$ such that $\psi(\frac{1}{2})$ is the critical point of f and $\psi(x) \to \infty$ as $x \to \infty$, $x \in \mathbb{R}$. It can be deduced from Lemma 2.4 (see [**1**, Lemma 6] for the details of this argument) that $\psi \circ g = f \circ \psi$. This implies that $\psi(g^n(z)) = f^n(\psi(z))$ for $z \in S$ and $n \in \mathbb{N}$. Combining this with Lemmas 2.1 and 2.2 we see that $\psi(x)/x \to 1$ as $x \to \infty$, $x \in \mathbb{R}$. It follows from Lemma 2.3 that

$$g^n(1) \le |\psi(g^n(1))| = |f^n(\psi(1))|.$$

Lemma 2.1 implies that

$$n + \frac{1}{4}\log n \le |n + a\log n| + o(\log n).$$

It follows that $\operatorname{Re} a \ge \frac{1}{4}$. □

4. Proof of Theorem 1.2

Let $R > 0$ and $S_R := \{z \in S : |z| > R\} = \{z \in \mathbb{C} : |z| > R \text{ and } z \notin (-\infty, -R)\}$. The map $\phi : S_R \to S$ defined by $\phi(z) := z + R^2/z - 2R$ is biholomorphic with $\phi^{-1}(z) = R + \frac{1}{2}z + \frac{1}{2}\sqrt{4Rz + z^2}$. With g as in Lemma 2.4 we define $h := \phi^{-1} \circ g \circ \phi$. A computation shows that

$$h(z) = z + 1 + \frac{1}{4z} + \frac{2R^2 + R}{z^2} + O\left(\frac{1}{|z|^3}\right)$$

as $z \to \infty$. By Lemma 2.4, h is a proper selfmap of S_R.

We define $b := a - \frac{1}{4}$. Then $\operatorname{Re} b \ge 0$ by hypothesis. We claim that there exists $c \in \mathbb{R}$ such that

$$\psi(z) := z + b\log z + c\frac{\log z}{z}$$

is univalent in S_R for large R. (Here log denotes the principal branch of the logarithm.) To see this we note first that $\psi'(z) \to 1$ as $z \to \infty$ and thus it follows easily that ψ is univalent in any convex subdomain of S_R for sufficiently large R.

We define curves $\gamma_\pm : (R, \infty) \to \mathbb{C}$ by

$$\gamma_\pm(r) := \lim_{\theta \to \pm\pi} \psi(re^{i\theta}) = -r + b(\log r \pm i\pi) - c\frac{\log r \pm i\pi}{r}$$

and put $\beta := \operatorname{Im} b$. We shall show that $\operatorname{Im}\gamma_+(r) > \beta\log|\operatorname{Re}\gamma_+(r)|$ and $\operatorname{Im}\gamma_-(r) < \beta\log|\operatorname{Re}\gamma_-(r)|$ for $r > R$, if c and R suitably chosen. This means that γ_+ is "above" the curve given by $\operatorname{Im} z = \beta\log|\operatorname{Re} z|$ and γ_- is "below" this curve. The univalence of ψ then follows from this.

If $\operatorname{Re} b = 0$ so that $b = i\beta$, then we choose $c \in \mathbb{R}$ with $c < -\beta^2$. Without loss of generality we may assume that $\beta \geq 0$. We then have

$$\begin{aligned}
\beta \log|\operatorname{Re}\gamma_+(r)| &= \beta\log|-r-\beta\pi - c\frac{\log r}{r}| \\
&\leq \beta\log(r+\beta\pi) \\
&= \beta\left(\log r + \log\left(1+\frac{\beta\pi}{r}\right)\right) \\
&\leq \beta\left(\log r + \frac{\beta\pi}{r}\right) \\
&< \beta\log r - \frac{c\pi}{r} \\
&= \operatorname{Im}\gamma_+(r)
\end{aligned}$$

for $r \geq 1$. Choosing $\beta', \beta'' \in \mathbb{R}$ with $\beta < \beta' < \beta''$ and $c < -\beta\beta''$ we also have

$$\begin{aligned}
\beta \log|\operatorname{Re}\gamma_-(r)| &= \beta\log|-r+\beta\pi - c\frac{\log r}{r}| \\
&\geq \beta\log(r-\beta'\pi) \\
&= \beta\left(\log r + \log\left(1-\frac{\beta'\pi}{r}\right)\right) \\
&\geq \beta\left(\log r - \frac{\beta''\pi}{r}\right) \\
&> \beta\log r + \frac{c\pi}{r} \\
&= \operatorname{Im}\gamma_-(r)
\end{aligned}$$

for large r.

If $\operatorname{Re} b > 0$, then we can choose $c = 0$ and obtain, for large R, the desired estimates by a similar (and in fact simpler) computation. It follows that ψ is univalent if c is chosen as above and R is sufficiently large.

We now define $U := \psi(S_R)$ and $f := \psi \circ h \circ \psi^{-1}$. Then f is a proper holomorphic selfmap of U. A lenghty computation shows that

$$\psi^{-1}(z) = z - b\log z + (b^2 - c)\frac{\log z}{z} + O\left(\left|\frac{\log z}{z}\right|^2\right)$$

as $z \to \infty$ and this, again with a lenghty computation, implies that f has the required asymptotics. □

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