On the Julia set of analytic self-maps of the punctured plane

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ABSTRACT Let f be a non-constant and non-linear entire function, g an analytic self-map of $\mathbb{C}\backslash\{0\}$, and suppose that $\exp \circ f = g \circ \exp$. It is shown that z is in the Julia set of f if and only if e^z is in the Julia set of g.

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1 Introduction and main result

Let f be an analytic self-map of a domain $D \subset \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. The main objects studied in complex dynamics are the Fatou set $F(f)$ which is defined as the set where the family $\{f^n\}$ of iterates of f is normal and the Julia set $J(f) := D\backslash F(f)$. By Montel's theorem $J(f) = \emptyset$ if $\widehat{\mathbb{C}}\backslash D$ contains more than two points. Thus it suffices to consider the cases $D = \widehat{\mathbb{C}}$, $D = \mathbb{C}$, and $D = \mathbb{C}^* := \mathbb{C}\backslash\{0\}$. If $D = \widehat{\mathbb{C}}$, then f is rational. This case was studied in long memoirs by Fatou [15] and Julia [19] between 1918 and 1920 and has been the object of much research in recent years, see the books [5, 9, 29] for an introduction. The case that $D = \mathbb{C}$ so that f is entire was considered first by Fatou [16] in 1926 and since then by many other authors, see [6] and [11, §4] for surveys. This paper is concerned with the case $D = \mathbb{C}^*$ which was studied first by Radström [27] in 1953 and more recently in [2, 8, 12–14, 20–26].

Given an analytic self-map g of $\mathbb{C}^*$ there exists an entire function f satisfying

$$\exp f(z) = g(e^z) \tag{1}$$

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for all $z \in \mathbb{C}$. This function f is unique up to an additive constant which is a multiple of $2\pi i$. It follows from (1) that

$$\exp f^n(z) = g^n(e^z) \tag{2}$$

for $n \in \mathbb{N}$. From (2) we can easily deduce that if $\exp z_0 \in F(g)$, then $z_0 \in F(f)$; that is,

$$\exp^{-1} F(g) \subset F(f). \tag{3}$$

In fact, let U be a neighborhood of z_0 and $V = \exp U$. If $g^{n_k} \to 0$ or $g^{n_k} \to \infty$ in V as $k \to \infty$, then $\Re f^{n_k} \to -\infty$ or $\Re f^{n_k} \to \infty$ and hence $|f^{n_k}| \to \infty$ in U as $k \to \infty$. If $g^{n_k} \to \varphi \not\equiv 0, \infty$ in V, then $|f^{n_k}(z) - \psi(e^z)| = 2\pi i m_k + o(1)$ for $z \in U$ as $k \to \infty$, where $m_k \in \mathbb{Z}$ and $\exp \psi = \varphi$. Again we find that $\{f^{n_k}\}$ has a convergent subsequence. We thus conclude that $\{f^n\}$ is normal in U if $\{g^n\}$ is normal in V. Hence (3) holds.

It is less obvious that we also have

$$\exp^{-1} J(g) \subset J(f) \tag{4}$$

and hence, together with (3),

$$\exp^{-1} J(g) = J(f). \tag{5}$$

Consider for example $f(z) = 2z$ and $g(z) = z^2$. Then (1) holds and $\{f^n\}$ is normal in $\mathbb{C}^*$ so that $J(f) = \{0\}$, but $J(g) = \{z : |z| = 1\}$. Note, however, that in this example f is linear, a case which is usually excluded in complex dynamics.

Theorem *Let f be entire, g an analytic self-map of $\mathbb{C}^*$, and suppose that* (1) *holds. If f is not linear or constant, then* (5) *holds.*

This result is stated in [12, Lemma 1.2] and [20, Lemma 2.2], but I have been unable to follow the arguments for (4) given there. The proofs of (3) in [12] and [20] are different from the one given above. On the other hand, the question whether (5) holds was raised in [24] and certain partial results were obtained. In particular, the above theorem answers the question asked in [24] whether $J(g) = \mathbb{C}^*$ implies that $J(f) = \mathbb{C}$.

Our theorem may be useful to obtain results for analytic self-maps of $\mathbb{C}^*$ from those for entire functions. For example, it was proved in [2, 24] that if g is an analytic self-map of $\mathbb{C}^*$, then the components of $F(g)$ are simply or doubly connected. This result follows immediately from our theorem and Lemma 3 below. As another example we mention the results of Baker and Weinreich [4] on the boundary of unbounded invariant components of the Fatou set of transcendental entire functions. Our theorem immediately yields analogous results for analytic self-maps of $\mathbb{C}^*$. A further application concerns the Lebesgue measure and the Hausdorff dimension of Julia sets of entire functions and analytic self-maps of $\mathbb{C}^*$, see [13]. We also mention that it was used in [4, 7, 18] that (5) holds for certain particular examples of functions f and g satisfying (1).

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2 Lemmas

Lemma 1 *Let f be a (non-constant and non-linear) rational function, entire function, or analytic self-map of $\mathbb{C}^*$. Then $J(f)$ is the closure of the set of repelling periodic points of f.*

Recall that z_0 is called a repelling periodic point of f if $f^n(z_0) = z_0$ and $|(f^n)'(z_0)| > 1$ for some $n \in \mathbb{N}$, with a slight modification if f is rational and $z_0 = \infty$. Lemma 1 is due to Fatou [15, §30, p. 69] and Julia [19, p. 99, p. 118] for rational functions, Baker [1] for entire functions, and Bhattacharyya [8, Theorem 5.2] for analytic self-maps of $\mathbb{C}^*$. A different proof (that applies to all three cases) has recently been given by Schwick [28].

We note that Lemma 1 also gives a short proof of (3) because if z_0 is a repelling periodic point of f, then $\exp z_0$ is a repelling periodic point of g. This is the proof of (3) given in [12].

Lemma 2 *Let f and g be as in the theorem. Then there exists $\ell \in \mathbb{Z}$ such that $f(z+2\pi i) = f(z) + \ell 2\pi i$. For $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ we have $f^n(z + m2\pi i) = f^n(z) + \ell^n m2\pi i$.*

The first claim follows easily from (1). The second claim is deduced from the first one by induction.

Lemma 3 *Let f and g be as in the theorem. Then all components of $F(f)$ are simply connected.*

To prove Lemma 3 we note that Lemma 2 implies that $|f(it)| = O(t)$ as $t \to \infty$, $t > 0$. The conclusion now follows from [6, Theorem 10].

3 Points that tend to infinity under iteration

Eremenko [10] considered for transcendental entire f the set

$$I(f) = \{z : \lim_{n\to\infty} |f^n(z)| = \infty\}$$

and proved that

$$J(f) = \partial I(f). \tag{6}$$

The main difficulty is to prove that $I(f) \neq \emptyset$. Once this is kown, (6) is not difficult to deduce. Eremenko's proof that $I(f) \neq \emptyset$ is based on the theory of Wiman and Valiron on the behavior of entire functions near points of maximum modulus [17, 30]. His proof shows that there exists $z \in I(f)$ such that $|f^{n+1}(z)| \sim M(|f^n(z)|, f)$ as $n \to \infty$, where $M(r, f) = \max_{|\zeta|=r} |f(\zeta)|$. Since $\log M(r, f)/\log r \to \infty$ as $r \to \infty$ for transcendental entire f it follows that $\log |f^{n+1}(z)|/\log |f^n(z)| \to \infty$ as $n \to \infty$. We define

$$I'(f) = \left\{ z : \lim_{n\to\infty} \frac{\log |f^{n+1}(z)|}{\log |f^n(z)|} = \infty \right\}$$

and deduce that $I'(f) \neq \emptyset$. Again it follows that $J(f) = \partial I'(f)$.

The theory of Wiman-Valiron and the arguments of Eremenko based on it do not require that f is transcendental entire but only that f is analytic in a neighborhood of ∞ and that ∞ is an essential singularity of f. In particular, the above arguments remain valid for analytic self-maps of $\mathbb{C}^*$ with an essential singularity at ∞. We summarize the above discussion as follows.

Lemma 4 *Let f be a transcendental entire functions or an analytic self-map of $\mathbb{C}^*$ with an essential singularity at ∞. Then $I'(f) \neq \emptyset$ and $J(f) = \partial I'(f)$.*

4 Proof of the theorem

If g is rational, then $g(z) = cz^k$ where $c \in \mathbb{C}^*$ and $k \in \mathbb{Z}$. It follows that f is linear or constant, contradicting the hypothesis. Thus g is transcendental and there is no loss of generality in assuming that ∞ is an essential singularity of g.

We have already shown in the introduction (and also after Lemma 1) that (3) holds. It remains to prove (4). We thus assume that $w_0 = \exp z_0 \in J(g)$ and have to show that $z_0 \in J(f)$. Let U be a neighborhood of z_0. We shall show that $U \cap J(f) \neq \emptyset$. The conclusion then follows since $J(f)$ is closed.

By Lemma 1 and Lemma 4 there exist $z_1, z_2 \in U$ such that $w_1 = \exp z_1$ is a repelling periodic point of g, say $g^k(w_1) = w_1$, and $w_2 = \exp z_2 \in I'(g)$. If $z_1 \in J(f)$ or $z_2 \in J(f)$, then we are done. If z_1 and z_2 lie in different components of $F(f)$, then we connect them by a path in U. This path meets $J(f)$ and again we have $U \cap J(f) \neq \emptyset$. Thus we may assume that z_1 and z_2 lie in the same component of $F(f)$. Since $w_2 \in I'(g) \subset I(g)$ we deduce from (2) that $z_2 \in I(f)$ and hence $z_1 \in I(f)$.

By a result of Baker ([3, Lemma 1], see also [6, Lemma 7]) and Lemma 3 there exists a constant C such that

$$|f^n(z_2)| \leq C|f^n(z_1)| \tag{7}$$

for all large n.

Since $g^k(w_1) = w_1$ we have $\exp f^k(z_1) = \exp z_1$ and hence $f^k(z_1) = z_1 + m2\pi i$ for some $m \in \mathbb{Z}$. By Lemma 4 we have

$$f^{2k}(z_1) = f^k(z_1 + m2\pi i) = f^k(z_1) + \ell^k m2\pi i = z_1 + m(1 + \ell^k)2\pi i$$

and induction shows that

$$f^{nk}(z_1) = z_1 + m\left(\sum_{j=0}^{n-1} \ell^{jk}\right) 2\pi i.$$

We deduce that if $M > \max\{1, |\ell|\}$, then

$$|f^{nk}(z_1)| = o(M^{nk})$$

as $n \to \infty$. Combining this with (7) we find that

$$|f^{nk}(z_2)| = o(M^{nk}) \tag{8}$$

as $n \to \infty$.

On the other hand, $|g^n(w_2)| \to \infty$ and $\log|g^{n+1}(w_2)|/\log|g^n(w_2)| \to \infty$ as $n \to \infty$ by the choice of w_2. Hence $\Re f^n(z_2) \to \infty$ and $\Re f^{n+1}(z_2)/\Re f^n(z_2) \to \infty$ as $n \to \infty$ by (2). We deduce that

$$|f^n(z_2)| \geq \Re f^n(z_2) \geq M^n$$

for all large n, contradicting (8). Thus z_1 and z_2 cannot lie in the same component of $F(f)$ and the proof of the theorem is complete.

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