Some remarks on Picard's theorem and Nevanlinna theory*

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1 Introduction and result

Let f be a transcendental entire function. Picard's theorem [11] from 1879 says that f takes every complex value infinitely often, with at most one exception. Denote by $n(r,h)$ the number of poles of a meromorphic function h in $|z| \leq r$, counted according to multiplicity. Thus $n\left(r, \frac{1}{f-a}\right)$ counts the a-points of f. Let $M(r,f) = \max_{|z|=r} |f(z)|$ be the maximum modulus of f. Borel [5] proved that

$$\limsup_{r\to\infty} \frac{\log n\left(r, \frac{1}{f-a}\right)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \tag{1}$$

for all $a \in \mathbb{C}$, with at most one exception. The expression on the right side of (1) is denoted by ρ and called the order of f. Roughly speaking, Borel's theorem says that $n(r, \frac{1}{f-a})$ and $\log M(r,f)$ are of the same order of magnitude for all except possibly one value of a. Borel's theorem is thus a quantitative version of Picard's theorem, at least if $\rho > 0$.

It was Nevanlinna [9] who in 1925 gave a quantitative version of Picard's theorem which is much more precise than Borel's. For a meromorphic function h, define

$$N(r,h) = \int_0^r \frac{n(t,h) - n(0,h)}{t} dt + n(0,h) \log r.$$

The Nevanlinna characteristic $T(r,h)$ is then defined by

$$T(r,h) = N(r,h) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h\left(re^{i\theta}\right)| d\theta.$$

One advantage of Nevanlinna's theory is that it applies not only to entire functions, but also to meromorphic functions. We shall restrict, however, to entire functions here. Note that $N(r,f) \equiv 0$ if f is entire. Specialized to an entire function f, Nevanlinna's second fundamental theorem [7, 9, 10] says that if $q \in \mathbb{N}$ and $a_1, \ldots, a_q \in \mathbb{C}$ are pairwise distinct, then

$$\sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) - N\left(r, \frac{1}{f'}\right) \geq (q - 1 - o(1))\, T(r,f) \tag{2}$$

as $r \to \infty$, possibly outside an exceptional set of finite measure. Here the exceptional set is bounded if $\rho < \infty$.

In this note, we prove an inequality similar to (2), but with $N(r,\cdot)$ and $T(r,f)$ replaced by the perhaps more natural quantities $n(r,\cdot)$ and $\log M(r,f)$.

*1991 Mathematics Subject Classification: 30D20, 30D35

Theorem *Let f be an entire function of order ρ, where $\frac{1}{2} \leq \rho \leq \infty$. Let $q \in \mathbb{N}$ and $a_1, \ldots, a_q \in \mathbb{C}$ pairwise distinct. Then*

$$\sum_{j=1}^{q} n\left(r, \frac{1}{f-a_j}\right) - n\left(r, \frac{1}{f'}\right) \geq \frac{q-1-o(1)}{\pi} \log M(r,f) \tag{3}$$

on some unbounded sequence of r-values.

For $q = 2$ and without the term $n(r, 1/f')$ this was proved in [3]. One of the tools used there is Ahlfors's theory of covering surfaces [1, 7, 10] which implies that

$$n(r, a_1) + n(r, a_2) \geq (1 - o(1))A(r, f) \tag{4}$$

on a sequence of r-values. Here

$$A(r,f) = \frac{1}{\pi} \iint_{|z| \leq r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} dxdy.$$

The other main ingredient in [3] is a result proved in [2] which says that

$$\log M(r,f) \leq (1 + o(1))\pi A(r,f) \tag{5}$$

on a sequence of r-values if $\frac{1}{2} \leq \rho \leq \infty$. It is shown in [3] that there exist values of r such that (4) and (5) hold simultaneously.

The arguments of this paper do employ some ideas from [3], but avoid the use of Ahlfors's theory. Instead they use an improved version of the growth lemma introduced in [2] in order to prove (5).

The constant $1/\pi$ occuring in (3) is sharp. Given ρ satisfying $\frac{1}{2} \leq \rho \leq \infty$ and $q \in \mathbb{N}$ there exists an entire function of order ρ and $a_1, \ldots, a_q \in \mathbb{C}$ such that we have equality in (3) as $r \to \infty$. In fact, if $\frac{1}{2} \leq \rho < \infty$, then Mittag-Leffler's function $E_{1/\rho}$ has this property, provided we choose $a_1 = 0$ if $\rho = 1$. For examples of infinite order we refer to [2, 3, 6].

If $0 < \rho < \frac{1}{2}$, then the constant $1/\pi$ occuring in (3) has to be replaced by $\sin \pi\rho/\pi$.

2 Lemmas

The next lemma is a generalization of similar results proved in [2, Lemma 1] and [4, Lemma 1].

Lemma 1 *Let $\Phi(x)$ be positive and continuous for $x \geq x_0 > 0$ and define*

$$\rho = \limsup_{x \to \infty} \frac{\Phi(x)}{x}$$

Then there exist sequences (x_j), (M_j), (μ_j) and (ε_j) satisfying $x_j \to \infty$, $M_j \to \infty$, $\mu_j \to \rho$ and $\varepsilon_j \to 0$ as $j \to \infty$ such that

$$\Phi(x_j + h) \leq \Phi(x_j) + \mu_j h + \varepsilon_j$$

for $|h| \leq M_j/\mu_j$. Here

$$\limsup_{t \to 0+} \frac{\Phi(x_j + t) - \Phi(x_j)}{t} \leq \mu_j \leq \liminf_{t \to 0-} \frac{\Phi(x_j + t) - \Phi(x_j)}{t}.$$

Moreover, if $\Psi(x)$ *is positive and continuously differentiable for* $x \geq x_0$ *such that* $\Psi(x)/x$ *is non-decreasing,* $\Psi'(x) \leq \sqrt{\Psi(x)}$*, and* $\int_{x_0}^{\infty} dx/\Psi(x) < \infty$*, then* (x_j) *and* (μ_j) *can be chosen such that*

$$\mu_j = o(\Psi(\Phi(x_j))$$

as $j \to \infty$*.*

Proof First we consider the case that $\rho = \infty$. If Φ is increasing and differentiable, then the conclusion was proved in [4, Lemma 1]. In this case we clearly have $\mu_j = \Phi'(x_j)$. Here we only note that if we do not assume that Φ is differentiable, then the argument used in [4] gives the above conclusion. Moreover, this remains true if Φ is only assumed to be non-decreasing (instead of increasing).

The general case where Φ is not necessarily non-decreasing can be reduced to the case where Φ is non-decreasing. In fact, let $\overline{\Phi}(x) = \max_{x_0 \leq t \leq x} \Phi(t)$. Then $\overline{\Phi}$ is non-decreasing and $\Phi(x) \leq \overline{\Phi}(x)$. For $\overline{\Phi}$ we now obtain sequences (x_j), (M_j), (μ_j) and (ε_j) with the properties described in the lemma. But because $\mu_j \to \infty$, we have

$$\liminf_{t \to 0-} \frac{\overline{\Phi}(x_j + t) - \overline{\Phi}(x_j)}{t} \to \infty$$

as $j \to \infty$ and thus $\Phi(x_j) = \overline{\Phi}(x_j)$ for large j. This shows that the sequences (x_j), (M_j), (μ_j) and (ε_j) also have the desired properties with respect to Φ.

Now we consider the case that $\rho < \infty$. In this case we consider a strong proximate order $\rho^*(r)$ for $f(r) = \exp \Phi(\log r)$ and choose $x_j = \log r_j$ where $f(r_j) = r_j^{\rho^*(r_j)}$, see [8, §I.12]. It can then be shown (cf. [3]) that x_j has the desired properties. We omit the details.

The next lemma follows from an inspection of the proof of [2, Theorem 1].

Lemma 2 *Let* f *be an entire function and let* (r_j)*,* (M_j)*,* (μ_j)*,* (γ_j) *and* (ε_j) *be sequences satisfying* $r_j \to \infty$*,* $M_j \to \infty$*,* $\mu_j \to \rho$*,* $\gamma_j \to \infty$ *and* $\varepsilon_j \to 0$ *as* $j \to \infty$*. Suppose that*

$$T(r, f) \leq (1 + \varepsilon_j) \left(\frac{r}{r_j} \right)^{\mu_j} \gamma_j \tag{6}$$

for $|\log r/r_j| \leq M_j/\mu_j$*. If* $\frac{1}{2} \leq \rho \leq \infty$*, then*

$$\log M(r_j, f) \leq (1 + o(1))\pi \mu_j \gamma_j$$

as $j \to \infty$*.*

3 Proof of the theorem

We put

$$N(r) = \frac{1}{q-1} \left(\sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) - N\left(r, \frac{1}{f'}\right) \right),$$

replacing $\frac{1}{q-1}$ by a (large) constant if $q = 1$ here, and, for $\varepsilon > 0$, we define

$$\gamma(r) = \max\{T(r, f), (1 + \varepsilon)N(r)\}.$$

We apply Lemma 1 to $\Phi(x) = \log \gamma(e^x)$ and obtain sequences (x_j), (M_j), (μ_j) and (ε_j) with the properties described there. We put $r_j = \exp x_j$. In terms of γ we then have

$$\gamma(r) \leq e^{\varepsilon_j} \left(\frac{r}{r_j}\right)^{\mu_j} \gamma(r_j) \tag{7}$$

for $|\log r/r_j| \leq M_j/\mu_j$. Noting that $T(r,f) \leq \gamma(r)$ we obtain (6) with $\gamma_j = \gamma(r_j)$. Thus

$$\log M(r_j, f) \leq (1 + o(1))\pi \mu_j \gamma_j$$

by Lemma 2.

Suppose first that $\rho < \infty$. Then $\gamma(r) = (1+\varepsilon)N(r)$ for all large r by (2). (Note that the exceptional set of r-values occuring there is bounded for functions of finite order.) In particular, $\gamma(r_j) = (1+\varepsilon)N(r_j)$ and

$$\mu_j \leq \lim_{t \to 0-} \frac{n(r_j + t)}{N(r_j)}$$

where

$$n(r) = \frac{1}{q-1} \left(\sum_{j=1}^{q} n\left(r, \frac{1}{f - a_j}\right) - n\left(r, \frac{1}{f'}\right) \right).$$

We thus obtain

$$\log M(r_j, f) \leq (1+o(1))(1+\varepsilon)\pi \lim_{t \to 0-} n(r_j + t)$$

and hence

$$\log M(r_j', f) \leq (1+o(1))(1+\varepsilon)\pi n(r_j')$$

for $r_j' = r_j - \delta_j$, provided δ_j is sufficiently small. Letting $\varepsilon \to 0$ we obtain the conclusion.

Suppose now that $\rho = \infty$. We shall show that

$$T(r_j, f) < \gamma(r_j). \tag{8}$$

Then we again have $\gamma_j = \gamma(r_j) = (1+\varepsilon)N(r_j)$ and in fact $\gamma(r) = (1+\varepsilon)N(r)$ in some open interval containing r_j. The proof can then be completed as before.

To prove (8) we need a more precise form of the error term "$o(1)$" in (2). We have [7, §2.2]

$$T(r,f) \leq N(r) + C_1 \log \frac{1}{R - r} + C_2 \log T(R, f)$$

with constants C_1, C_2 if $R > r \geq r_0$. We choose $r = r_j$ and $R_j = r_j e^{1/\mu_j}$. Then

$$\log T(R_j, f) \leq \log \gamma(R_j) \leq \log \gamma(r_j) + O(1)$$

by (7) and

$$\log \frac{1}{R_j - r_j} = \log \frac{1}{r_j \left(e^{1/\mu_j} - 1\right)} \leq \log \mu_j + O(1)$$

as $j \to \infty$. Choosing $\Psi(x) = x^{3/2}$ in Lemma 2 we may achieve that

$$\mu_j = o\left((\log \gamma(r_j))^{3/2} \right)$$

so that

$$\log \mu_j \le \frac{3}{2} \log\log \gamma(r_j) + O(1).$$

We thus obtain

$$T(r_j, f) \le N(r_j) + \frac{3}{2} C_1 \log\log \gamma(r_j) + C_2 \log \gamma(r_j) + O(1) \le N(r_j) + O\left(\log \gamma(r_j)\right). \quad (9)$$

Now, if (8) does not hold, then $T(r_j, f) = \gamma(r_j) \ge (1+\varepsilon)N(r_j)$. Thus (9) yields

$$\gamma(r_j) \le \frac{1}{1+\varepsilon}\gamma(r_j) + O\left(\log \gamma(r_j)\right),$$

which is impossible. Therefore (8) holds, and this completes the proof.

Acknowledgement I would like to thank the Deutsche Forschungsgemeinschaft for a travel grant giving me the opportunity to attend the conference and for supporting me with a Heisenberg Fellowship during part of the time when this paper was written.

References

[1] L. V. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* **65** (1935), 157-194.

[2] W. Bergweiler, Maximum modulus, characteristic, and area on the sphere, *Analysis* **10** (1990), 163-176. Erratum: *Analysis* **12** (1992), 67-69.

[3] W. Bergweiler, A quantitative version of Picard's theorem, *Ark. Mat.* **34** (1996), 225-229.

[4] W. Bergweiler and H. Bock, On the growth of meromorphic functions of infinite order, *J. Analyse Math.* **64** (1994), 327-336.

[5] É. Borel, Sur les zéros des fonctions entières, *Acta Math.* **20** (1897), 357-396.

[6] C. J. Dai, D. Drasin, and B. Q. Li, On the growth of entire and meromorphic functions of infinite order, *J. Analyse Math.* **55** (1990), 217-228. Correction: *J. Analyse Math.* **57** (1991), 299.

[7] W. K. Hayman, *Meromorphic Functions*, Oxford University Press 1964.

[8] B. J. Levin, *Nullstellenverteilung ganzer Funktionen*, Akademie-Verlag, Berlin 1972.

[9] R. Nevanlinna, Zur Theorie der meromorphen Funktionen, *Acta Math.* **46** (1925), 1-99.

[10] R. Nevanlinna, *Analytic Functions*, Springer, Berlin - Heidelberg - New York 1970.

[11] É. Picard, Sur une propriété des fonctions entières, *C. R. Acad. Sci. Paris* **88** (1879), 1024-1027.

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