On a theorem of Gol'dberg concerning meromorphic solutions of algebraic differential equations

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Dedicated to Professor A. A. Gol'dberg

Abstract

We give a new proof of a generalization of Gol'dberg's theorem that meromorphic solutions of first order algebraic differential equations have finite order of growth. The main tool is a result of Zalcman concerning normal families.

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1 Introduction and result

A theorem of Gol'dberg [4] says that if a function meromorphic in the plane satisfies a first order algebraic differential equation, then it has finite order. Different proofs of this result have been given by Bank and Kaufman ([1], see also [8, §11]) and by Barsegian [2]. In this note we give a proof of Gol'dberg's theorem which is different from (and perhaps simpler than) the proofs mentioned. The proof is based on a result of Zalcman [10] concerning normal families.

Barsegian [3] has extended Gol'dberg's result to certain differential equations of higher order. Our method also gives his result, and thus we shall in fact prove his generalization of Gol'dberg's theorem. In order to state it, let f be meromorphic, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $r_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for $j = 1, 2, \ldots, n$, and put $r = (r_1, r_2, \ldots, r_n)$. Define $M_r[f]$ by

$$M_r[f](z) = f'(z)^{r_1} f''(z)^{r_2} \ldots f^{(n)}(z)^{r_n},$$

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with the convention that $M_{(0)}[f]=1$. We call $w(r)=r_1+2r_2+\ldots+nr_n$ the *weight* of $M_r[f]$. A differential polynomial $P[f]$ is an expression of the form

$$P[f](z)=\sum_{r\in I} a_r(z,f(z))M_r[f](z) \tag{1}$$

where the a_r are rational in two variables and I is a finite index set. The *weight* $w(P)$ of $P[f]$ is given by $w(P)=\max_{r\in I} w(r)$. Now Barsegian's result may be stated as follows.

Theorem *Let f be meromorphic in $\mathbb{C}$ and let $P[f]$ be a differential polynomial. If f satisfies the differential equation $(f')^n=P[f]$ where $n\in\mathbb{N}$ and $n>w(P)$, then f has finite order.*

It is clear that any first order differential equation can be written in the above form. Thus Gol'dberg's theorem is contained in this result.

2 Proof of the theorem

We assume that f has infinite order. It follows from the Ahlfors-Shimizu form of the Nevanlinna characteristic that there exists a sequence (a_k) such that $\log f^{\#}(a_k)/\log|a_k|\to\infty$ as $k\to\infty$, where $f^{\#}$ denotes the spherical derivative of f. This implies that the family $\{f(a_k+z)\}_{k\in\mathbb{N}}$ is not normal at zero. By a result of Zalcman [10], there exist sequences (b_k) and (ρ_k) such that $|b_k-a_k|<1$, $\rho_k\to 0$, and $h_k(z)=f(b_k+\rho_k z)$ converges locally uniformly to a non-constant meromorphic function h. It follows from Zalcman's proof that we may assume that $\rho_k=1/f^{\#}(b_k)$ and $f^{\#}(b_k)\geq f^{\#}(a_k)$. This implies that $b_k^M\rho_k\to 0$ for any fixed constant M.

In the differential equation $f'(z)^n=P[f](z)$ we now replace z by $b_k+\rho_k z$. Assuming that $P[f]$ has the form (1) we obtain

$$\rho_k^{-n}h_k'(z)^n=\sum_{r\in I} a_r(b_k+\rho_k z,h_k(z))\rho_k^{-w(r)}M_r[h_k](z).$$

We multiply by ρ_k^n and take the limit as $k\to\infty$. Because $w(r)<n$ for all $r\in I$ by hypothesis and $b_k^M\rho_k\to 0$ for any fixed constant M we obtain $h'(z)^n=0$. It follows that h is constant, a contradiction.

3 Remark

The method can also be applied to certain equations where $n=w(P)$. We illustrate this with an example. Consider the differential equation

$$f''(z)=L(f(z),z)f'(z)^2+M(f(z),z)f'(z)+N(f(z),z) \tag{2}$$

with rational functions L, M, and N. Suppose that it has a meromorphic solution of infinite order. The method used in the proof of the theorem shows that there exists a non-constant meromorphic function h satisfying $h''(z)=l(h(z))h'(z)^2$, with $l(w)=L(w,\infty)$. It follows (cf. [5, p. 322] or [7, Satz 6.1 (b)]) that h also satisfies a differential equation of the form

$$h'(z)^n=P(h(z)) \tag{3}$$

and that $l(w) = P'(w)/nP(w)$. Here P is a polynomial and $n \in \mathbb{N}$. The differential equations of the form (3) which admit meromorphic solutions are known (see e. g. [6]), they coincide with those which do not have movable singularities. The corresponding functions $l(w)$ are listed in [5, p. 323]. In particular, it follows that we may assume that $\deg(P) \leq 4$ and $n \leq 6$. We thus see that the differential equation (2) can have meromorphic solutions of infinite order only if $l(w) = L(w, \infty)$ is of a very special form.

On the other hand, if h satisfies (3), then $f(z) = h(e^z)$ satisfies (2) with $L(w, z) = l(w) = P'(w)/nP(w)$, $M(w, z) = 1$, and $N(w, z) = 0$. Moreover, if h is transcendental, then it is an exponential, trigonometric or elliptic function, and it follows that f has infinite order.

For other examples of differential equations of the form (2) with meromorphic solutions of infinite order we refer to [9, §4].

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