RESCALING PRINCIPLES IN FUNCTION THEORY

WALTER BERGWEILER

ABSTRACT. We survey some applications of Zalcman's Lemma and the Wiman-Valiron method. We point out that some of the underlying ideas are similar.

1. INTRODUCTION

Let f be a function meromorphic in some domain and let L and M be affine functions. Then we call $L \circ f \circ M$ a *rescaling* of f. We shall be interested in the case where for a sequence (f_k), and associated sequences (L_k) and (M_k), the sequence $(L_k \circ f_k \circ M_k)$ converges to a function f which is meromorphic in the complex plane $\mathbb{C}$. Having information about the sequence (f_k) we may then draw some conclusions about f and vice versa.

This kind of reasoning occurs at various instances in function theory. Here we shall survey some of these, focussing on Zalcman's Lemma and the Wiman-Valiron method, but also including a brief discussion of Poincaré functions. We discuss applications to complex differential equations, value distribution and the Ahlfors theory of covering surfaces. We shall see that rescaling is always one of the underlying ideas.

2. THREE RESCALING PRINCIPLES

2.1. Zalcman's Lemma. We shall discuss the following result.

Zalcman's Lemma. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \mathbb{C}$ and let $m \in \mathbb{N}$, $K \geq 0$ and $\alpha \in \mathbb{R}$ with $-m \leq \alpha < 1$. Suppose that the zeros of the functions in $\mathcal{F}$ have multiplicity at least m; that is, if $f \in \mathcal{F}$ and $\xi \in D$ with $f(\xi) = 0$, then $f^{(k)}(\xi) = 0$ for $1 \leq k \leq m-1$. If $\alpha = -m$, then suppose in addition that $|f^{(m)}(\xi)| \leq K$ if $f \in \mathcal{F}$, $\xi \in D$ and $f(\xi) = 0$.*

Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then there exist a sequence (f_k) in $\mathcal{F}$, a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers and a non-constant function f which is meromorphic in $\mathbb{C}$ such that $z_k \to z_0$, $\rho_k \to 0$ and

$$\rho_k^{\alpha} f_k(z_k + \rho_k z) \to f(z)$$

locally uniformly in $\mathbb{C}$. Moreover, the spherical derivative $f^{\#} := |f'|/(1+|f|^2)$ of f satisfies $f^{\#}(z) \leq f^{\#}(0) = mK + 1$ for all $z \in \mathbb{C}$. In particular, f has finite order.

It follows from the statement of the lemma that if $\alpha = K = 0$, then $\rho_k f_k^{\#}(z_k) \to f^{\#}(0) = 1$. We note, however, that we can actually achieve that $\rho_k f_k^{\#}(z_k) = 1$ for all $k \in \mathbb{N}$. In fact, in the proof f_k and z_k are choosen first, and then ρ_k is defined by this equation.

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We also note that $\{1/f : f \in \mathcal{F}\}$ is normal if and only if $\mathcal{F}$ is normal. Thus we obtain an analogous result for $-1 < \alpha \leq \ell$, if the poles of the functions in $\mathcal{F}$ have multiplicity at least ℓ, with an additional hypotheses if $\alpha = \ell$. Note that no hypothesis on the zeros or poles is required if $-1 < \alpha < 1$.

The case $\alpha = 0$ of this lemma was first formulated by Zalcman [60], and thus the result is – as here – often named after him. The analogous result for normal functions had been proved earlier by Lohwater and Pommerenke [38], and Zalcman [60] gives major credit to Pommerenke for the idea. Pommerenke told me that he got the idea from reading Landau [35]. Landau in turn credits Valiron [53] with this idea. Of course, what can be found in Valiron's paper is still far away from the statement of the lemma in Zalcman's paper.

Having discussed the history of the Zalcman's lemma for the case $\alpha = 0$, we now turn to the extensions concerned with the case $\alpha \neq 0$. The idea to consider such an extension seems to be due to Pang [43, 44], who proved that one can always take $-1 < \alpha < 1$. Pang and Xue [59] showed that $\alpha < 0$ is admissible if the functions in $\mathcal{F}$ have no zeros. Then Chen and Gu [20, Theorem 2] proved that one can take $-m < \alpha \leq 0$ if the zeros of the functions in $\mathcal{F}$ have multiplicity at least m. Finally, the case $\alpha = -m$ is due to Pang and Zalcman [45, Lemma 2].

We see that Zalcman's Lemma is a beautiful example to show how mathematics is the work of many people, each one adding a new idea to the ideas of the predecessors.

Zalcman's motivation for the above lemma was a heuristic principle which states that if P is a "property" such that every function meromorphic in $\mathbb{C}$ which has property P is constant, then the family of all functions meromorphic in some domain which have property P is likely to be normal. The prime example here is of course the property to omit the values $0, 1$ and ∞, or any other fixed three values. Then the statement about functions in the plane is Picard's Theorem and the one about normal families is Montel's Theorem. We mention that the heuristic principle also works in the other direction: from a normality criterion one can often obtain a result about functions meromorphic in the plane. In fact, this is for many properties a rather simple argument.

The heuristic principle is usually attributed to Bloch, but does not seem to have been stated explicitly by him. (I would be reluctant to interpret his statement "Nihil est in infinito quod non prius fuerit in finito" in [14] this way.) The first explicit formulation of the heuristic principle seems to be due to Valiron [54, p. 2]. In [55, p. 4] Valiron mentions Bloch in this context.

Of course, the principle is not a rigorous statement, since the term "property" is not precisely defined. In fact, there are various counterexamples to the principle. On the other hand, Zalcman's lemma makes the heuristic principle a rigorous theorem for many properties. Roughly speaking, this is the case for properties which are preserved under appropriate rescaling and taking limits.

The fact that the limit function in Zalcman's Lemma has bounded spherical derivative can also be exploited successfully: assuming that there exists a nonconstant function meromorphic in the plane with a certain property, we find that the family of all such functions is not normal, and then obtain a function meromorphic in the plane which has this property and bounded spherical derivative. Thus to prove a result about functions meromorphic in the plane it sometimes suffices to prove it for functions with bounded spherical derivative. The first instance where Zalcman's Lemma was used this way – independently and almost simultaneously in [13, 19, 61] – seems to be the proof of Hayman's conjecture that if f is meromorphic in the plane and $f'f$ omits a finite non-zero value, then f is constant.

An excellent discussion of the heuristic principle and other applications of Zalcman's Lemma like the one just mentioned can be found in [62]. Some further applications not covered in [62] will be given in §4 and §5 below.

2.2. **The Wiman-Valiron method.** Let $f(z)=\sum_{k=0}^{\infty}a_k z^k$ be a transcendental entire function, $r>0$, $M(r,f):=\max_{|z|=r}|f(z)|$ the *maximum modulus*, $\mu(r,f):=\max_{k\geq 0}|a_k|r^k$ the *maximum term*, and $\nu(r,f):=\max\{k:\mu(r,f)=|a_k|r^k\}$ the *central index*. Note that $\mu(r,f)\leq M(r,f)$ by Cauchy's inequalities.

Wiman-Valiron Theorem. *There exists a set $F\subset[1,\infty)$ satisfying*

$$\int_F \frac{dt}{t}<\infty \tag{1}$$

with the following property: if (z_k) is a sequence in $\mathbb{C}$ with $|f(z_k)|=M(|z_k|,f)$, $|z_k|\notin F$, and $z_k\to\infty$, and if $\nu_k:=\nu(|z_k|,f)$, then

$$\frac{f\left(z_k+\frac{z_k}{\nu_k}z\right)}{f(z_k)}\to e^z \tag{2}$$

as $k\to\infty$.

The result goes back to Wiman [56, 57] and Valiron [51, 52]. In their papers, as well as in more recent accounts of the theory [30, 32, 34], the result is usually stated somewhat differently, but the above form easily follows from the statements there. In fact, the usual statements are more precise in the sense that they give explicit estimates for the difference between the left and right side of (2). Also, instead of assuming that $|f(z_k)|=M(|z_k|,f)$ they only require that $|f(z_k)|$ is sufficiently large compared to $M(|z_k|,f)$. The sharpest estimates seem to be due to Hayman [30]. While such estimates are important for some applications of the Wiman-Valiron method, they are not required for the applications considered here.

The result can be interpreted as follows: near points of maximum modulus an entire function looks like a rescaled exponential function.

With a look at the applications in §3 we note that [32, Satz 4.3]

$$\log\mu(r,f)=\int_0^r\frac{\nu(t,f)}{t}dt+\log|f(0)|,$$

with a slight modification if $f(0)=0$. Hence

$$\int_r^\infty\frac{\nu(t,f)}{t(\log\mu(t,f))^{1+\varepsilon}}dt=\frac{1}{\varepsilon(\log\mu(r,f))^{\varepsilon}}<\infty$$

for $\varepsilon>0$, which implies that [32, Satz 21.2]

$$\nu(t,f)\leq(\log\mu(t,f))^{1+\varepsilon}$$

and hence

$$\log\nu(t,f)\leq(1+\varepsilon)\log\log\mu(t,f)\leq(1+\varepsilon)\log\log M(t,f)=o(\log M(t,f))$$

for t outside some set F satisfying (1). Thus we may assume in the Wiman-Valiron Theorem that

$$\log\nu_k=o(\log|f(z_k)|). \tag{3}$$

Finally, it is not difficult so see (cf. [32, Satz 4.5]) that if f has infinite order, then there exists $E \subset [1, \infty)$ with $\int_E dt/t = \infty$ such that $\log \nu(r)/\log r \to \infty$ as $r \to \infty$, $r \in E$. Choosing $|z_k| \in E \cap F$ in the Wiman-Valiron theorem we thus find a sequence (z_k) which besides (2) and (3) also satisfies

$$\log |z_k| = o(\log \nu_k). \tag{4}$$

2.3. **Poincaré functions.** We denote the k-th iterate of a function f by f^k. The following result is well-known; see, e. g., [39, Corollary 8.10].

Schröder-Kœnigs-Poincaré Theorem. *Let f be a rational or entire function and let $z_0 \in \mathbb{C}$ be a fixed point of f. Define $\lambda := f'(z_0)$ and suppose that $|\lambda| > 1$. Then*

$$f^k\left(z_0 + \lambda^{-k} z\right) \to \phi(z)$$

locally uniformly in $\mathbb{C}$ for some function ϕ which is meromorphic in $\mathbb{C}$ and satisfies the functional equation

$$\phi(\lambda z) = f(\phi(z)) \tag{5}$$

for $z \in \mathbb{C}$. If f is entire, then so is ϕ.

The equation (5) was considered first by Schröder [50], then Kœnigs [33] proved that it has a solution ϕ analytic in a neighborhood of 0 if $\lambda \neq 0$ and $|\lambda| \neq 1$, and Poincaré [46] observed that (5) permits analytic continuation of ϕ to the whole plane if $|\lambda| > 1$. Thus ϕ is sometimes called a *Poincaré function* to f, but it is also common to name these functions after Kœnigs or Schröder.

Poincaré functions are important in complex dynamics; see, e. g., [22] for an application concerning accessibility of periodic points of polynomials from the basin of attraction of ∞. The value distribution of Poincaré functions is studied in [23].

3. Complex differential equations

3.1. **Results of Valiron and Gol'dberg.** Already Wiman and Valiron themselves applied their method to complex differential equations; see [57, § 5] and [52, §IV.6]. Today the applications of the Wiman-Valiron method to complex differential equations are numerous; see the books by Jank and Volkmann [32] and Laine [34] for a thorough treatment. Here we only mention the following result already obtained by Valiron himself [52, §IV.6].

Valiron's Theorem. *An entire function which satisfies a first order algebraic differential equation has finite order.*

This result was extended by Gol'dberg [26] to functions meromorphic in the plane.

Gol'dberg's Theorem. *A function meromorphic in the plane which satisfies a first order algebraic differential equation has finite order.*

Below we give proofs of these results, and in fact of extensions thereof, using the Wiman-Valiron method and Zalcman's Lemma.

For $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, $r = (r_0, r_1, r_2, \dots, r_n) \in \mathbb{N}_0^{n+1}$ and f meromorphic we define $M_r[f]$ by

$$M_r[f](z) = f(z)^{r_0} f'(z)^{r_1} f''(z)^{r_2} \dots f^{(n)}(z)^{r_n}.$$

We call $d(r) = r_0 + r_1 + r_2 + \cdots + r_n$ the *degree* and $w(r) = r_1 + 2r_2 + \cdots + nr_n$ the *weight* of $M_r[f]$. An algebraic differential equation of order n is an equation of the form

$$\sum_{r\in I} p_r(z)M_r[f](z) = 0, \tag{6}$$

where the p_r are polynomials, $p_r \not\equiv 0$, and $I \subset \mathbb{N}_0^{n+1}$ is a finite index set.

3.2. **Rescaling differential equations.** Suppose now that a meromorphic function f satisfies (6) and that $g_k(z) := \mu_k f(z_k + \rho_k z) \to g(z)$, with $z_k \to \infty$ and $\rho_k = o(|z_k|)$. Then $g_k^{(j)}(z) = \mu_k\rho_k^j f^{(j)}(z_k + \rho_k z) \to g^{(j)}(z)$ for $j \in \mathbb{N}$ and thus $M_r[g_k](z) = \mu_k^{d(r)}\rho_k^{w(r)}M_r[f](z_k + \rho_k z) \to M_r[g](z)$. Let $p_r(z) \sim a_r z^{q(r)}$ as $z \to \infty$. Then $p_r(z_k + \rho_k z) = (1 + \varepsilon_{r,k}(z))\, a_r z_k^{q(r)}$ with $\varepsilon_{r,k}(z) \to 0$ as $k \to \infty$. We write (6) in the form

$$\sum_{r\in I} (1 + \varepsilon_{r,k}(z))\, a_r z_k^{q(r)} \mu_k^{-d(r)} \rho_k^{-w(r)} M_r[g_k](z) = 0. \tag{7}$$

For abbreviation we put $b_{r,k} = a_r z_k^{q(r)}\mu_k^{-d(r)}\rho_k^{-w(r)}$. Passing to a subsequence if necessary we may assume that there exists $t \in I$ with $|b_{t,k}| = \max_{r\in I} |b_{r,k}|$ for all $k \in \mathbb{N}$. Moreover, we may assume that $(b_{r,k}/b_{t,k})$ converges for all $r \in I$, say $b_{r,k}/b_{t,k} \to c_r$, with $c_t = 1$ and $|c_r| \le 1$ for all $r \in I$. Dividing (7) by $b_{t,k}$ and taking the limit as $k \to \infty$ we obtain

$$\sum_{r\in I} c_r M_r[g] = 0. \tag{8}$$

3.3. **Wiman-Valiron theory and differential equations.** We now consider the consequences of the above reasoning if f is entire transcendental and rescaling is done according to the Wiman-Valiron Theorem; that is, $\mu_k = 1/f(z_k)$, $\rho_k = z_k/\nu_k$ and $g(z) = e^z$. Assuming that f has infinite order we have $\log|z_k| = o(\log|\rho_k|)$ and $\log|\rho_k| = o(\log|\mu_k|)$ by (3) and (4). This means that μ_k tends to zero faster than any power of ρ_k, and ρ_k tends to zero faster than any negative power of z_k. This implies that

$$\frac{b_{r,k}}{b_{s,k}} = \frac{a_r}{a_s} z_k^{q(r)-q(s)} \mu_k^{d(s)-d(r)} \rho_k^{w(s)-w(r)} \to 0$$

if $d(r) < d(s)$. Similarly, $b_{r,k}/b_{s,k} \to 0$ if $d(r) = d(s)$ and $w(r) < w(s)$ or if $d(r) = d(s)$, $w(r) = w(s)$ and $q(r) < q(s)$. In other words, $b_{r,k}/b_{s,k} \to 0$ if $(d(r), w(r), q(r))$ is less than $(d(s), w(s), q(s))$ with respect to the lexicographical order. Let $J := \{r \in I : d(r) = d(t), w(r) = w(t) \text{ and } q(r) = q(t)\}$; that is, J consists of the indices $r \in I$ for which $(d(r), w(r), q(r))$ is maximal with respect to the lexicographical order. Then $c_r = 0$ for $r \in I\backslash J$ and $c_r = a_r/a_t$ for $r \in J$. Thus (8) takes the form

$$\sum_{r\in J} \frac{a_r}{a_t} M_r[\exp](z) = \sum_{r\in J} \frac{a_r}{a_t} e^{d(r)z} = 0.$$

Hence

$$\sum_{r\in J} a_r = 0. \tag{9}$$

To prove Valiron's Theorem, we now only have to note that if $n = 1$, then the set J contains only one element. This clearly contradicts (9). □

There are many other algebraic differential equations where J has only one element. For example, this is the case for linear differential equations. The above argument thus also shows that entire solutions of linear differential equations have finite order of growth. We note, however, that the Wiman-Valiron method not only yields the finiteness of the order, but also gives precise information on the possible orders; see, e. g., [32, 52].

Using estimates of the remainder term in the Wiman-Valiron method one can improve (9). In fact, Hayman [31, Theorem C] has shown that if (6) has an entire solution of infinite order and if $\Lambda := \{r \in I : d(r) = d(t) \text{ and } w(r) = w(t)\}$, then $\sum_{r\in\Lambda} p_r \equiv 0$. Clearly, this is stronger than (9).

3.4. **Zalcman's Lemma and differential equations.** Following [9] we now consider the case that the rescaling in §3.2 is done according to Zalcman's Lemma. We assume that f meromorphic in $\mathbb{C}$ satisfies (6) and has infinite order. It follows from the Ahlfors-Shimizu form of the characteristic that there does not exist $M > 0$ such that $f^{\#}(z) = O(|z|^M)$. In particular, the family $\{f^{\#}(z+c)\}_{c\in\mathbb{C}}$ is not normal at 0. Applying Zalcman's Lemma with $\alpha = K = 0$ to this family we obtain a sequence (z_k) tending to ∞ and satisfying $\log|z_k| = o(\log f^{\#}(z_k))$ such that $f(z_k + \rho_k z) \to g(z)$ for some non-constant function g meromorphic in the plane, with $\rho_k = 1/f^{\#}(z_k)$. We conclude that (7) holds with $\mu_k = 1$. Similarly as before we find that $b_{r,k}/b_{s,k} \to 0$ if $w(r) < w(s)$ or if $w(r) = w(s)$ and $q(r) < q(s)$. This means that $b_{r,k}/b_{s,k} \to 0$ if $(w(r), q(r))$ is less than $(w(s), q(s))$ with respect to the lexicographical order. With $J := \{r \in I : w(r) = w(t) \text{ and } q(r) = q(t)\}$ we thus find that (8) takes the form

$$\sum_{r\in J} \frac{a_r}{a_t} M_r[g](z) = 0. \tag{10}$$

Suppose now that if $r = (r_0, r_1, r_2, \dots, r_n) \in J$ and $j \geq 2$, then $r_j = 0$. In other words, suppose that the terms in the differential equations that carry the highest weight contain only the first, but no higher derivative. (This condition is trivially satisfied if the differential equation is of the first order.) Then (10) takes the form $P(g)(g')^{w(t)} = 0$ for some polynomial P. This implies that g is constant, a contradiction.

We have thus obtained the following generalization of Gol'dberg's Theorem, first proved by Barsegian ([4, 5], see also [3]).

Barsegian's Theorem. *Let f be a function meromorphic in the plane which satisfies* (6). *Let $w := \max\{w(r) : r \in I\}$ and suppose that if $r = (r_0, r_1, r_2, \dots, r_n) \in I$ with $w(r) = w$, then $r_j = 0$ for $j \geq 2$. Then f has finite order.*

Some refinements of this result have been given by Frank and Wang [25] and Wulan [58].

3.5. **Poincaré functions and differential equations.** We sketch the results concerning differential equations and the Schröder-Kœnigs-Poincaré Theorem only briefly. Following Boshernitzan and Rubel [15] we call a family of meromorphic functions *coherent*, if there exists an algebraic differential equations which is satisfied by all functions in the family. It is not difficult to see that if a family $\mathcal{F}$ is coherent, then so is the family of all rescalings of functions in $\mathcal{F}$. Together with the Schröder-Kœnigs-Poincaré Theorem this implies that if f is rational or entire, and $\{f^k\}_{k\in\mathbb{N}}$ is coherent, then every Poincaré function to f satisfies an algebraic differential equation. Conversely, if a Poincaré function to f satisfies an algebraic differential equation, then $\{f^k\}_{k\in\mathbb{N}}$ is coherent. This can be seen from the equation $f^k(z) = \phi(\lambda^k(\phi^{-1}(z))$ obtained from (5). For example, we can now immediately conclude that if one Poincaré function to f satisfies an algebraic differential equation, then so do all Poincaré functions to f.

Ritt [47] has classified the Poincaré functions to rational functions that satisfy an algebraic differential equation. Thus we obtain a classification of the rational functions whose iterates are coherent. It was shown in [7] that Poincaré functions to transcendental entire functions do not satisfy algebraic differential equations; equivalently, the iterates of a transcendental entire function cannot be coherent. Analogous results hold not only for Poincaré functions, which are solutions of the Schröder functional equation (5), but also for the solutions of the related functional equations of Abel and Böttcher. One obtains a complete classification of the cases when a solution of such an equation satisfies an algebraic differential equation; see [6] for details.

4. Exceptional values of derivatives

Recall that Picard's Theorem says that if an entire function omits two values, say $f(z) \neq 0$ and $f(z) \neq 1$ for all $z \in \mathbb{C}$, then f is constant. We note here that both the Wiman-Valiron method and Zalcman's Lemma yield rather easy proofs of this result. As far as the Wiman-Valiron method is concerned, this was already noted by Wiman [57, § 6] and Valiron [52, §IV.9] themselves. A proof of Picard's theorem using Zalcman's Lemma can be found in [62, p. 218].

Saxer [49] used the Wiman-Valiron method to prove a variant of Picard's Theorem.

Saxer's Theorem. *Let f be an entire function. Suppose that $f(z) \neq 0$ and $f'(z) \neq 1$ for all $z \in \mathbb{C}$. Then f is constant.*

The idea of the proof is as follows: if f omits 0, then f has the form $f = e^g$ for some entire function g. We may assume that g is transcendental. Applying the Wiman-Valiron Theorem to g we find with $\mu_k := g(z_k)$ and $\rho_k := z_k/\nu_k$ that

$$f'(z_k + \rho_k z) = g'(z_k + \rho_k z)e^{g(z_k+\rho_k z)} = (1 + o(1))\frac{\mu_k}{\rho_k}e^z e^{(1+o(1))\mu_k e^z}$$

and thus

$$h_k(z) := \frac{\log f'(z_k + \rho_k z)}{\mu_k} \to e^z$$

for a suitable branch of the logarithm. Let $Q := \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1, |\operatorname{Im} z| \leq \pi\}$. Then $\exp Q = A := \{z \in \mathbb{C} : 1 \leq |z| \leq e\}$. If U is open with $Q \subset U$ we have $h_k(U) \supset A$ and thus $\{\log f'(z_k + \rho_k z) : z \in U\} \supset \{z \in \mathbb{C} : |\mu_k| \leq |z| \leq e|\mu_k|\}$ for sufficiently large k. For large k there also exists $m_k \in \mathbb{N}$ with $|\mu_k| \leq 2\pi m_k \leq e|\mu_k|$ and thus there exists $\zeta_k \in U$ with $\log f'(z_k + \rho_k\zeta_k) = 2\pi m_k i$. With $\xi_k := z_k + \rho_k\zeta_k$ we have $f'(\xi_k) = 1$. □

It is also possible to prove Saxer's Theorem with Zalcman's Lemma; see §5 below, where a generalization is proved.

Saxer obtained the following consequence of his result.

Corollary. *Let h be entire and suppose that h, h' and h'' do not have zeros. Then h has the form $h(z) = e^{az+b}$ where $a, b \in \mathbb{C}$, $a \neq 0$.*

To prove the corollary, we consider $f := h/h'$ and observe that f is entire, $f \neq 0$ and $f' = 1 - hh''/(h')^2 \neq 1$. Thus f is constant. □

Hayman [28] extended Saxer's Theorem to meromorphic functions.

Hayman's Theorem. *Let f be meromorphic in the plane. Suppose that $f(z) \neq 0$ and $f'(z) \neq 1$ for all $z \in \mathbb{C}$. Then f is constant.*

This result implies that the conclusion of the above corollary remains valid if we only assume that h and h'' have no zeros. In fact, the hypotheses $h' \neq 0$ was only used to ensure that $f = h/h'$ is entire.

Hayman also made a conjecture about meromorphic functions which together with their second derivatives have no zeros. It took more than thirty years until this conjecture was finally confirmed by Langley [37].

Langley's Theorem. *Let h be meromorphic in $\mathbb{C}$ and suppose that h and h'' do not have zeros. Then h has the form $h(z) = e^{az+b}$ or $h(z) = (az+b)^{-n}$, where $a, b \in \mathbb{C}$, $a \neq 0$, and $n \in \mathbb{N}$.*

We note that the case that h has finite order was settled already much earlier by Mues [41].

We sketch an alternative proof [11] of Langley's Theorem using Zalcman's Lemma. We consider again the auxiliary function $f = h/h'$. We still have $f' \neq 1$, but f may have zeros now. In fact, if z is a pole of h of multiplicity m, then z is a simple zero of f with $f'(z) = -1/m$. We find that Langley's Theorem follows from and is in fact equivalent to the following generalisation of Hayman's Theorem.

Proposition 1. *Let f be meromorphic in $\mathbb{C}$ and suppose that $f'(z) \neq 1$ for all $z \in \mathbb{C}$. Suppose also that if $f(z) = 0$, then $f'(z) = -1/m$ for some $m \in \mathbb{N}$. Then f is constant or f has the form $f(z) = (c - z)/m$ for some $c \in \mathbb{C}$ and $m \in \mathbb{N}$.*

This result is also easier to prove in the special case that f has finite order. In fact, in this case the conclusion follows directly from an earlier result of Langley [36, Corollary]. Note, however, that this special case does not follow from the result of Mues [41], since $f = h/h'$ may finite order even if h has infinite order.

The special case that f has finite order in Proposition 1 can also be obtained with the methods of [13]. In fact, the results obtained there lead to the following more general result, cf. [11, Lemma 5].

Proposition 2. *Let g be meromorphic in $\mathbb{C}$ and of finite order. Suppose that $g'(z) \neq 1$ for all $z \in \mathbb{C}$ and that there exists $K > 0$ such that if $g(z) = 0$, then $|g'(z)| \leq K$. Then g is rational and has the form*

$$g(z) = z + a + \frac{b}{(z+c)^{\ell}}, \tag{11}$$

with $a, b, c \in \mathbb{C}$, $b \neq 0$, $\ell \in \mathbb{N}$, or the form $g(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 1$, $|\alpha| \leq K$.

We now show how Zalcman's Lemma can be used to obtain Proposition 1*, and thus Langley's Theorem, from Proposition* 2. Suppose that the conclusion of Proposition 1 is false. Then there exists a function f satisfying the hypotheses of Proposition 1 which is not constant and not a linear polynomial. This implies that the family $\{f(nz)/n\}_{n\in\mathbb{N}}$ is not normal at 0. We can now apply Zalcman's Lemma with $K = m = 1$ and $\alpha = -1$ to obtain sequences (n_k), (z_k), (ρ_k) and a function g meromorphic in $\mathbb{C}$ such that $n_k \to \infty$, $z_k \to 0$, $\rho_k \to 0$ and

$$\frac{f(n_k z_k + n_k \rho_k z)}{n_k \rho_k} \to g(z). \tag{12}$$

Moreover, $g^\#(z) \le g^\#(0) = K+1 = 2$ for all $z \in \mathbb{C}$. In particular, g has finite order. Also, if g has the form $g(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$, then $|\alpha| \ge |\alpha|/(1+|\beta|^2) = g^\#(0) = 2$. In particular, g does not have the form $g(z) = (c-z)/m$ with $c \in \mathbb{C}$ and $m \in \mathbb{N}$ or the form $g(z) = z + c$ with $c \in \mathbb{C}$. Thus $g'(z) \not\equiv 1$.

Since $f'(z) \ne 1$ and

$$f'(n_k z_k + n_k \rho_k z) \to g'(z) \tag{13}$$

by (12), we conclude that $g'(z) \ne 1$ for all $z \in \mathbb{C}$. Similarly, we find that if z is a simple zero of g, then $g'(z) = -1/m$ for some $m \in \mathbb{N}$.

However, g might have multiple zeros, so we cannot simply apply the finite order version of Proposition 1. But we can apply Proposition 2 and find that g has the form (11). Now we note that if $f(z) = 0$ and $f'(z) = -1/m$, then z is a (simple) pole of $1/f$ with $\operatorname{res}(1/f, z) = -m$. (Here $\operatorname{res}(\cdot,\cdot)$ denotes the residue of a function at a pole.) We find that if z is a zero of g, and thus a (possibly multiple) pole of $1/g$, then $\operatorname{res}(1/g, z) = -m$ for some $m \in \mathbb{N}$. On the other hand, we have

$$\sum_{z \in g^{-1}(0)} \operatorname{res}\left(\frac{1}{g}, z\right) = 1$$

by (11) and the residue theorem. This is a contradiction. □

We remark that the hypothesis that $f'(z) = -1/m$ for some $m \in \mathbb{N}$ if $f(z) = 0$ may be replaced by more general conditions. For example, the method is applicable if there exists $K > 0$ such that $\operatorname{Re} f'(z) < 0$ and $|f'(z)| \le K$ if $f(z) = 0$; see [11] for a discussion of this and even more general conditions.

We note that the method sketched here also leads to a normal family analogue of Langley's Theorem; cf. [11, Theorem 3]. We also mention that Zalcman's Lemma easily leads to normal family analogue's of Saxer's and Hayman's Theorem. These analogues had first been obtained by Bureau [16, 17, 18] and Miranda [40] for holomorphic functions and by Gu [27] in the meromorphic case.

5. The Ahlfors theory of covering surfaces

Let $D, V \subset \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be domains and let f be meromorphic in D. We say that f has an *island* over V if $f^{-1}(V)$ has a connected component whose closure (with respect to $\widehat{\mathbb{C}}$) is contained in D. Note that if U is such a component, then $f|_U : U \to V$ is a proper map so that, in particular, $V = f(U) \subset f(D)$. If there is a component U as above for which, in addition, $f|_U$ is univalent, then we say that f has a *simple island* over V.

Ahlfors [1, 2] obtained a striking generalization of Picard's theorem.

Ahlfors's Theorem. *Let $D_1, D_2, D_3 \subset \widehat{\mathbb{C}}$ be Jordan domains with pairwise disjoint closure and let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. Suppose that f has no island over any of the domains D_1, D_2, D_3. Then f is constant.*

This result can be considered as one of the central results of the Ahlfors theory of covering surfaces [2, 29, 42].

Following [8] we describe an argument how Ahlfors's Theorem can be deduced from Picard's Theorem using Zalcman's Lemma. (Actually [8] is mainly concerned with Ahlfors's Five Islands Theorem – another central result of the Ahlfors theory – but, as already pointed out in [8, §5.1] and [10, §7.2], the method used in [8] also gives the above result. In fact, it yields a more general result called "Scheibensatz" by Ahlfors [2, p. 190].)

To deduce Ahlfors's Theorem from Picard's Theorem, let $a_1, a_2, a_3 \in \mathbb{C}$ be distinct. It follows from Picard's Theorem that for each non-constant function f meromorphic in $\mathbb{C}$ there exists $\varepsilon_f > 0$ and $j \in \{1,2,3\}$ such that f has an island over $D(a_j, \varepsilon_f)$, where $D(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ for $a \in \mathbb{C}$ and $r > 0$. We show first that $\varepsilon_f > 0$ can in fact be choosen to be independent of f; that is, there exists $\varepsilon > 0$ such that the conclusion of Ahlfors's Theorem holds if $D_j = D(a_j, \varepsilon)$.

Otherwise there exists for all $n \in \mathbb{N}$ a non-constant function f_n meromorphic in $\mathbb{C}$ which has no island over any of the three disks $D(a_j, 1/n)$. We may assume that $\{f_n\}_{n\in\mathbb{N}}$ is not normal at 0. (This can be achieved by replacing $f_n(z)$ by $f_n(K_n z)$ with a large constant K_n.) Applying Zalcman's Lemma with $\alpha = K = 0$ we obtain sequences (n_k), (z_k), (ρ_k) and a function f meromorphic in $\mathbb{C}$ such that $n_k \to \infty$, $z_k \to 0$, $\rho_k \to 0$ and $f_{n_k}(z_k + \rho_k z) \to f(z)$. It follows that for the limit function f there does not exist $\varepsilon_f > 0$ with the property described above, contradicting Picard's Theorem.

It remains to show that the general case of Ahlfors's Theorem can be reduced to the case where $D_j = D(a_j, \varepsilon)$. To do this we note that there exists a quasiconformal map $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\phi(D_j) \subset D(a_j, \varepsilon)$ for $j \in \{1, \ldots, 3\}$. For $f : \mathbb{C} \to \widehat{\mathbb{C}}$ meromorphic the quasiregular map $\phi \circ f$ can be factored as $\phi \circ f = g \circ \psi$ with a meromorphic function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ and a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$. As we know already that the conclusion of the Ahlfors's Theorem holds for the function g and the domains $D(a_j, \varepsilon)$, we conclude that it also holds for f and the D_j. □

One may ask whether the theorems of Saxer and Hayman discussed in §4 have extensions similar to Ahlfors's extension of Picard's Theorem. Here we state only one result of this type. (For a more detailed discussion of this type of question we refer to [12].)

Theorem. *Let $D \subset \mathbb{C}$ be a Jordan domain with $\overline{D} \subset \mathbb{C}\backslash\{0\}$ and let f be entire. Suppose that $f(z) \neq 0$ for all $z \in \mathbb{C}$ and that f' has no simple island over D. Then f is constant.*

It turns out that the two rescaling principles mainly discussed in this paper both yield proofs of this theorem, and thus we shall give two proofs.

Proof using the Wiman-Valiron method. Suppose that f is a non-constant entire function which omits 0. We proceed as in the proof of Saxer's theorem in §4 and define μ_k and h_k as there. As the exponential function has a simple island over any Jordan domain not containing 0, we deduce that h_k has a simple island over $S := \{z \in \mathbb{C} : 1 < |z| < 2, |\arg z| < \frac{3}{4}\pi\}$ for sufficiently large k. Let $V := \log D$, for a fixed branch of the logarithm, and define $V_m := V + 2\pi m i$ for $m \in \mathbb{Z}$. For sufficiently large k there exists $m_k \in \mathbb{Z}$ such that $V_{m_k} \subset \mu_k S$. Thus $\log f'$ has a simple island over V_{m_k} and hence f' has a simple island over $D = \exp V_{m_k}$. □

Proof using Zalcman's Lemma [12]. Suppose again that f is a non-constant entire function which omits 0. As in the proof of Proposition 1 in §4 we apply Zalcman's Lemma with $K = 1$ and $\alpha = -1$ to obtain sequences (n_k), (z_k), (ρ_k) with $n_k \to \infty$, $z_k \to 0$, $\rho_k \to 0$ and a function g meromorphic in $\mathbb{C}$ such that (12) holds and

$g^{\#}(z) \le g^{\#}(0) = K + 1 = 2$ for all $z \in \mathbb{C}$. By a result of Clunie and Hayman [21], the boundedness of $g^{\#}$ implies that g has exponential type.

Since $f(z) \ne 0$ for all $z \in \mathbb{C}$ we conclude that $g(z) \ne 0$ for all $z \in \mathbb{C}$. This implies that g has the form $g(z) = e^{az+b}$ with $a, b \in \mathbb{C}$, $a \ne 0$. It follows that g' has a simple island over any Jordan domain not containing 0. From (13) we conclude that f' has a simple island over D. □

We remark that the conclusion of the above Theorem (and hence Saxer's Theorem) also holds if f' is replaced by a higher derivative. This can be shown by the same methods of proof. Hayman's Theorem also holds for higher derivatives, and was in fact already stated by Hayman this way. Finally, the conclusion of Langley's Theorem also remains valid if f'' is replaced by a higher derivative. This result, which had also been conjectured by Hayman, was obtained by Frank [24] before Langley proved his theorem. Frank's methods are quite different from Langley's and the ones employed here. It would be of interest whether his result can also be obtained by using the rescaling principles discussed here.

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Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany
E-mail address: bergweiler@math.uni-kiel.de