Some remarks on Picard’s theorem and Nevanlinna theory∗

Walter Bergweiler

1 Introduction and result

Let $f$ be a transcendental entire function. Picard’s theorem [11] from 1879 says that $f$ takes every complex value infinitely often, with at most one exception. Denote by $n(r, h)$ the number of poles of a meromorphic function $h$ in $|z| \leq r$, counted according to multiplicity. Thus $n\left(r, \frac{1}{f-a}\right)$ counts the $a$-points of $f$. Let $M(r, f) = \max_{|z|=r} |f(z)|$ be the maximum modulus of $f$. Borel [5] proved that

$$\limsup_{r \to \infty} \frac{\log n\left(r, \frac{1}{f-a}\right)}{\log r} = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}$$

for all $a \in \mathbb{C}$, with at most one exception. The expression on the right side of (1) is denoted by $\rho$ and called the order of $f$. Roughly speaking, Borel’s theorem says that $n\left(r, \frac{1}{f-a}\right)$ and $\log \log M(r, f)$ are of the same order of magnitude for all except possibly one value of $a$. Borel’s theorem is thus a quantitative version of Picard’s theorem, at least if $\rho > 0$.

It was Nevanlinna [9] who in 1925 gave a quantitative version of Picard’s theorem which is much more precise than Borel’s. For a meromorphic function $h$, define

$$N(r, h) = \int_0^r n(t, h) - n(0, h) \frac{dt}{t} + n(0, h) \log r.$$ 

The Nevanlinna characteristic $T(r, h)$ is then defined by

$$T(r, h) = N(r, h) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| d\theta.$$ 

One advantage of Nevanlinna’s theory is that it applies not only to entire functions, but also to meromorphic functions. We shall restrict, however, to entire functions here. Note that $N(r, f) \equiv 0$ if $f$ is entire. Specialized to an entire function $f$, Nevanlinna’s second fundamental theorem [7, 9, 10] says that if $q \in \mathbb{N}$ and $a_1, \ldots, a_q \in \mathbb{C}$ are pairwise distinct, then

$$\sum_{j=1}^q N\left(r, \frac{1}{f-a_j}\right) - N\left(r, \frac{1}{f}\right) \geq (q - 1 - o(1)) T(r, f)$$

as $r \to \infty$, possibly outside an exceptional set of finite measure. Here the exceptional set is bounded if $\rho < \infty$.

In this note, we prove an inequality similar to (2), but with $N(r, \cdot)$ and $T(r, f)$ replaced by the perhaps more natural quantities $n(r, \cdot)$ and $\log M(r, f)$.

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Theorem Let $f$ be an entire function of order $\rho$, where $\frac{1}{2} \leq \rho \leq \infty$. Let $q \in \mathbb{N}$ and $a_1, \ldots, a_q \in \mathbb{C}$ pairwise distinct. Then

$$\sum_{j=1}^{q} n\left(r, \frac{1}{f - a_j}\right) - n\left(r, \frac{1}{f'}\right) \geq \frac{q - 1 - o(1)}{\pi} \log M(r, f)$$

(3)
on some unbounded sequence of $r$-values.

For $q = 2$ and without the term $n(r, 1/f')$ this was proved in [3]. One of the tools used there is Ahlfors’s theory of covering surfaces [1, 7, 10] which implies that

$$n(r, a_1) + n(r, a_2) \geq (1 - o(1)) A(r, f)$$

(4)
on a sequence of $r$-values. Here

$$A(r, f) = \frac{1}{\pi} \iint_{|z| \leq r} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} dx dy.$$  

The other main ingredient in [3] is a result proved in [2] which says that

$$\log M(r, f) \leq \left(1 + o(1)\right) \frac{\pi}{A(r, f)}$$

(5)
on a sequence of $r$-values if $\frac{1}{2} \leq \rho \leq \infty$. It is shown in [3] that there exist values of $r$ such that (4) and (5) hold simultaneously.

The arguments of this paper do employ some ideas from [3], but avoid the use of Ahlfors’s theory. Instead they use an improved version of the growth lemma introduced in [2] in order to prove (5).

The constant $1/\pi$ occurring in (3) is sharp. Given $\rho$ satisfying $\frac{1}{2} \leq \rho \leq \infty$ and $q \in \mathbb{N}$ there exists an entire function of order $\rho$ and $a_1, \ldots, a_q \in \mathbb{C}$ such that we have equality in (3) as $r \to \infty$. In fact, if $\frac{1}{2} \leq \rho < \infty$, then Mittag-Leffler’s function $E_{1/\rho}$ has this property, provided we choose $a_1 = 0$ if $\rho = 1$. For examples of infinite order we refer to [2, 3, 6].

If $0 < \rho < \frac{1}{2}$, then the constant $1/\pi$ occurring in (3) has to be replaced by $\sin \pi \rho/\pi$.

2 Lemmas

The next lemma is a generalization of similar results proved in [2, Lemma 1] and [4, Lemma 1].

Lemma 1 Let $\Phi(x)$ be positive and continuous for $x \geq x_0 > 0$ and define

$$\rho = \limsup_{x \to \infty} \frac{\Phi(x)}{x}$$

Then there exist sequences $(x_j)$, $(M_j)$, $(\mu_j)$ and $(\varepsilon_j)$ satisfying $x_j \to \infty$, $M_j \to \infty$, $\mu_j \to \rho$ and $\varepsilon_j \to 0$ as $j \to \infty$ such that

$$\Phi(x_j + h) \leq \Phi(x_j) + \mu_j h + \varepsilon_j$$

for $|h| \leq M_j/\mu_j$. Here

$$\limsup_{t \to 0^+} \frac{\Phi(x_j + t) - \Phi(x_j)}{t} \leq \mu_j \leq \liminf_{t \to 0^-} \frac{\Phi(x_j + t) - \Phi(x_j)}{t}.$$
Moreover, if \( \Psi(x) \) is positive and continuously differentiable for \( x \geq x_0 \) such that \( \Psi(x)/x \) is non-decreasing, \( \Psi'(x) \leq \sqrt{\Psi(x)} \), and \( \int_{x_0}^{\infty} dx/\Psi(x) < \infty \), then \((x_j)\) and \((\mu_j)\) can be chosen such that
\[
\mu_j = o(\Psi(\Phi(x_j)))
\]
as \( j \to \infty \).

**Proof**  First we consider the case that \( \rho = \infty \). If \( \Phi \) is increasing and differentiable, then the conclusion was proved in [4, Lemma 1]. In this case we clearly have \( \mu_j = \Phi'(x_j) \). Here we only note that if we do not assume that \( \Phi \) is differentiable, then the argument used in [4] gives the above conclusion. Moreover, this remains true if \( \Phi \) is only assumed to be non-decreasing (instead of increasing).

The general case where \( \Phi \) is not necessarily non-decreasing can be reduced to the case where \( \Phi \) is non-decreasing. In fact, let \( \Phi(x) = \max_{x_0 \leq t \leq x} \Phi(t) \). Then \( \Phi \) is non-decreasing and \( \Phi(x) \leq \Phi(x) \). For \( \Phi \) we now obtain sequences \((x_j), (M_j), (\mu_j)\) and \((\epsilon_j)\) with the properties described in the lemma. But because \( \mu_j \to \infty \), we have
\[
\liminf_{t \to 0^-} \frac{\Phi(x_j + t) - \Phi(x_j)}{t} \to \infty
\]
as \( j \to \infty \) and thus \( \Phi(x_j) = \Phi(x_j) \) for large \( j \). This shows that the sequences \((x_j), (M_j), (\mu_j)\) and \((\epsilon_j)\) also have the desired properties with respect to \( \Phi \).

Now we consider the case that \( \rho < \infty \). In this case we consider a strong proximate order \( \rho^*(r) \) for \( f(r) = \exp \Phi(\log r) \) and choose \( x_j = \log r_j \) where \( f(r_j) = \rho^*(r_j) \); see [8, §I.12]. It can then be shown (cf. [3]) that \( x_j \) has the desired properties. We omit the details.

The next lemma follows from an inspection of the proof of [2, Theorem 1].

**Lemma 2**  Let \( f \) be an entire function and let \((r_j), (M_j), (\mu_j), (\gamma_j)\) and \((\epsilon_j)\) be sequences satisfying \( r_j \to \infty, M_j \to \infty, \mu_j \to \rho, \gamma_j \to \infty \) and \( \epsilon_j \to 0 \) as \( j \to \infty \). Suppose that
\[
T(r,f) \leq (1 + \epsilon_j) \left( \frac{r}{r_j} \right)^{\mu_j} \gamma_j \tag{6}
\]
for \( |\log r/r_j| \leq M_j/\mu_j \). If \( \frac{1}{2} \leq \rho \leq \infty \), then
\[
\log M(r_j, f) \leq (1 + o(1)) \pi \mu_j \gamma_j
\]
as \( j \to \infty \).

### 3  Proof of the theorem

We put
\[
N(r) = \frac{1}{q-1} \left( \sum_{j=1}^{q} N \left( r, \frac{1}{f - a_j} \right) - N \left( r, \frac{1}{f} \right) \right),
\]
replacing \( \frac{1}{q-1} \) by a (large) constant if \( q = 1 \) here, and, for \( \epsilon > 0 \), we define
\[
\gamma(r) = \max\{T(r,f), (1 + \epsilon)N(r)\}.
\]
We apply Lemma 1 to $\Phi(x) = \log(e^x)$ and obtain sequences $(x_j), (M_j), (\mu_j)$ and $(\varepsilon_j)$ with the properties described there. We put $r_j = \exp x_j$. In terms of $\gamma$ we then have

$$\gamma(r) \leq e^{\varepsilon_j} \left( \frac{r}{r_j} \right)^{\mu_j} \gamma(r_j)$$

(7)

for $|\log r/r_j| \leq M_j/\mu_j$. Noting that $T(r, f) \leq \gamma(r)$ we obtain (6) with $\gamma_j = \gamma(r_j)$. Thus

$$\log M(r_j, f) \leq (1 + o(1))\pi \mu_j \gamma_j$$

by Lemma 2.

Suppose first that $\rho < \infty$. Then $\gamma(r) = (1 + \varepsilon)N(r)$ for all large $r$ by (2). (Note that the exceptional set of $r$-values occurring there is bounded for functions of finite order.) In particular, $\gamma(r_j) = (1 + \varepsilon_j)N(r_j)$ and

$$\mu_j \leq \lim_{t \to 0^-} \frac{n(r_j + t)}{N(r_j)}$$

where

$$n(r) = \frac{1}{q-1} \left( \sum_{j=1}^q n \left( r, \frac{1}{f - a_j} \right) - n \left( r, \frac{1}{f^j} \right) \right).$$

We thus obtain

$$\log M(r_j, f) \leq (1 + o(1))(1 + \varepsilon)\pi \lim_{t \to 0^-} n(r_j + t)$$

and hence

$$\log M(r_j', f) \leq (1 + o(1))(1 + \varepsilon)\pi n(r_j')$$

for $r_j' = r_j - \delta_j$, provided $\delta_j$ is sufficiently small. Letting $\varepsilon \to 0$ we obtain the conclusion.

Suppose now that $\rho = \infty$. We shall show that

$$T(r_j, f) < \gamma(r_j).$$

(8)

Then we again have $\gamma_j = \gamma(r_j) = (1 + \varepsilon_j)N(r_j)$ and in fact $\gamma(r) = (1 + \varepsilon)N(r)$ in some open interval containing $r_j$. The proof can then be completed as before.

To prove (8) we need a more precise form of the error term $"o(1)"$ in (2). We have [7, §2.2]

$$T(r, f) \leq N(r) + C_1 \log \frac{1}{R - r} + C_2 \log T(R, f)$$

with constants $C_1, C_2$ if $R > r \geq r_0$. We choose $r = r_j$ and $R_j = r_j e^{1/\mu_j}$. Then

$$\log T(R_j, f) \leq \log \gamma(R_j) \leq \log \gamma(r_j) + O(1)$$

by (7) and

$$\log \frac{1}{R_j - r_j} = \log \frac{1}{r_j \left( e^{1/\mu_j} - 1 \right)} \leq \log \mu_j + O(1)$$

as $j \to \infty$. Choosing $\Psi(x) = x^{3/2}$ in Lemma 2 we may achieve that

$$\mu_j = o \left( (\log \gamma(r_j))^{3/2} \right)$$

and

$$\log M(r_j', f) \leq (1 + o(1))(1 + \varepsilon)\pi n(r_j')$$

for $r_j' = r_j - \delta_j$, provided $\delta_j$ is sufficiently small. Letting $\varepsilon \to 0$ we obtain the conclusion.
so that
\[ \log \mu_j \leq \frac{3}{2} \log \gamma(r_j) + O(1). \]

We thus obtain
\[ T(r_j, f) \leq N(r_j) + \frac{3}{2} C_1 \log \gamma(r_j) + C_2 \log \gamma(r_j) + O(1) \leq N(r_j) + O(\log \gamma(r_j)). \]  \hspace{1cm} (9)

Now, if (8) does not hold, then \( T(r_j, f) = \gamma(r_j) \geq (1 + \varepsilon) N(r_j) \). Thus (9) yields
\[ \gamma(r_j) \leq \frac{1}{1 + \varepsilon} \gamma(r_j) + O(\log \gamma(r_j)), \]
which is impossible. Therefore (8) holds, and this completes the proof.

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References


Walter Bergweiler
Mathematisches Seminar
Christian–Albrechts–Universität zu Kiel
Ludewig–Meyn–Str. 4
D–24098 Kiel