

A quantitative version of Picard's theorem ^{*†}

Walter Bergweiler[‡]

Abstract

Let f be an entire function of order at least $\frac{1}{2}$, $M(r) = \max_{|z|=r} |f(z)|$, and $n(r, a)$ the number of zeros of $f(z) - a$ in $|z| \leq r$. It is shown that $\limsup_{r \rightarrow \infty} n(r, a) / \log M(r) \geq 1/2\pi$ for all except possibly one $a \in \mathbb{C}$.

1 Introduction and results

Let f be a transcendental entire function. Picard's theorem [5] says that there exists at most one value $a \in \mathbb{C}$ such that $f(z) - a$ has only finitely many zeros. Borel's theorem [2] gives a quantitative version of this result by saying that

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, a)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} \quad (1)$$

for all $a \in \mathbb{C}$, with at most one exception. Here $n(r, a)$ denotes the number of zeros of $f(z) - a$ in $|z| \leq r$, counted according to multiplicity, and $M(r) = \max_{|z|=r} |f(z)|$ is the maximum modulus of f . The quantity on the right side of (1) is called the order of f and denoted by ρ .

The "true" quantitative version of Picard's theorem is of course given by Nevanlinna's theory on the distribution of values and by Ahlfors's theory of covering surfaces, see [3, 8]. Here we will use the theories of Nevanlinna and Ahlfors to prove a quantitative version of Picard's theorem whose statement uses only "pre-Nevanlinna" terminology.

Theorem 1 *Let f be an entire function of order at least $\frac{1}{2}$. Then*

$$\limsup_{r \rightarrow \infty} \frac{n(r, a)}{\log M(r)} \geq \frac{1}{2\pi} \quad (2)$$

for all $a \in \mathbb{C}$, with at most one exception.

The proof will show that there exists an unbounded sequence (r_j) depending only on f such that $n(r_j, a) \geq (1/2\pi - o(1)) \log M(r_j)$ for all except possibly one value of a and $\log \log M(r_j) / \log r_j \rightarrow \rho$ as $j \rightarrow \infty$. Thus Theorem 1 can be considered as a strong form of Borel's theorem for the case that $\rho \geq \frac{1}{2}$.

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We note that the character of the problem is different if $\rho < \frac{1}{2}$. A classical result of Pólya [6] and Valiron [9, 10] says that if $0 < \rho < 1$, then

$$\limsup_{r \rightarrow \infty} \frac{n(r, a)}{\log M(r)} \geq \frac{\sin \pi \rho}{\pi} \quad (3)$$

for all $a \in \mathbb{C}$. Simple examples show that if $0 \leq \rho \leq \frac{1}{2}$, then we may have equality for all $a \in \mathbb{C}$ here. Thus for $0 \leq \rho \leq \frac{1}{2}$ the constant $1/2\pi$ in (2) has to be replaced by $\sin \pi \rho / \pi$, and this bound is sharp. (For $\frac{1}{6} < \rho \leq \frac{1}{2}$ this bound is better than $1/2\pi$).

Theorem 1, however, which deals with the case $\rho \geq \frac{1}{2}$, is probably not sharp. Note that for $\frac{1}{2} \leq \rho < \frac{5}{6}$ the estimate (3) is better than (2). It seems likely that the constant $1/2\pi$ on the right side of (2) can be replaced by $1/\pi$. This would be best possible. In fact, we have

$$\limsup_{r \rightarrow \infty} \frac{n(r, a)}{\log M(r)} = \frac{1}{\pi} \quad (4)$$

if $f(z) = \exp z$ and $a \neq 0$. If $\frac{1}{2} \leq \rho < \infty$, $\rho \neq 1$, and $f(z) = E_{1/\rho}(z)$, then (4) holds for all $a \in \mathbb{C}$. Here E_α denotes Mittag-Leffler's function. Note that $E_{1/\rho}$ has order ρ . Another example (of order 2) is $f(z) = \int_0^z \exp(-t^2) dt$, where (4) holds for $a = \pm\sqrt{\pi}/2$. A function of infinite order satisfying (4) for all $a \in \mathbb{C}$ is the function E_0 considered by Hayman [3, p. 81] and Pólya and Szegő [7, Vol. I, p. 115]. Other examples of infinite order can be obtained from the functions constructed in [1, Theorem 3].

Theorem 1 is an immediate consequence of the following result.

Theorem 2 *Let f be an entire function of order at least $\frac{1}{2}$. Then*

$$\limsup_{r \rightarrow \infty} \frac{n(r, a) + n(r, b)}{\log M(r)} \geq \frac{1}{\pi} \quad (5)$$

if $a, b \in \mathbb{C}$, $a \neq b$.

In a certain sense Theorem 2 is sharp, because we have equality in (5) if $f(z) = \exp z$ and $a = 0$. But it seems likely that the constant $1/\pi$ on the right side of (5) can be replaced by $2/\pi$ if the order of f is sufficiently large and, in particular, if f has infinite order.

2 A growth lemma for real functions

Lemma *Let $\Phi(x)$ be increasing and twice differentiable for $x \geq x_0$ and assume that $\Phi(x) \geq cx$ for some positive constant c and arbitrarily large x . Then there exist sequences (x_j) , (M_j) , and (ε_j) such that $x_j \rightarrow \infty$, $M_j \rightarrow \infty$, and $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, $\Phi'(x_j) \geq c/8$, $\Phi''(x_j) \leq 2\Phi'(x_j)^2/\Phi(x_j)$, and $\Phi(x_j + h) \leq \Phi(x_j) + \Phi'(x_j)h + \varepsilon_j$ for $|h| \leq M_j/\Phi'(x_j)$.*

Without the claim about $\Phi''(x_j)$, this was proved in [1, Lemma 1]. The proof given there, however, also yields the above version. To see this, let F and v be as in [1, p. 168], and put $D(x) = F(x) - \Phi(x)$. Then D has a local minimum at v . Hence $D''(v) \geq 0$. Since $D(v) = D'(v) = 0$, we deduce that

$$\Phi''(v) \leq F''(v) = \frac{2F'(v)^2}{F(v)} = \frac{2\Phi'(v)^2}{\Phi(v)}$$

and the claim made about $\Phi''(x_j)$ follows.

3 Proof of Theorem 2

Let $\Phi(x) = \log T(e^x)$, where $T(r)$ denotes the Ahlfors-Shimizu characteristic of f . Let $A(r) = rT'(r)$, that is,

$$A(r) = \int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} dx dy.$$

Then

$$\Phi'(x) = \frac{A(r)}{T(r)} \tag{6}$$

and

$$\Phi''(x) = \frac{rA'(r)}{T(r)} - \left(\frac{A(r)}{T(r)}\right)^2 \tag{7}$$

where $r = e^x$.

Suppose first that f has infinite order. We choose x_j according to the lemma and define $r_j = \exp x_j$. As shown in [1, p. 171], we have

$$\log M(r_j) \leq (1 + o(1))\pi A(r_j). \tag{8}$$

From (6), (7), and the lemma we deduce that

$$\begin{aligned} r_j A'(r_j) &= \Phi''(x_j)T(r_j) + \frac{A(r_j)^2}{T(r_j)} \\ &\leq \frac{2\Phi'(x_j)^2}{\Phi(x_j)}T(r_j) + \frac{A(r_j)^2}{T(r_j)} \\ &= \left(\frac{2}{\log T(r_j)} + 1\right) \frac{A(r_j)^2}{T(r_j)}. \end{aligned}$$

Thus

$$r_j A'(r_j) \leq (1 + o(1)) \frac{A(r_j)^2}{T(r_j)} \tag{9}$$

as $j \rightarrow \infty$.

Suppose now that f has finite order $\rho \geq \frac{1}{2}$. We shall show that (8) and (9) hold again for a suitable sequence (r_j) . Let $\rho^*(r)$ be a strong proximate order for $T(r)$, cf. [4, §I.12], and define $\gamma(r) = r^{\rho^*(r)}$. Let r_j be an unbounded sequence satisfying $T(r_j) = \gamma(r_j)$. Then (r_j) is also a sequence of Pólya peaks (of order ρ) for $T(r)$ and the arguments of [1, p. 164] show that (8) holds again. Define $x_j = \log r_j$ and $\Psi(x) = \log \gamma(e^x)$. Then $\Phi(x) \leq \Psi(x)$ with equality for $x = x_j$ and thus $\Phi'(x_j) = \Psi'(x_j)$ and $\Phi''(x_j) \leq \Psi''(x_j)$. It follows from the properties of strong proximate orders that $\Psi'(x) \rightarrow \rho$ and $\Psi''(x) \rightarrow 0$ as $x \rightarrow \infty$. Thus $\Phi'(x_j) \rightarrow \rho$ and $\Phi''(x_j) \leq o(1)$ as $j \rightarrow \infty$. Combining this with (6) and (7) we see that (9) also holds again.

Define (cf. [3, p. 144], [8, p. 348])

$$L(r) = \int_{|z|=r} \frac{|f'(z)|}{1+|f(z)|^2} |dz|.$$

Then

$$L(r)^2 \leq 2\pi^2 r A'(r)$$

by Schwarz's inequality (cf. [3, p. 144], [8, p. 349]). Together with (9) it follows that

$$L(r_j) \leq (1 + o(1))\sqrt{2\pi} \frac{A(r_j)}{\sqrt{T(r_j)}} = o(A(r_j)).$$

Hence

$$n(r_j, a) + n(r_j, b) \geq (1 - o(1))A(r_j) \tag{10}$$

by the main result of Ahlfors's theory of covering surfaces (cf. [3, p. 148], [8, p. 349, inequality (II')]). Combining (8) and (10) we obtain (5).

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Fachbereich Mathematik
Technische Universität Berlin
Straße des 17. Juni 136
D-10623 Berlin
Germany

Email: bergweil@math.tu-berlin.de