COVERING PROPERTIES OF DERIVATIVES OF MEROMORPHIC FUNCTIONS

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Dedicated to the memory of Professor Chuang Chi-Tai

ABSTRACT. The Ahlfors theory of covering surfaces yields a striking generalization of Picard's theorem. We discuss whether Picard type theorems where certain derivatives omit values admit similar generalizations.

1. Introduction and results

One of the central results in complex function theory is

Picard's Theorem. *Let $a_1, a_2, a_3 \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be distinct and let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. Suppose that $f(z) \neq a_j$ for all $z \in \mathbb{C}$ and $j \in \{1, 2, 3\}$. Then f is constant.*

Various results of Picard type where instead of f some derivative $f^{(k)}$ omits a value were obtained by Milloux [17], Hiong [15], and others. The following result was proved by Hayman [13, Theorem 3].

Hayman's Theorem. *Let $k \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{C}\backslash\{0\}$. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. Suppose that $f(z) \neq a$ and $f^{(k)}(z) \neq b$ for all $z \in \mathbb{C}$. Then f is constant.*

The case that f is entire and $k = 1$ is due to Saxer [25, p. 210, Hilfssatz] and the case that f is entire and $k \in \mathbb{N}$ is due to Bureau [5, 6].

Ahlfors [1, 2] obtained a striking generalization of Picard's theorem. To state his result, let $D, V \subset \widehat{\mathbb{C}}$ be domains and let $f : D \to \widehat{\mathbb{C}}$ be meromorphic. We say that f has an *island* over V if $f^{-1}(V)$ has a connected component whose closure (with respect to $\widehat{\mathbb{C}}$) is contained in D. Note that if U is such a component, then $f|_U : U \to V$ is a proper map so that, in particular, $V = f(U) \subset f(D)$. If there is a component U as above for which, in addition, $f|_U$ is univalent, then we say that f has a *simple island* over V.

Ahlfors's Theorem. *Let $D_1, D_2, D_3 \subset \widehat{\mathbb{C}}$ be Jordan domains with pairwise disjoint closure and let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. Suppose that f has no island over any of the domains D_1, D_2, D_3. Then f is constant.*

This result can be considered as one of the central results of the Ahlfors theory of covering surfaces [2, 14, 19]. For a proof which is different from Ahlfors's proof we refer to [3].

In this paper we shall consider the question whether results concerning omitted values of derivatives also admit generalizations in the spirit of Ahlfors's generalization of Picard's theorem. Hayman's theorem suggests the following

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Conjecture. Let $k \in \mathbb{N}$ and let A, B be Jordan domains with $\overline{A} \subset \mathbb{C}$ and $\overline{B} \subset \mathbb{C} \backslash \{0\}$. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic. Suppose that f has no island over A and $f^{(k)}$ has no island over B. Then f is constant.

Note that we cannot replace "island" by "simple island" in this conjecture, as shown by the example

$$f(z) = \frac{(e^z + a)^2}{e^z + 1}$$

where $a^2 - 2a + 28 = 0$. Then f has no simple zeros, and nor has

$$f'(z) - 8 = \frac{(e^z - 2)^3}{(e^z + 1)^2}.$$

I thank Jim Langley for communicating this example to me.

I have been unable to prove (or disprove) the above conjecture, and thus this paper contains only a small step towards this problem. For $c \in \mathbb{C}$ and $r > 0$ we use the notation $D(c, r) := \{z \in \mathbb{C} : |z - c| < r\}$. We also put $\mathbb{D} := D(0, 1)$.

Theorem 1. *For each $k \in \mathbb{N}$ there exists a positive constant η_k with the following property: if $a \in \mathbb{C}$, $b \in \mathbb{C} \backslash \{0\}$ and $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic such that $f(z) \neq a$ for all $z \in \mathbb{C}$ and $f^{(k)}$ has no island over $D(b, \eta_k |b|)$, then f is constant.*

Theorem 2. *Let $k \in \mathbb{N}$, $a \in \mathbb{C}$ and let $B \subset \mathbb{C}$ be a Jordan domain with $\overline{B} \subset \mathbb{C} \backslash \{0\}$. Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Suppose that $f(z) \neq a$ for all $z \in \mathbb{C}$ and that $f^{(k)}$ has no simple island over B. Then f is constant.*

We shall also discuss analogues of Theorems 1 and 2 in the theory of normal families. Recall that Bloch's principle says that a family of functions meromorphic in a domain and having a property P there is likely to be normal if there is no non-constant function meromorphic in the plane which has property P. While this is a heuristic principle only – and there are interesting counterexamples to it – Bloch's principle does hold in the case of Picard's, Hayman's and Ahlfors's theorem. The normality criterion corresponding to Picard's theorem is Montel's theorem, of course. The normality criterion corresponding to Hayman's theorem was proved by Gu [12]. (For holomorphic families it had been obtained by Bureau [5, 6] and Miranda [18].) And the Ahlfors theory yields not only the result stated above, but also the corresponding normality criterion. For a thorough discussion of Bloch's principle we refer to [23, Chapter 4] and [29, §7].

Theorem 1'. *For each $k \in \mathbb{N}$ there exists a positive constant η_k with the following property: if $a \in \mathbb{C}$, $b \in \mathbb{C} \backslash \{0\}$, and if $\mathcal{F}$ is the family of all meromorphic functions $f : \mathbb{D} \to \widehat{\mathbb{C}}$ such that $f(z) \neq a$ for all $z \in \mathbb{D}$ and $f^{(k)}$ has no island over $D(b, \eta_k |b|)$, then $\mathcal{F}$ is normal.*

Theorem 2'. *Let $k \in \mathbb{N}$, $a \in \mathbb{C}$ and let $B \subset \mathbb{C}$ be a Jordan domain with $\overline{B} \subset \mathbb{C} \backslash \{0\}$. Let $\mathcal{F}$ be the family of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that $f(z) \neq a$ for all $z \in \mathbb{D}$ and $f^{(k)}$ has no simple island over B. Then $\mathcal{F}$ is normal.*

We note that Zalcman [28] has given a formalization of the Bloch principle which makes this heuristic principle a rigorous theorem in many cases. In particular it applies in the case of Picard's and Ahlfors's theorem. A generalization of Zalcman's result due to Xue and Pang [24] (see Lemma 1 below) allows to obtain Gu's result from Hayman's theorem. This result would also allow to deduce Theorems 1' and 2' from Theorems 1 and 2. We shall, however, proceed differently. First we shall use the result of Xue and Pang to deduce Theorem 1' from Hayman's result, and then we shall deduce Theorem 1 from Theorem 1'. Similarly we shall first prove Theorem 2' and then deduce Theorem 2 from it.

2. Rescaling lemmas

The result of Xue and Pang [24] already mentioned in the introduction is the following.

Lemma 1. *Let $\mathcal{F}$ be a family of functions meromorphic in $\mathbb{D}$ such that $f(z) \neq 0$ for all $z \in \mathbb{D}$ and all $f \in \mathcal{F}$. Let $\alpha > 0$. If $\mathcal{F}$ is not normal, then there exist a sequence (z_n) in $\mathbb{D}$, a sequence (ρ_n) of positive real numbers, a sequence (f_n) in $\mathcal{F}$ and a non-constant meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $\rho_n \to 0$ and $\rho_n^{-\alpha} f_n(z_n + \rho_n z) \to f(z)$ locally uniformly in $\mathbb{C}$. Moreover, f can be chosen such that the spherical derivative $f^\#$ of f satisfies $f^\#(z) \leq f^\#(0) = 1$ for all $z \in \mathbb{C}$.*

We note that we may also allow $-1 < \alpha \leq 0$ in Lemma 1, but we do not need this result here. In fact, as proved by Pang [20, 21], the conclusion of Lemma 1 holds for $-1 < \alpha < 1$ even if the functions in $\mathcal{F}$ are allowed to have zeros. The case $\alpha = 0$ is due to Zalcman [28]. A similar result for normal functions had been proved earlier by Lohwater and Pommerenke [16]. If we assume that the zeros of the functions in $\mathcal{F}$ have multiplicity at least m, then the conclusion of Lemma 1 holds for $-1 < \alpha < m$. This last result was proved by Chen and Gu [8, Theorem 2]. In the papers cited, the condition $f^\#(z) \leq f^\#(0) = 1$ is usually not mentioned, but it follows immediately from the proof. For a further generalization of Lemma 1 (and a complete proof) we refer to a recent paper by Pang and Zalcman [22, Lemma 2].

For a discussion of Lemma 1, and its versions due to Zalcman, Pang, Chen and Gu, we also refer to [9, §3.2], [23, Chapter 4] and [29].

We note here that Zalcman's result was the main tool used in the proof of Ahlfors's theorem given in [3]. Here we shall employ ideas similar to those developed in [3].

While Lemma 1 allows to obtain normality criteria from results about functions in the plane, the following lemma works the other way round.

Lemma 2. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function and let $\alpha > 0$. Suppose that $f(z) \neq 0$ for all $z \in \mathbb{C}$. For $n \in \mathbb{N}$ define $h_n : \mathbb{C} \to \widehat{\mathbb{C}}$ by $h_n(z) := n^{-\alpha} f(nz)$. Then the family $\{h_n\}_{n \in \mathbb{N}}$ is not normal at 0.*

Proof of Lemma 2. We show that there exist sequences (a_n) and (b_n) tending to 0 such that $h_n(a_n) \to 0$ and $h_n(b_n) \to \infty$. From this the conclusion clearly follows.

First we note that we can simply take $a_n = w/n$ for any $w \in \mathbb{C}$ which is not a pole of f. If f has a pole p, then we can also take $b_n = p/n$.

It remains to show the existence of (b_n) if f is entire. Since f has no zeros, we easily see that the maximum modulus $M(r, f)$ of f satisfies $M(r, f) \geq e^{cr}$ for some $c > 0$ and all large r. It follows that $M(1/\sqrt{n}, h_n) = n^{-\alpha} M(\sqrt{n}, f) \geq n^{-\alpha} e^{c\sqrt{n}} \geq n$ for large n, and thus a sequence (b_n) with the required properties exists. □

3. Proof of Theorems 1 and 1'

Without loss of generality we may restrict to the case that $a = 0$ and $b = 1$ because otherwise we can consider $(f(z) - a)/b$ instead of $f(z)$.

Proof of Theorem 1'. We assume that the conclusion is false for some $k \in \mathbb{N}$. Let $\varepsilon > 0$ and let $\mathcal{F}(\varepsilon)$ be the family of all meromorphic functions $f : \mathbb{D} \to \widehat{\mathbb{C}}$ such that $f(z) \neq 0$ for all $z \in \mathbb{D}$ and $f^{(k)}$ has no island over $D(1, \varepsilon)$. Then, for all $\varepsilon > 0$, the family $\mathcal{F}(\varepsilon)$ is not normal. Applying Lemma 1 with $\alpha = k$ we see that there exist a sequence (z_n) in $\mathbb{D}$, sequences (ε_n) and (ρ_n) tending to zero and a sequence (f_n) with $f_n \in \mathcal{F}(\varepsilon_n)$ such that $\rho_n^{-k} f_n(z_n + \rho_n z) \to f(z)$ locally uniformly in $\mathbb{C}$ for some non-constant meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$. We have $f(z) \neq 0$ for all $z \in \mathbb{C}$ and, since $f_n^{(k)}(z_n + \rho_n z) \to f^{(k)}(z)$, we also see that $f^{(k)}$ has no island over $D(1, \varepsilon)$, for all $\varepsilon > 0$. This implies that $f^{(k)}(z) \neq 1$ for all $z \in \mathbb{C}$. Thus f is constant by Hayman's theorem, a contradiction. □

Proof of Theorem 1. Let $k \in \mathbb{N}$ and choose η_k according to Theorem 1'. Suppose that f has the properties stated in Theorem 1, but is non-constant. For $n \in \mathbb{N}$ define $h_n : \mathbb{D} \to \widehat{\mathbb{C}}$ by $h_n(z) := n^{-k} f(nz)$. Then, by Lemma 2, the family $\{h_n\}_{n \in \mathbb{N}}$ is not normal at 0. This contradicts Theorem 1'. □

4. Proof of Theorems 2 and 2'

The following result is a special case of results due to Clunie and Hayman [11, Theorems 2 and 3].

Lemma 3. *Let g be an entire function whose spherical derivative is bounded. Then g is of exponential type.*

Proof of Theorem 2'. Let k, a, B and $\mathcal{F}$ be as in Theorem 2' and suppose that $\mathcal{F}$ is not normal. Without loss of generality we may assume that $a = 0$. Applying Lemma 1 for $\alpha = k$ we obtain sequences (z_n), (ρ_n) and (f_n) as there such that $\rho_n^{-k} f_n(z_n + \rho_n z) \to f(z)$ locally uniformly in $\mathbb{C}$ for some non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ satisfying $f^{\#}(z) \leq f^{\#}(0) = 1$ for all $z \in \mathbb{C}$. Lemma 3 implies that f has exponential type. Since $f(z) \neq 0$ for all $z \in \mathbb{C}$ this implies that f has the form $f(z) = e^{cz+d}$ where $c, d \in \mathbb{C}$, $c \neq 0$. Hence $f^{(k)}(z) = c^k e^{cz+d}$, and we conclude that $f^{(k)}$ has a simple island over any Jordan domain D with $\overline{D} \subset \mathbb{C} \backslash \{0\}$. Applying this to a Jordan domain D containing $\overline{B}$ we conclude that $f_n^{(k)}$ has a simple island over B for sufficiently large n, a contradiction. □

The deduction of Theorem 2 from Theorem 2' is analogous to that of Theorem 1 from Theorem 1'.

5. Differential polynomials omitting values

There are a number of results concerning differential polynomials omitting values; see, e. g., [10, §6] or [23, §§4.4-4.5]. Again one may ask whether these results have generalizations to differential polynomials having no islands over certain domains.

The following result, which confirmed a conjecture of Hayman [13], can be found in [4, Theorem 2], [7, §1], [27] and [29, §8].

Theorem A. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic, $n, k \in \mathbb{N}$, $n > k$, and $a \in \mathbb{C} \backslash \{0\}$. If $(f^n)^{(k)}(z) \neq a$ for all $z \in \mathbb{C}$, then f is constant.*

This leads to the following

Conjecture. Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic, $n, k \in \mathbb{N}$, $n > k$, and let D be a Jordan domain with $\overline{D} \subset \mathbb{C} \backslash \{0\}$. If $(f^n)^{(k)}$ has no island over D, then f is constant.

Again we can give only a partial answer. To do this we note that Wang and Fang [26] have obtained the following generalization of Theorem A, as well as the corresponding normality criterion.

Theorem B. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic, $n, k \in \mathbb{N}$, $n > k$, and $a \in \mathbb{C}\backslash\{0\}$. Suppose that the zeros of f have multiplicity at least n and the poles of f have multiplicity at least 2. If $f^{(k)}(z) \neq a$ for all $z \in \mathbb{C}$, then f is constant.*

With the methods used above we can deduce the following results from Theorem B.

Theorem 3. *For $n, k \in \mathbb{N}$ with $n > k$ there exists a positive constant $\sigma_{n,k}$ with the following property: if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic with zeros of multiplicity at least n and poles of multiplicity at least 2, and if $f^{(k)}$ has no island over $D(a, \sigma_{n,k}|a|)$ for some $a \in \mathbb{C}\backslash\{0\}$, then f is constant.*

Theorem 3'. *For $n, k \in \mathbb{N}$ with $n > k$ there exists a positive constant $\sigma_{n,k}$ with the following property: if $a \in \mathbb{C}\backslash\{0\}$ and $\mathcal{F}$ is the family of all meromorphic functions $f : \mathbb{D} \to \widehat{\mathbb{C}}$ which have zeros of multiplicity at least n and poles of multiplicity at least 2 and for which $f^{(k)}$ has no island over $D(a, \sigma_{n,k}|a|)$, then $\mathcal{F}$ is normal.*

To prove these results we proceed as in the proof of Theorems 1 and 1'. But instead of Lemma 1 we use its generalization due to Chen and Gu [8] mentioned in §2, and we also use the corresponding generalization of Lemma 2. We omit the details.

We note that we cannot replace $D(a, \sigma_{n,k}|a|)$ by an arbitrary Jordan domain D with $\overline{D} \subset \mathbb{C}\backslash\{0\}$ in the above results. In fact, for the example $f(z) = (z^2-1)^2/z^3$ we see that f' takes the value 1 only at $\pm\sqrt{6}/2$, and $|f'(x)| \leq 4/3$ for $|x| \geq \sqrt{6}/2$, $x \in \mathbb{R}$. Thus f' has no island over $D(1, r)$ if $r > 1/3$ and hence $\sigma_{2,1} \leq 1/3$ in Theorem 3. Considering the (non-normal) family $\{f(nz)/n\}_{n\in\mathbb{N}}$, with f as above, we obtain an analogous remark concerning Theorem 3'. As far as Theorem 3 is concerned, however, the situation may be different if we exclude rational functions f.

It seems that a systematic study of the questions considered here will require new methods.

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