Composite meromorphic functions and growth of the spherical derivative

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Let f and h be transcendental meromorphic and g a transcendental entire function. It is shown that if h grows slower than g in a suitable sense, then there exists an unbounded sequence (z_n) such that $f(g(z_n)) = h(z_n)$.

1. INTRODUCTION AND RESULTS

This paper is concerned with the following

CONJECTURE. Let f be a transcendental meromorphic, g a transcendental entire, and h a nonconstant meromorphic function. Suppose that

$$T(r,h) = o(T(r,g)) \tag{1}$$

as $r \to \infty$. Then $f(g) - h$ has infinitely many zeros.

Here and in the following, unless stated otherwise, "meromorphic" is understood to mean "meromorphic in the complex plane $\mathbb{C}$", and $T(r,\cdot)$ denotes the Nevanlinna characteristic of a meromorphic function; see [11, 12, 17] for an introduction to Nevanlinna theory.

The above conjecture appears e. g. in [4, p. 43] or [28]. We recall some background of this conjecture. Gross [10] had conjectured that the composition of two transcendental entire functions has infinitely many fixed points. This was proved in [3] where it was actually shown that if f and g are transcendental entire functions and if h is a nonconstant polynomial, then $f(g) - h$ has infinitely many zeros. These results were later extended

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to the case that f is meromorphic and h is rational; see [5, 6, 7] and Example 3.4 of this paper. The case that h is a transcendental function satisfying (1) is studied in [13, 14, 33]. In particular, it follows from the results obtained there that the above conjecture is true if f has finite order and g has finite lower order. Actually these papers contain stronger results by giving lower bounds for the counting function of the zeros of $f(g) - h$. In the special case that f is entire and h is a polynomial, but without restrictions on the order or lower order, such estimates can be found in [29, 30]. There are a number of further papers devoted to these and similar topics; here we only refer to the references of the papers cited.

The condition (1) says that h grows slower than g, when the growth is measured by the Nevanlinna characteristic. For us it will be convenient to measure the growth of a meromorphic function f in terms of its spherical derivative

$$f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2}$$

and

$$\mu(r, f) := \max_{|z|=r} f^{\#}(z).$$

It is apparent already from the Ahlfors-Shimizu form of the characteristic that there are relations between $T(r, f)$ and $\mu(r, f)$, and in fact such relations have been studied in detail by various authors; see e. g. [1, 2, 9, 15, 18, 20, 21, 22, 23, 24, 25, 26].

Here we only note that Clunie and Hayman [9] proved that if f is entire transcendental, then

$$\limsup_{r\to\infty} \frac{r\mu(r, f)}{\log M(r, f)} > A \tag{2}$$

for some absolute constant $A > 0$. In particular,

$$\limsup_{r\to\infty} r\mu(r, f) = \infty. \tag{3}$$

This had been proved before by Lehto [15] who had also shown that

$$\limsup_{r\to\infty} r\mu(r, f) \geq \frac{1}{2} \tag{4}$$

for transcendental meromorphic f.

Theorem 1.1. *Let f be a transcendental meromorphic, g a transcendental entire, and h a meromorphic function. Suppose that f takes every* *value* $c \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ *at least twice. Suppose also that*

$$\limsup_{r\to\infty} \frac{r\mu(r,g)}{1+\max_{|t-r|\le K/\mu(r,g)} t\mu(t,h)} = \infty \tag{5}$$

for all $K > 0$. *Then there exists an unbounded sequence* (ζ_n) *such that* $f(g(\zeta_n)) = h(\zeta_n)$.

To discuss condition (5), let $\varphi(r)$ be positive and nondecreasing for $r \ge r_0 > 0$. A classical result of Borel ([8, pp. 375-377], see also [16]) implies that if $K > 0$ and $C > 1$, then the set F of all $r \ge r_0$ satisfying

$$\varphi\left(r + \frac{Kr}{\varphi(r)}\right) > C\varphi(r) \tag{6}$$

has finite logarithmic measure; that is,

$$\int_F \frac{dr}{r} < \infty.$$

Applying this to

$$\varphi(r) := \max_{t\le r} t\mu(t,h) = \max_{|z|\le r} |z| h^{\#}(z)$$

yields the following result.

PROPOSITION 1.1. *Let* g *and* h *be as in Theorem* 1.1. *If there exists a set* $F \subset [1,\infty)$ *of infinite logarithmic measure such that*

$$\lim_{r\to\infty, r\in F} \frac{r\mu(r,g)}{\max_{t\le r} t\mu(t,h)} = \infty,$$

then (5) *holds.*

For "nice" functions like $\varphi(r) = r^\alpha$ or $\varphi(r) = \exp r^\alpha$, $\alpha > 0$, the set of r-values where (6) holds is bounded. Sometimes it is convenient to compare $\mu(r,h)$ and $\mu(r,g)$ with such functions.

PROPOSITION 1.2. *Let* g *and* h *be as in Theorem* 1.1 *and and let* $\varphi(r)$ *be positive and nondecreasing for* $r \ge r_0 > 0$. *Suppose that for all* $K > 0$ *there exists* $C > 0$ *such that the set where* (6) *holds is bounded. If*

$$\limsup_{r\to\infty} \frac{r\mu(r,h)}{\varphi(r)} < \infty \quad \textit{and} \quad \limsup_{r\to\infty} \frac{r\mu(r,g)}{\varphi(r)} = \infty \tag{7}$$

or if

$$\limsup_{r\to\infty} \frac{r\mu(r,h)}{\varphi(r)} = 0 \quad \textit{and} \quad \limsup_{r\to\infty} \frac{r\mu(r,g)}{\varphi(r)} > 0, \tag{8}$$

then (5) *holds.*

Theorem 1.1 will be deduced from the following theorem, which yields the desired conclusion under a more abstract condition.

THEOREM 1.2. *Let f be a transcendental meromorphic, g a transcendental entire, and h a meromorphic function. Suppose that f takes every value $c \in \widehat{\mathbb{C}}$ at least twice. Suppose also that there exists a sequence (T_n) of linear transformations such that $(g \circ T_n)$ is not normal at 0 and $(h \circ T_n)$ is normal at 0. Then there exists an unbounded sequence (ζ_n) such that $f(g(\zeta_n)) = h(\zeta_n)$.*

In §3 we shall discuss some examples. The emphasis is not on obtaining very general results, but rather on illustrating the method. The examples will show that our results apply even in some cases where condition (1) is not satisfied. Conversely, we do not know whether the hypotheses of Theorem 1.2 are always satisfied if (1) holds.

2. PROOF OF THE THEOREMS

The following lemma is a local adaption of a lemma due to Zalcman [31]. A proof can be found in [19, Lemma 1.5].

LEMMA 2.1. *Let $\mathcal{F}$ be a family of functions meromorphic in a neighbourhood U of 0. If $\mathcal{F}$ is not normal at 0, then there exist a sequence (f_k) in $\mathcal{F}$, a sequence (M_k) of linear transformations, and a non-constant meromorphic function f such that $M_k \to 0$ and $f_k \circ M_k \to f$ locally uniformly in $\mathbb{C}$.*

A discussion of this lemma and a survey of its various applications is given in [32].

Proof of Theorem 1.2. The conclusion follows from Picard's Theorem (applied to g) if h is constant. We may thus assume that h is non-constant.

According to Lemma 2.1 there exists a subsequence (T_{n_k}) of (T_n), a sequence (M_k) of linear transformations and a non-constant entire function G such that

$$M_k \to 0 \quad \text{and} \quad g \circ T_{n_k} \circ M_k \to G$$

locally uniformly on $\mathbb{C}$. We define $L_k := T_{n_k} \circ M_k$ and conclude that

$$f \circ g \circ L_k \to f \circ G$$

locally uniformly on $\mathbb{C}$. Passing over to a subsequence if necessary we may assume that $(h \circ T_{n_k})$ converges uniformly on a neighbourhood of 0 to some function H which is meromorphic there. Since $M_k \to 0$ locally uniformly on $\mathbb{C}$ we conclude that $h \circ L_k \to c := H(0)$ locally uniformly on $\mathbb{C}$. We may assume that $c \in \mathbb{C}$ because otherwise we can consider $1/f$ and $1/h$ instead of f and h. Thus

$$(f \circ g - h) \circ L_k \to f \circ G - c$$

locally uniformly on $\mathbb{C}$. Since G is non-constant and f takes the value c at least twice, we conclude from Picard's theorem that $f \circ G - c$ has at least one zero. Thus Hurwitz's theorem implies that there exists $k_0 \in \mathbb{N}$ and a bounded sequence (x_k) of complex numbers such that

$$((f \circ g - h) \circ L_k)(x_k) = 0$$

for $k \geq k_0$. With $\zeta_k = L_k(x_k)$ we thus have $f(g(\zeta_k)) = h(\zeta_k)$.

To prove that $\zeta_k \to \infty$ we note that $h \circ L_k \to c$ locally uniformly in $\mathbb{C}$ and h is non-constant. We conclude that (L_k) is normal such that each limit function is constant. Since $g \circ L_k \to G$ locally uniformly on $\mathbb{C}$ and G is non-constant we see that $L_k \to \infty$ locally uniformly in $\mathbb{C}$. Hence $\zeta_k = L_k(x_k) \to \infty$.

Proof of Theorem 1.1. It follows from (5) and the continuity of $\mu(r, \cdot)$ that there exists a sequence (r_n) tending to ∞ such that

$$r_n \mu(r_n, g) = n \left(1 + \max_{|t - r_n| \leq n/\mu(r_n, g)} t\mu(t, h) \right).$$

Choose z_n such that $|z_n| = r_n$ and $g^{\#}(z_n) = \mu(r_n, g)$ and define

$$\rho_n := \frac{n}{g^{\#}(z_n)} \quad \text{and} \quad T_n(z) := z_n + \rho_n z.$$

Then

$$(g \circ T_n)^{\#}(0) = g^{\#}(z_n)\rho_n = n \to \infty,$$

which by Marty's theorem implies that $(g \circ T_n)$ is not normal at 0. On the other hand, if $|z| < 1$, then

$$r_n - \rho_n \leq |z_n + \rho_n z| \leq r_n + \rho_n$$

which implies that

$$|z_n + \rho_n z| h^{\#}(z_n + \rho_n z) \leq \max_{|t - r_n| \leq \rho_n} t\mu(t, h) = \frac{r_n \mu(r_n, g)}{n} - 1 = \frac{r_n}{\rho_n} - 1$$

and hence

$$\begin{aligned}(h\circ T_n)^{\#}(z) &= \rho_n h^{\#}(z_n+\rho_n z)\\ &\leq \frac{\rho_n}{|z_n+\rho_n z|}\left(\frac{r_n}{\rho_n}-1\right)\\ &\leq \frac{\rho_n}{r_n-\rho_n}\frac{r_n-\rho_n}{\rho_n}\\ &= 1.\end{aligned}$$

Marty's Theorem implies that $(h\circ T_n)$ is normal at 0. The conclusion follows from Theorem 1.2.

3. EXAMPLES

EXAMPLE 3.1. If $h(z)=\exp p(z)$ or $h(z)=\cos p(z)$ for a polynomial p of degree $d\geq 1$, then we have $\mu(r,h)\leq ar^{d-1}$ for some $a>0$ and sufficiently large r, and $T(r,h)\sim br^d$ for some $b>0$ as $r\to\infty$. From (2) we can deduce that if g is an entire function satisfying (1), then $\limsup_{r\to\infty}\mu(r,g)/r^{d-1}=\infty$. It follows that (7) is satisfied for $\varphi(r)=r^{d-1}$. Thus Proposition 1.2 implies that (5) is always satisfied if (1) is satisfied. A similar argument can be made for more general functions h.

EXAMPLE 3.2. If $h(z)=H(p(z))$ for an elliptic function H and a polynomial p of degree $d\geq 1$, then we again have $\mu(r,h)\leq ar^{d-1}$ for some $a>0$ and sufficiently large r, but $T(r,h)\sim br^{2d}$ for some $b>0$ as $r\to\infty$. The arguments used before show that (5) not only holds for all entire functions g satisfying (1), but even for all entire g satisfying $\sqrt{T(r,h)}=o(T(r,g))$.

EXAMPLE 3.3. If $g(z)=e^z\sin z$, then $T(r,g)\sim br$ as $r\to\infty$ with $b=(1+\sqrt{2})/\pi$ (see, e. g., [27]), and $g^{\#}(\pi k)=\exp\pi k$ for $k\in\mathbb{N}$. Let h be an entire function satisying (1). Then

$$\begin{aligned}\mu(r,h)\leq M(r,h') &= \frac{1}{2\pi}\max_{|z|=r}\left|\int_{|\zeta|=2r}\frac{h(\zeta)}{(\zeta-z)^2}d\zeta\right|\\ &\leq \frac{2M(2r,h)}{r}\\ &\leq \frac{2\exp 3T(4r,h)}{r}.\end{aligned}$$

Thus $r\mu(r,h)\leq\exp r$ for large r. We deduce that (7) holds for $\varphi(r)=\exp r$. From Proposition 1.2 we conclude that (5) holds for all entire h

satisfying (1). We also see that (5) holds for many functions h which do not satisfy (1).

EXAMPLE 3.4. If h is rational, then $t\mu(t,h) \to 0$ as $t \to \infty$, and from (3) we deduce that (5) always holds in this case. The proof shows that in this case the conclusion of Theorem 1.1 holds if f just takes the value $c = h(\infty)$ at least twice. This result was proved in [6] with a different method.

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