# An introduction to complex dynamics *

Walter Bergweiler

## Contents



*Textos de Matemática, Universidade de Coimbra, Série B, No. 6 (1995).

# 1 Introduction

Iteration occurs in many parts of mathematics. For example, many algorithms of numerical mathematics are based on it. Given a set $D$ and a function $f : D \to D$ the iterates $f^n$ of $f$ are defined by $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n \geq 2$. The main problem in iteration theory is to study the behavior of the sequence $(f^n)$ as $n \to \infty$. For example, one may ask for which $z \in D$ the sequence $(f^n(z))$ is convergent or periodic. Of course, the theory developed heavily depends on the assumptions that are made about $D$ and $f$.

In this course, we shall study the case that $D$ is a domain on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $f$ is meromorphic in $D$. It turns out that the limiting behavior of $(f^n)$ is relatively simple if $\widehat{\mathbb{C}} \backslash D$ contains more than two points. We shall mainly consider the remaining cases which, after a normalization, are $D = \widehat{\mathbb{C}}$, $D = \mathbb{C}$, and $D = \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. We shall see that here, after exclusion of certain trivial cases, the domain $D$ divides into two sets, the Fatou set $F$ and the Julia set $J$, such that the limiting behavior of $(f^n)$ is relatively tame in $F$ while it is "chaotic" in $J$. These sets turn out to be quite complicated, with a beauty also appealing to the non-mathematician. Computer graphics of Julia sets can be found in [30] and many other places. Much of these notes is devoted to a study of the various properties of Fatou and Julia sets.

Good introductions into the subject are the books by Beardon [6], Carleson and Gamelin [11], and Steinmetz [40], as well as the lecture notes by Milnor [27]. We also mention the articles of Blanchard [10] and Lyubich [26]. All of them are, however, only concerned with the case that $D = \widehat{\mathbb{C}}$ so that $f$ is rational. We shall consider this case as well as the two other cases $D = \mathbb{C}$ and $D = \mathbb{C}^*$ mentioned above. Surveys of the case $D = \mathbb{C}$ are given in [17, §4] and [8]. In [8] the more general case that $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is also considered, where $f^n$ may not be defined at certain points. We shall not consider this case here.

We finish this introduction with a few historical remarks. The iteration theory of rational functions was created by Julia [24] and Fatou [18], who wrote long memoirs on the subject between 1918 and 1920. Earlier work on iteration had mainly been concerned with the behavior of the iterates near fixpoints (§9 of these notes). In 1926 Fatou [19] extended some of the results to entire functions. The case of holomorphic self-maps of $\mathbb{C}^*$ was first considered by Radström [33] in 1953.

One can say that there was comparatively little work on iteration in the sixty years following the fundamental papers of Fatou and Julia, but some important contributions were made in this period. We mention the work of Cremer [13, 14] and Siegel [39] (recently completed by Yoccoz [43, 44]) on the "center problem" left open by Fatou and Julia as well as the many contributions by Baker (e. g. [3, 5]) and his students.

Around 1980 there was an explosion of interest in the subject. This was partially due to the already mentioned beautiful computer graphics related to it, but also to new mathematical methods introduced by Sullivan [41], Douady and Hubbard [15], and others.

# 2 Preliminaries

We shall assume some familiarity with complex analysis of one variable, but recall some results that are of particular importance for our purposes. Besides the results presented in this chapter we shall also need some results from Nevanlinna's theory on the distribution of values. These will be given in §6.

When dealing with convergent sequences of iterates we shall often use the following results.

**Theorem 1 (Weierstraß)** *Let* $(f_k)$ *be a sequence of functions holomorphic in a domain* $D \subset \mathbb{C}$ *and suppose that* $f_k$ *converges to a function* $f : D \to \mathbb{C}$ *as* $k \to \infty$*, locally uniformly in* $D$*. Then* $f$ *is holomorphic in* $D$ *and* $f_k'$ *converges to* $f'$ *as* $k \to \infty$*, locally uniformly in* $D$*.*

**Theorem 2 (Hurwitz)** *Let* $(f_k)$*,* $f$*, and* $D$ *be as in Theorem* 1 *and* $a \in \mathbb{C}$*. If* $f_k(z) \neq a$ *for all* $z \in D$ *and all* $k \in \mathbb{N}$*, then* $f(z) \neq a$ *for all* $z \in D$ *or* $f \equiv a$*.*

A consequence of Theorem 2, also referred to as Hurwitz's Theorem, is that if, under the above hypotheses, all $f_k$ are univalent, then $f$ is univalent or constant.

In the usual versions of these two theorems, the topology that defines terms like "convergence" is the one induced by the Euclidean metric, where the distance $d(z, w)$ between two points $z, w \in \mathbb{C}$ is given by $d(z, w) = |z-w|$. When dealing with subsets of $\widehat{\mathbb{C}}$ and meromorphic functions (with poles) it is more appropriate to use the chordal metric $\chi$ given by

$$\chi(z,w)=\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

for $z,w\in\mathbb{C}$, $\chi(z,\infty)=2/\sqrt{1+|z|^2}$ for $z\in\mathbb{C}$, and $\chi(\infty,\infty)=0$. Recall that if $z,w$ are mapped by stereographic projection onto $z',w'\in S=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2+x_3^3=1\}$, with $\infty$ being mapped to $(0,0,1)$, then $\chi(z,w)$ is the Euclidean distance in $\mathbb{R}^3$ between $z'$ and $w'$.

We note that if a sequence converges with respect to the Euclidean metric, then it also converges with respect to the chordal metric. The converse also holds, except when the limit is $\infty$.

From now on, if we do not specify the metric, we will always consider the chordal metric. We write $f_k\to f$ if $f_k$ converges to $f$ (with respect to the chordal metric) as $k\to\infty$, locally uniformly in the domain $D$ where the $f_k$ and $f$ are defined (or in a subdomain of $D$ if one is specified).

Weierstraß's theorem now takes the following form.

**Theorem 3 (Weierstraß)** *Let $(f_k)$ be a sequence of functions meromorphic in a domain $D\subset\widehat{\mathbb{C}}$ and suppose that $f_k\to f$ for some function $f:D\to\widehat{\mathbb{C}}$ as $k\to\infty$. Then $f$ is meromorphic in $D$ or $f\equiv\infty$. If $f\not\equiv\infty$, then $f_k'\to f'$ as $k\to\infty$. If all $f_k$ are holomorphic and if $f\not\equiv\infty$, then $f$ is holomorphic.*

One of the main tools used in complex dynamics is Montel's theory of normal families.

**Definition 4** Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D\subset\widehat{\mathbb{C}}$. Then $\mathcal{F}$ is called *normal in $D$* if any sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly in $D$ and $\mathcal{F}$ is called *normal at a point* $z_0\in D$ if $z_0$ has a neighborhood $U$ such that $\mathcal{F}$ is normal in $U$.

It is not difficult to show that $\mathcal{F}$ is normal in $D$ if and only if $\mathcal{F}$ is normal at each point of $D$. We note that the limit functions of the convergent subsequences need not be in $\mathcal{F}$. In particular, $\infty$ is a possible limit function. By Weierstraß's theorem, limit functions are meromorphic and they are even holomorphic if all functions in $\mathcal{F}$ are, unless the limit is $\infty$.

By $f^\sharp$ we denote the spherical derivative of $f$; that is,

$$f^\sharp(z)=\frac{|f'(z)|}{1+|f(z)|^2}$$

if $z \neq \infty$ and $f(z) \neq \infty$, and $f^\sharp(z) = \lim_{w\to z} f^\sharp(w)$ otherwise.

**Theorem 5 (Marty)** *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \widehat{\mathbb{C}}$. Then $\mathcal{F}$ is called normal in $D$ if and only if $\{f^\sharp\}_{f\in\mathcal{F}}$ is locally uniformly bounded in $D$.*

The main results that we need are the following two theorems of Montel.

**Theorem 6 (Montel)** *Let $\mathcal{F}$ be a family of functions holomorphic in a domain $D \subset \widehat{\mathbb{C}}$. Suppose that there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in D$ and all $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in $D$.*

**Theorem 7 (Montel)** *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \widehat{\mathbb{C}}$. Let $a, b, c \in \widehat{\mathbb{C}}$ be pairwise distinct and suppose that $f(z) \neq a, b, c$ for all $z \in D$ and all $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in $D$.*

Note that Theorem 6 is a special case of Theorem 7. The proof of Theorem 7 is, however, considerably more difficult than that of Theorem 6. We shall refer to Theorem 7 as Montel's theorem, although Theorem 6 is also due to him. A consequence of Montel's theorem are the theorems of Picard.

**Theorem 8 (Great Picard Theorem)** *Let $D \subset \widehat{\mathbb{C}}$ be a domain, $z_0 \in D$, and let $f$ be meromorphic in $D\backslash\{z_0\}$. Let $a, b, c \in \widehat{\mathbb{C}}$ be pairwise disjoint and suppose that $f(z) \neq a, b, c$ for all $z \in D$. Then $f$ has a meromorphic extension to $D$.*

It follows that if $z_0$ is an essential singularity of $f$, then one of the equations $f(z) = a$, $f(z) = b$, $f(z) = c$ has infinitely many solutions in each neighborhood of $z_0$. In the special case $D = \widehat{\mathbb{C}}$ and $c = z_0 = \infty$ we obtain

**Theorem 9 (Little Picard Theorem)** *A non-constant entire function assumes every finite value, with at most one exception.*

We shall not give the proofs of Theorem 5 – 9 as those can be found in many books on function theory, for example in [12]. That Theorem 8 (and hence Theorem 9) follows from Theorem 7 is due to Montel and can be found in [12, p. 300f] or [36, p. 60].

It is perhaps not so well-known that Theorem 7 can in turn be deduced from Theorem 9. This was done by Zalcman [45] using his Lemma 34 below. As we shall develop the Nevalinna theory (which immediately gives Theorem 9) later in §6, this yields an alternative approach to Theorems 6 – 8. After Lemma 34 we shall sketch the argument how Theorem 7 (and hence Theorems 6 and 8) follow from Theorem 9.

## 3 The case where the iterates are normal

**Theorem 10** *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $f : D \to D$ be meromorphic. Suppose that $\{f^n\}_{n\in\mathbb{N}}$ is normal in $D$. Then exactly one of the following possibilities holds:*

(i) *there exists $a \in D$ such that $f(a) = a$ and $f^n \to a$ as $n \to \infty$. If $a \neq \infty$, then $|f'(a)| < 1$,*

(ii) $\operatorname{dist}(f^n(z), \partial D) \to 0$ *as $n \to \infty$ (locally uniformly in $D$),*

(iii) *$f$ is a conformal automorphism of $D$.*

Here, by definition, a conformal automorphism of $D$ is a bijective meromorphic self-map of $D$, and $\operatorname{dist}(z, S)$ denotes the (chordal) distance of a point $z$ to a set $S$; that is, $\operatorname{dist}(z, S) = \inf_{w\in S} \chi(z, w)$.

**Remarks** 1. In view of Montel's Theorem, the hypothesis that $\{f^n\}_{n\in\mathbb{N}}$ be normal in $D$ is always satisfied if $\widehat{\mathbb{C}}\backslash D$ contains more than two points.

2. In case (ii) one can, in general, not conclude that $f^n \to a$ for some $a \in \partial D$. The theorem of Denjoy-Wolff says that this is the case if $D$ is the unit disk. As we shall see later in §10, this conclusion can also be obtained with suitable hypotheses concerning the extendability of $f$ to $\partial D$.

3. In case (iii) one can show that if $D$ is not simply or doubly connected, then there exists $n \in \mathbb{N}$ such that $f^n = id|_D$; that is, $f^n(z) = z$ for all $z \in D$.

*Proof of Theorem* 10. We first suppose that the family $\{f^n\}_{n\in\mathbb{N}}$ has a non-constant limit function, say $f^{n_k} \to \phi \not\equiv \mathit{const}$. Passing to a subsequence if necessary, we may assume that $m_k = n_{k+1} - n_k \to \infty$. There exists a subsequence $(m_{k_j})$ such that $f^{m_{k_j}} \to \psi$ for some function $\psi$. Since $f^{n_{k+1}} = f^{n_k} \circ f^{m_k} = f^{m_k} \circ f^{n_k}$ we have $\phi = \phi \circ \psi = \psi \circ \phi$ so that $\psi = id|_D$. Hence $f^{m_{k_j}} \to id|_D$ and this implies that $f$ is injective. By Hurwitz's Theorem, $f$ is also surjective and we have case (iii).

Now we assume that all limit functions of $\{f^n\}_{n\in\mathbb{N}}$ are constant. Suppose that (ii) does not hold. Then there exist $a \in D$ and a sequence $(n_k)$ such that $f^{n_k} \to a$ as $k \to \infty$. It follows, with $z \in D$, that $f(a) = f(\lim_{k\to\infty} f^{n_k}(z)) = \lim_{k\to\infty} f^{n_k}(f(z)) = a$. Assume first that $a \neq \infty$. Then $(f^{n_k})' \to 0$ by Weierstraß's Theorem. Thus $|(f^{n_k})'(a)| < 1$ for large $k$. But $(f^{n_k})'(a) =$

$f'(a)^{n_k}$ by the chain rule since $f(a) = a$. Hence $|f'(a)| < 1$. We deduce that there exists $\delta > 0$ such that $|f'(z)| \le 1 - \delta$ for $|z - a| \le \delta$. It follows that $|f(z) - a| = |f(z) - f(a)| = |\int_a^z f'(t)dt| \le (1 - \delta)|z - a|$ and hence $|f^n(z) - a| \le (1 - \delta)^n |z - a|$ for $|z - a| \le \delta$. Hence $f^n \to a$ in $D(a, \delta) = \{z : |z - a| < \delta\}$. From the normality of $\{f^n\}_{n \in \mathbb{N}}$ in $D$ we can now deduce that $f^n \to a$ in $D$ so that we have case (i).

Assume now that $a = \infty$. We consider $g(z) = 1/f(1/z)$ and note that $g^{n_k} \to 0$ in $D' = \{z : 1/z \in D\}$. As above we find that $g(0) = 0$, $|g'(0)| < 1$, and $g^n \to 0$ in $D'$. Hence $f^n \to \infty$ in $D$. ■

# 4 Fatou and Julia sets

In view of the results of §3 we shall assume from now on that $|\widehat{\mathbb{C}} \backslash D| \le 2$. Here and in the following $|S|$ denotes the cardinality of a set $S$. There is no loss of generality in assuming that $D \in \{\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*\}$ where $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. For example, if $f : \widehat{\mathbb{C}} \backslash \{a, b\} \to \widehat{\mathbb{C}} \backslash \{a, b\}$ where $a, b \in \mathbb{C}$, we consider $T(z) = (z - a)/(z - b)$ and $g = T \circ f \circ T^{-1}$. Then $g : \mathbb{C}^* \to \mathbb{C}^*$ and $g^n = T \circ f^n \circ T^{-1}$ so that the dynamics of $f$ can be described by those of $g$.

By $\text{Aut}(D)$ we denote the group of conformal automorphisms of $D$. The following lemma is well-known.

**Lemma 11**

(i) $\text{Aut}(\widehat{\mathbb{C}}) = \{T : T(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc \ne 0\}$

(ii) $\text{Aut}(\mathbb{C}) = \{T : T(z) = az + b, a, b \in \mathbb{C}, a \ne 0\}$

(iii) $\text{Aut}(\mathbb{C}^*) = \{T : T(z) = az \text{ or } T(z) = \frac{a}{z}, a \in \mathbb{C}^*\}$

The iteration of conformal automorphisms of $D$ is relatively simple, see e. g. [6, §1.2]. The iteration of constant functions is trivial. Thus, from now on, we shall always make the following

**Assumption:** *$D \in \{\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*\}$, $f : D \to D$ is meromorphic, $f \notin$ Aut$(D)$, $f \not\equiv$ const.*

We note that $f$ is rational if $D = \widehat{\mathbb{C}}$ and $f$ is entire if $D = \mathbb{C}$.

The basic definition of the theory is the following

**Definition 12** The *Fatou set* $F = F(f)$ is the set where the family $\{f^n\}_{n\in\mathbb{N}}$ is normal. The *Julia set* $J = J(f)$ is the complement of the Fatou set; that is, $J = D\backslash F$.

We begin with some simple properties of $F$ and $J$.

**Lemma 13**

(i) *$F$ is open and $J$ is closed.*

(ii) *$F$ and $J$ are completely invariant; that is, $z \in F$ if and only if $f(z) \in F$ and $z \in J$ if and only if $f(z) \in J$.*

(iii) *$F(f^n) = F(f)$ and $J(f^n) = J(f)$ for all $n \in \mathbb{N}$.*

(iv) *If $g = T \circ f \circ T^{-1}$ where $T \in \mathrm{Aut}(D)$, then $J(g) = T(J(f))$.*

If $f$ and $g$ are as in (iv), then we say that $f$ and $g$ are *conjugate* (by $T$). The map $T$ is called the *conjugacy.*

*Proof of Lemma* 13. (i) This is so by definition.

(ii) If $U$ is a neighborhood of $z_0 \in D$, then $V = f(U)$ is a neighborhood of $f(z_0)$. Conversely, if $V$ is a sufficiently small neighborhood of $f(z_0)$, then there exists a neighborhood $U$ of $z_0$ such that $f(U) = V$. The conclusion now follows since $(f^{n_k}|_V)$ converges if and only if $(f^{n_k+1}|_U)$ converges.

(iii) First we note that $F(f) \subset F(f^n)$ because $\{f^k\}_{k\in\mathbb{N}} \supset \{(f^n)^k\}_{k\in\mathbb{N}}$. To prove that $F(f^n) \subset F(f)$ let $z_0 \in F(f^n)$ and let $(f^{n_k})$ be a sequence of iterates of $f$. There exists $l \in \{0, 1, \ldots, n-1\}$ and a subsequence $(n_{k_j})$ of $(n_k)$ such that $n_{k_j} = nm_j - l$, $m_j \in \mathbb{N}$. By assumption there exists a neighborhood $U$ of $z_0$ such that a subsequence of $(f^n)^{m_j}$ converges locally uniformly in $U$. Without loss of generality we may assume that $(f^n)^{m_j}|_U \to \phi$; that is, $f^{l+n_{k_j}}|_U \to \phi$. If $\phi \equiv a \in \widehat{\mathbb{C}}\backslash D$, then $\mathrm{dist}(f^{n_{k_j}}(z), \widehat{\mathbb{C}}\backslash D) \to 0$ as $j \to \infty$, locally uniformly for $z \in U$, and hence some subsequence of $(f^{n_k})$ converges in $U$. If $\phi \not\equiv \textit{const}$ or if $\phi \equiv a \in D$, then we have either again $\mathrm{dist}(f^{n_{k_j}}(z), \widehat{\mathbb{C}}\backslash D) \to 0$ as $j \to \infty$, or some subsequence of $(f^{n_{k_j}})$ converges to a function $\psi$ satisfying $f^l(\psi) = \phi$. In any case we find that some subsequence of $(f^{n_k})$ converges in $U$ so that $z_0 \in F(f)$.

(iv) This (simple) proof is left as an exercise. ■

**Definition 14** For $z_0 \in D$ we call $O^+(z_0) = \{f^n(z_0) : n \in \mathbb{N}\}$ the *forward orbit* of $z_0$, $O^-(z_0) = \bigcup_{n=1}^{\infty} f^{-n}(z_0) = \bigcup_{n=1}^{\infty}\{z : f^n(z) = z_0\}$ the *backward orbit* of $z_0$, and $O(z_0) = O^+(z_0) \cup \{z_0\} \cup O^-(z_0)$ the *orbit* of $z_0$. For $S \subset D$ we define $O^{(\pm)}(S) = \bigcup_{z \in S} O^{(\pm)}(z)$.

With this definition, Lemma 13 (ii) takes the form $O(J) = J$ and $O(F) = F$.

**Definition 15** A point $z_0 \in D$ is called *exceptional point* if $O^-(z_0)$ is finite. The set of exceptional points is denoted by $E = E(f)$.

It is easy to see that if $f$ and $g$ are conjugate by $T$, then $E(g) = T(E(f))$. Quite generally we can say that most dynamical properties are preserved under conjugation. We will use this principle often without always mentioning it explicitly.

**Theorem 16**

(i) *If $D = \widehat{\mathbb{C}}$, then $|E| \le 2$ and $E \subset F$. If $|E| = 1$, then $f$ is conjugate to a polynomial, with the exceptional point being mapped to $\infty$ by the conjugacy. If $|E| = 2$, then $f$ is conjugate to $g(z) = z^d$ for some $d \in \mathbb{Z}\backslash\{0, \pm 1\}$, with the exceptional set being mapped to $\{0, \infty\}$ by the conjugacy.*

(ii) *If $D = \mathbb{C}$, then $|E| \le 1$. If $|E| = 1$, then there exists $m \ge 0$ and an entire function $h$ such that $f$ is conjugate to $g(z) = z^m e^{h(z)}$, with the exceptional point being mapped to $0$ by the conjugacy.*

(iii) *If $D = \mathbb{C}^*$, then $E = \emptyset$.*

It follows that $|\widehat{\mathbb{C}}\backslash D \cup E| \le 2$ in all three cases. The theorem says, in particular, that $f$ may also be considered as a self-map of $D\backslash E$.

*Proof of Theorem* 16. (i) Let $d = \deg(f)$, the degree of $f$. For simplicity we assume that $\infty \notin E(f)$ because this can be achieved by conjugation. Each point in $\mathbb{C}$ has $d$ preimages under $f$, counted according to multiplicity. It is easy to see that $f'$ has at most $2d - 2$ zeros, again counted according to multiplicity. We conclude that for any finite set $S \subset \mathbb{C}$ we have $|f^{-1}(S)| \ge d|S| - (2d-2) = d(|S|-2)+2$. Applying this to $f^n$, which has degree $d^n$, we find that $|f^{-n}(S)| \ge d^n(|S| - 2) + 2$. We deduce that if $|S| \ge 3$, then there exists $z_0 \in S$ such that $|O^-(z_0)| = \infty$. Hence $|E| \le 2$.

Assume now that $|E| = |E(f)| = 1$. A suitable $g$ conjugate to $f$ then satisfies $E(g) = \{\infty\}$. Clearly $g^{-1}(E(g)) \subset E(g)$. We deduce that $g(z) = \infty$ only if $z = \infty$. Thus $g$ is a polynomial. It is easy to see that $\infty \in F(g)$ for any polynomial $g$. Hence $E(f) \subset F(f)$.

Assume now that $|E(f)| = 2$. Then we can find $g$ conjugate to $f$ such that $E(g) = \{0, \infty\}$. Again we have $g^{-1}(E(g)) \subset E(g)$ and thus either $g(0) = 0$ and $g(\infty) = \infty$ or $g(0) = \infty$ and $g(\infty) = 0$. In the first case $g$ is a polynomial whose only zero is at the origin so that $g(z) = cz^d$ for some $c \in \mathbb{C}^*$. In the second case we find similarly that $g(z) = cz^{-d}$ for some $c \in \mathbb{C}^*$. In both cases we may assume that $c = 1$ because this can be achieved by a suitable further conjugation. Again it is not difficult to see that $\{0, \infty\} \subset F(g)$ and thus $E(f) \subset F(f)$.

(ii) We may assume that $f$ is transcendental as the case of rational functions is covered by (i). Picard's Theorem says that $|f^{-1}(a)| = \infty$ and hence $|O^-(a)| = \infty$ for all $a \in \mathbb{C}$ with at most one exception. Thus $|E| \leq 1$. If $E(f) = \{a\}$, then $E(g) = \{0\}$ if $g = T \circ f \circ T^{-1}$ and $T(z) = z - a$. It follows that $g(z) \neq 0$ for $z \neq 0$. It is not difficult to deduce now that $g$ has the form stated. Vice versa, any function $g$ of the above form satisfies $E(g) = \{0\}$. If $m = 1$, then $g(0) = 0$ and $g'(0) = e^{h(0)}$. It will follow immediately from the results of §5 (Theorems 21 and 23) that $0 \in F(g)$ and $0 \in J(g)$ may be achieved by a suitable choice of $h$.

(iii) Again we may assume that $f$ is transcendental so that 0 or $\infty$ is an essential singularity of $f$. By the Great Picard Theorem, $|f^{-1}(a)| = \infty$ and hence $|O^-(a)| = \infty$ for all $a \in \mathbb{C}^*$. Hence $E = \emptyset$. ■

**Theorem 17** *If $z_0 \in J$ and $z_0 \notin E$, then $J = \overline{O^-(z_0)}$.*

By Theorem 16 the hypothesis that $z_0 \notin E$ is always satisfied if $z_0 \in J$ and $D = \widehat{\mathbb{C}}$ or $D = \mathbb{C}^*$.

We shall see in the next section that in many cases a point $z_0 \in J \backslash E$ is easily found. Thus Theorem 16 leads in an obvious way to an algorithm to produce computer pictures of Julia sets, at least for functions where the inverse function is easy to compute, e. g. for polynomials of low degree. Some difficulties do occur, however, see [30, §2] for a discussion.

*Proof of Theorem* 17. We have $\overline{O^-(z_0)} \subset J$ since $J$ is closed and completely invariant. To prove that $J \subset \overline{O^-(z_0)}$, let $z_1 \in J$, $U$ a neighborhood of $z_1$, and suppose that $U \cap O^-(z_0) = \emptyset$. Then $f^n(z) \neq w$ for all $n \in \mathbb{N}$, $z \in U$, and $w \in O^-(z_0)$. Because $|O^-(z_0)| = \infty$, Montel's Theorem yields that

$\{f^n\}_{n\in\mathbb{N}}$ is normal in $U$, contradicting the choice of $z_1$. Thus $U\cap O^-(z_0)\neq\emptyset$. It follows that $z_1\in\overline{O^-(z_0)}$ so that $J\subset\overline{O^-(z_0)}$. ■

**Theorem 18** *Let $U\subset D$ be open, $U\cap J\neq\emptyset$. Then $O^+(U)\supset D\backslash E$ and $O^+(U\cap J)\supset J\backslash E$. If $D=\widehat{\mathbb{C}}$ or $D=\mathbb{C}^*$, then $O^+(U\cap J)=J$.*

*Proof.* If $z_0\notin O^+(U)$, then $O^-(z_0)\cap O^+(U)=\emptyset$. Since $\{f^n|_U\}_{n\in\mathbb{N}}$ is not normal, $|\widehat{\mathbb{C}}\backslash O^+(U)|\leq 2$ by Montel's Theorem. Thus $|O^-(z_0)|\leq 2$ and hence $z_0\in E$. It follows that $O^+(U)\supset D\backslash E$. Since $J$ is completely invariant this implies that $O^+(U\cap J)\supset J\backslash E$. The last claim follows since $E\subset F$ if $D=\widehat{\mathbb{C}}$ and $E=\emptyset$ if $D=\mathbb{C}$. ■

It follows from Theorem 18 that if $K\subset D$ is compact and $K\cap E=\emptyset$, then there exists $N$ such that $K\subset\bigcup_{n=1}^N f^n(U)$ and $K\cap J\subset\bigcup_{n=1}^N f^n(U\cap J)$. If $f$ is rational, then we may take $K=J$ and obtain $J=\bigcup_{n=1}^N f^n(U\cap J)$. A stronger result will be proved in §8.

These results can be considered as an explanation of the "self-similarity" often found in Julia sets.

**Theorem 19** *Either $J=D$ or $J$ has empty interior.*

*Proof.* If the interior of $J$ is not empty, then we may take $U\subset J$ in Theorem 18 and obtain $J\supset O^+(U)\supset D\backslash E$ and thus $J=D$ because $J$ is closed. ■

We mention that the case $J=D$ can actually occur. Rational functions with this property can be obtained from the multiplication theorems of elliptic functions. Another example is $f(z)=(z-2)^2/z^2$, see [6, §11.9] for a fairly elementary proof that $J(f)=\widehat{\mathbb{C}}$. The first example of an entire function with $J=\mathbb{C}$ was given by Baker [4]. Actually the exponential function has this property. This was conjectured by Fatou [19] and proved by Misiurewicz [28].

# 5 Periodic points

**Definition 20** We say that $z_0\in D$ is a *periodic point* of $f$ if $f^n(z_0)=z_0$ for some $n\in\mathbb{N}$. The smallest $n$ with this property is called the *period* of $z_0$. Let $z_0$ be a periodic point of period $n$. We say that $\{z_0,f(z_0),\ldots,f^{n-1}(z_0)\}$ is a *cycle* of periodic points. If $z_0\neq\infty$, then $\lambda=(f^n)'(z_0)$ is called the *multiplier* of $z_0$. If $z_0=\infty$, then the multiplier is defined by $\lambda=\frac{d}{dz}1/f^n(\frac{1}{z})|_{z=0}$. We say that $z_0$ is *attracting*, *indifferent*, or *repelling* depending on whether $|\lambda|<1$, $|\lambda|=1$, or $|\lambda|>1$. If $\lambda=0$, then $z_0$ is called *superattracting*. If $z_0$ is indifferent, then $\lambda = e^{2\pi i\alpha}$ where $0 \le \alpha < 1$. We say that $z_0$ is *rationally indifferent* if $\alpha \in \mathbb{Q}$ and *irrationally indifferent* if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Finally, periodic points of period 1 are called *fixpoints*.

We note that it follows immediately from the chain rule that all points of a cycle of periodic points have the same multiplier. We also note that if $f$ and $g$ are conjugate by $T$ and if $z_0$ is a periodic point of $f$, then $T(z_0)$ is a periodic point of $g$, with the same period and multiplier. (For $z_0 = \infty$ and $T(z) = 1/z$ this is part of the definition of the multiplier.)

**Theorem 21** *Repelling and rationally indifferent periodic points are in $J$.*

*Proof.* Because of Lemma 13 (iii) it suffices to consider fixpoints. Thus let $z_0$ be a fixpoint and assume without loss of generality that $z_0 \neq \infty$. Suppose that $f^{n_k} \to \phi$ in a neighborhood of $z_0$. Then $\phi(z_0) = z_0$ and thus, in particular, $\phi \not\equiv \infty$.

If $z_0$ is repelling, then $|(f^{n_k})'(z_0)| = |f'(z_0)|^{n_k} \to \infty$. But $(f^{n_k})'(z_0) \to \phi'(z_0)$ by Weierstraß's Theorem, a contradiction. Thus $f^{n_k} \to \phi$ is impossible and $z_0 \in J$.

If $z_0$ is rationally indifferent, then we may assume that $f'(z_0) = 1$ because otherwise, again by Lemma 13 (iii), we can replace $f$ by a suitable iterate. We have

$$f(z) = z + a_m(z - z_0)^m + O\left((z - z_0)^{m+1}\right)$$

as $z \to z_0$, where $m \ge 2$ and $a_m \neq 0$. Induction shows that

$$f^n(z) = z + na_m(z - z_0)^m + O\left((z - z_0)^{m+1}\right)$$

as $z \to z_0$. It follows that $|(f^{n_k})^{(m)}(z_0)| = n_k|a_m|m! \to \infty$ as $k \to \infty$. But $(f^{n_k})^{(m)}(z_0) \to \phi^{(m)}(z_0)$ by Weierstraß's Theorem, a contradiction. Thus $z_0 \in J$. ■

**Definition 22** Let $z_0$ be an attracting fixpoint of $f$. Then we call $A(z_0) = \{z : \lim_{n\to\infty} f^n(z) = z_0\}$ the *basin of attraction* of $z_0$. More generally, for an attracting periodic point $z_0$ of period $p$ we define $A(z_0) = \{z : \lim_{n\to\infty} f^{pn}(z) = z_0\}$.

**Theorem 23** *Let $z_0$ be an attracting periodic point. Then $A(z_0)$ is open, $z_0 \in A(z_0) \subset F$, and $\partial A(z_0) = J$.*

If $f$ has three attracting periodic points $z_1, z_2, z_3$, then $J = \partial A(z_1) = \partial A(z_2) = \partial A(z_3)$. If we think of the basins of attractions as countries on a map, the result says that all countries have the same border. This shows that $J$ must have a complicated structure in this case.

*Proof of Theorem* 23. Because of Lemma 13 we may assume that $z_0$ is a fixpoint and $z_0 \neq \infty$. As in the proof of Theorem 10 we find that there exists $\delta > 0$ such that $f^n|_{D(z_0,\delta)} \to z_0$ so that $z_0 \in D(z_0, \delta) \subset F$. Since $A(z_0) = O^-(D(z_0, \delta))$ and since $F$ is completely invariant we conclude that $A(z_0) \subset F$ and that $A(z_0)$ is open.

In order to prove that $\partial A(z_0) = J$, let $z_1 \in \partial A(z_0)$. Then $z_1 \notin A(z_0)$ because $A(z_0)$ is open. Let $U$ be a neighborhood of $z_1$ and suppose that $f^{n_k}|_U \to \phi$. Then $\phi|_{U \cap A(z_0)} \equiv z_0$ and thus $\phi|_U \equiv z_0$. But $\phi(z_1) \neq z_0$ because $z_1 \notin A(z_0)$. This is a contradiction. Thus $f^{n_k}|_U \to \phi$ is impossible and we deduce that $z_1 \in J$. Hence $\partial A(z_0) \subset J$.

Let now $z_1 \in J$. Then $z_1 \notin A(z_0)$. Suppose that $z_1 \notin \partial A(z_0)$. Then $z_1$ has a neighborhood $V$ such that $V \cap A(z_0) = \emptyset$. This implies that $O^+(V) \cap D(z_0, \delta) = \emptyset$, with $\delta$ as above. We conclude that $1/|f^n(z) - z_0| \leq 1/\delta$ for all $z \in V$ and $n \in \mathbb{N}$. By Theorem 6 the family $\{1/(f^n - z_0)\}_{n \in \mathbb{N}}$ and hence the family $\{f^n\}_{n \in \mathbb{N}}$ is normal in $V$, which is impossible since $z_1 \in J$. This contradiction shows that $J \subset \partial A(z_0)$. Altogether we find that $J = \partial A(z_0)$. ■

So far we have not proved that yet that $J \neq \emptyset$. Of course, this follows from the previous results if $f$ has a repelling or rationally indifferent periodic point or if $f$ has at least two attracting periodic points. But, with the assumption that $f \notin \text{Aut}(D)$ and $f \not\equiv$ *const* being made, this is true in general.

**Theorem 24** $|J| = \infty$.

In this section, we will prove Theorem 24 only for the case that $D = \widehat{\mathbb{C}}$ so that $f$ is rational. For transcendental functions we will base the proof on some results of Nevanlinna theory developed in the next section.

*Proof of Theorem* 24 *for rational* $f$. It suffices to prove that $J \neq \emptyset$ because if once a point $z_0 \in J$ is found, then $O^-(z_0) \subset J$ by Lemma 13 (ii) and $|O^-(z_0)| = \infty$ by Theorem 16 (i).

One way to prove that $J \neq \emptyset$ is as follows. Suppose that $J = \emptyset$ so that $F = \widehat{\mathbb{C}}$. Then $f^{n_k} \to \phi$ in $\widehat{\mathbb{C}}$ for some sequence $(f^{n_k})$ and some function $\phi$ which is meromorphic in $\widehat{\mathbb{C}}$ and hence rational. It follows that $\deg(f^{n_k}) \to \deg(\phi)$. But $\deg(f^{n_k}) = \deg(f)^{n_k} \geq 2^{n_k}$, a contradiction. ■

Another way to prove that $J \neq \emptyset$ is to combine Lemma 21 with the following

**Lemma 25** *A rational function has a repelling fixpoint or a fixpoint of multiplier* 1.

*Proof.* Suppose that the conclusion is false. We may assume that $f(\infty) \neq \infty$ so that if $f = p/q$ with relatively prime polynomials $p$ and $q$, then $d = \deg(f) = \deg(q) \geq \deg(p)$. As the fixpoints of $f$ are the solutions of $p(z) = zq(z)$ we deduce that $f$ has $d+1$ fixpoints $z_1, z_2, \ldots, z_{d+1}$. From the assumption that no fixpoint has multiplier 1 we can deduce that the $z_j$ are pairwise distinct. Now let $R > \max_j |z_j|$. Then

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z-f(z)} = \sum_{j=1}^{d+1}\text{res}\left(\frac{1}{z-f(z)}, z_j\right) = \sum_{j=1}^{d+1}\frac{1}{1-f'(z_j)}$$

by the residue theorem. On the other hand, because $f(\infty) \neq \infty$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z-f(z)} - 1\right| &= \left|\frac{1}{2\pi i}\int_{|z|=R}\left(\frac{1}{z-f(z)} - \frac{1}{z}\right)dz\right| \\ &= \frac{1}{2\pi}\left|\int_{|z|=R}\frac{f(z)}{(z-f(z))z}dz\right| \\ &\leq \max_{|z|=R}\frac{|f(z)|}{|z-f(z)|} \\ &= o(1)\end{aligned}$$

as $R \to \infty$. Thus

$$1 = \sum_{j=1}^{d+1}\frac{1}{1-f'(z_j)}.$$

A simple computation shows that $|f'(z_j)| \leq 1$ and $f'(z_j) \neq 1$ is equivalent to $\Re\left(\frac{1}{1-f'(z_j)}\right) \geq \frac{1}{2}$. Hence

$$1 = \sum_{j=1}^{d+1}\Re\left(\frac{1}{1-f'(z_j)}\right) \geq \frac{d+1}{2},$$

which is a contradiction since $d \geq 2$. ∎

Once Theorem 24 is known, the following stronger result is not difficult to deduce.

**Theorem 26** *$J$ is perfect.*

Recall that a set is called perfect if it is non-empty, closed, and does not contain isolated points.

*Proof of Theorem* 26. We only have to prove that $J$ does not contain isolated points. First we show that there exists a point $z_1 \in J$ which is not periodic. To do this we note that because $|J| = \infty$ and $|E| \leq 2$ there exists $z_0 \in J \backslash E$. Hence $|O^-(z_0)| = \infty$. If $z_0$ is not periodic, we may take $z_1 = z_0$. But if $z_0$ is periodic, then $|O^+(z_0)| < \infty$ and we can choose $z_1 \in O^-(z_0) \backslash O^+(z_0)$. It is easy to see that $z_1$ has the desired properties. From the choice of $z_1$ it follows that $z_1 \notin O^-(z_1)$. Since $z_1 \in J = \overline{O^-(z_1)}$ we deduce that $z_1$ is not an isolated point of $J$. Since $J = \overline{O^-(z_1)}$ we conclude that no point in $J$ is isolated. ■

We discuss the theory developed so far at three

**Examples** 1. Consider $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $f(z) = z^2$. There are three fixpoints: 0, 1, and $\infty$. We see that 0 and $\infty$ are superattracting while 1 has multiplier 2 and is thus repelling. We have $A(0) = \{z \in \mathbb{C} : |z| < 1\}$, $A(\infty) = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, and $J(f) = \partial A(0) = \partial A(\infty) = \{z \in \mathbb{C} : |z| = 1\}$.

2. Newton's method of finding the zeros of a polynomial $p$ consists of iterating the function $N(z) = z - p(z)/p'(z)$. If $z_0$ is a zero of $p$ of multiplicity $m$, then $z_0$ is a fixpoint of $f$ of multiplier $1 - \frac{1}{m}$. Thus $z_0$ is an attracting fixpoint of $f$. If $p(z) = z^2 - 1$, then $N(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$. To determine $J(N)$ and the basins of attraction $A(\pm 1)$ we note that $N = T \circ f \circ T^{-1}$ with $T(z) = \frac{z+1}{z-1}$ and $f(z) = z^2$. It follows that $J(N) = T(J(f)) = \{iy : y \in \mathbb{R}\} \cup \{\infty\}$, $A(1) = \{z \in \mathbb{C} : \Re z > 0\}$, and $A(-1) = \{z \in \mathbb{C} : \Re z < 0\}$. Theorem 23 shows that for polynomials of degree greater than two the Julia set of the corresponding Newton function is much more complicated.

3. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be defined by $f(z) = z^2 - 2$. The fixpoints are $-1$, 2, and $\infty$. Again $\infty$ is superattracting, as it is the case for any polynomial. The other two fixpoints are repelling. Now $f$ maps $[-2, 0]$ bijectively onto $[-2, 2]$ and also maps $[0, 2]$ bijectively onto $[-2, 2]$. Thus $z \in [-2, 2]$ if and only if $f(z) \in [-2, 2]$. Hence $f(\widehat{\mathbb{C}} \backslash [-2, 2]) \subset \widehat{\mathbb{C}} \backslash [-2, 2]$ and hence, by Montel's Theorem, $\widehat{\mathbb{C}} \backslash [-2, 2] \subset F(f)$. We conclude that $F(f) = \widehat{\mathbb{C}} \backslash [-2, 2] = A(\infty)$ and $J(f) = [-2, 2]$.

## 6 Nevanlinna theory

We shall give the basic definitions and results of Nevanlinna theory. For more details and complete proofs we refer to [20, 23, 29].

Let $w : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function. By $n(r, w)$ we denote the number of poles of $w$ in $|z| \leq r$, counted according to multiplicity. We use the notation $\overline{n}(r, w)$ if multiplicities are ignored. We call

$$N(r, w) = \int_0^r \frac{n(t, w) - n(0, w)}{t} dt + n(0, w) \log r$$

the *counting function* of the poles of $w$. Note that $N(r, w) = \int_0^r n(t, w)/t \, dt$ if $w(0) \neq \infty$. Analogously we define $\overline{N}(r, w)$ in terms of $\overline{n}(r, w)$.

Next we put $\log^+ x = \log x$ for $x \geq 1$ and $\log^+ x = 0$ for $x < 1$. We call

$$m(r, w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\theta})| d\theta$$

the *proximity function* and

$$T(r, w) = N(r, w) + m(r, w)$$

the *Nevanlinna characteristic* of $w$. If $w$ is a rational function of degree $d$, then $T(r, w) \sim d \log r$ as $r \to \infty$. If $w$ is transcendental, then

$$\lim_{r \to \infty} \frac{T(r, w)}{\log r} = \infty. \tag{1}$$

For further use later we also note that if $w_1, w_2$ are meromorphic in $\mathbb{C}$, then

$$T(r, w_1 w_2) \leq T(r, w_1) + T(r, w_2) \tag{2}$$

and

$$T(r, w_1 + w_2) \leq T(r, w_1) + T(r, w_2) + O(1). \tag{3}$$

Jensen's formula says that if $a_j$ are the zeros and $b_j$ are the poles of $w$, then

$$\log |w(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |w(re^{i\theta})| d\theta - \sum_{|a_j| < r} \log \frac{r}{|a_j|} + \sum_{|b_j| < r} \log \frac{r}{|b_j|},$$

with a modification if $w(0) = 0$ or $w(0) = \infty$. Using integration by parts we find that

$$\begin{aligned}
\sum_{|b_j|<r} \log \frac{r}{|b_j|} &= \int_0^r \log \frac{r}{t} dn(t,w) \\
&= -\int_0^r n(t,w) d \log \frac{r}{t} \\
&= \int_0^r n(t,w) d \log t \\
&= N(r,w)
\end{aligned}$$

and

$$\sum_{|a_j|<r} \log \frac{r}{|a_j|} = N\left(r, \frac{1}{w}\right).$$

Using this and the identity $\log x = \log^+ x - \log^+ \frac{1}{x}$ we obtain from Jensen's formula

$$\log |w(0)| = T(r,w) - T\left(r, \frac{1}{w}\right),$$

again with a modification if $w(0) = 0$ or $w(0) = \infty$. From this identity the following result is not difficult to deduce.

**Theorem 27 (First Fundamental Theorem)** *For $a \in \mathbb{C}$ we have*

$$T\left(r, \frac{1}{w-a}\right) = T(r,w) + O(1)$$

*as $r \to \infty$.*

The theorem says that $N(r, \frac{1}{w-a}) + m(r, \frac{1}{w-a})$ does, apart from a bounded term, not depend on $a$. Noting that $N(r, \frac{1}{w-a})$ is measure for the frequency of the $a$-points of $f$ while $m(r, \frac{1}{w-a})$ measures how close $w(re^{i\theta})$ and $a$ are, Theorem 27 can be considered as a quantitative version of the statement that if $w$ has "few" $a$-points, then $w$ is "close" to $a$. For example, the exponential function has no zeros and is close to zero in the left half-plane.

The First Fundamental Theorem does not say which of the two terms in the sum $N(r, \frac{1}{w-a}) + m(r, \frac{1}{w-a})$ is dominant. The Second Fundamental Theorem says that for most values of $a$ it is the term $N(r, \frac{1}{w-a})$.

**Theorem 28 (Second Fundamental Theorem)** *Let $a_1, \dots, a_q \in \mathbb{C}$ be pairwise disjoint. Then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{w-a_j}\right) + m(r,w) \le 2T(r,w) - N_1(r) + S(r,w)$$

*where*

$$N_1(r) = N\left(r, \frac{1}{w'}\right) + 2N(r,w) - N(r,w') \ge 0$$

*and $S(r,w) = o(1)$ as $r \to \infty$, $r \notin E$, for some set $E \subset [0,\infty)$ of finite measure.*

Thus

$$\sum_{j=1}^{q} m\left(r, \frac{1}{w-a_j}\right) + m(r,w) \le (2+o(1))T(r,w) - N_1(r), \quad r \notin E.$$

Adding

$$\sum_{j=1}^{q} N\left(r, \frac{1}{w-a_j}\right) + N(r,w)$$

on both sides and using the First Fundamental Theorem we obtain

$$(q-1-o(1))T(r,w) \le \sum_{j=1}^{q} N\left(r, \frac{1}{w-a_j}\right) + N(r,w) - N_1(r), \quad r \notin E.$$

The term $N_1(r)$ counts the multiple points. Using this we can conclude that

$$(q-1-o(1))T(r,w) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{w-a_j}\right) + \overline{N}(r,w), \quad r \notin E. \tag{4}$$

It is convenient to define $N(r,a) = N(r, \frac{1}{w-a})$ for $a \in \mathbb{C}$ and $N(r,\infty) = N(r,w)$, with an analogous definition of $\overline{N}(r,a)$, $a \in \widehat{\mathbb{C}}$. Then the value $\infty$ does not play a special role anymore and instead of the previous inequalities we obtain

$$(q-2-o(1))T(r,w) \le \sum_{j=1}^{q} N(r,a_j) - N_1(r), \quad r \notin E \tag{5}$$

and

$$(q-2-o(1))T(r,w) \leq \sum_{j=1}^{q} \overline{N}(r,a_j), \quad r \notin E, \tag{6}$$

if $a_1, \ldots, a_q \in \widehat{\mathbb{C}}$ are pairwise disjoint.

If the equation $w(z) = a$ has only finitely many solutions, then $N(r,a) = O(\log r)$. Hence, if $w$ takes the values $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ only finitely many times, then we deduce, choosing $q = 3$ in (5), that $(1-o(1))T(r,w) = O(\log r)$, $r \notin E$, a contradiction to (1) if $w$ is transcendental. Thus we obtain the Little Picard Theorem. We may view the Second Fundamental Theorem (and the inequalities deduced from it) as a quantitative version of Picard's Theorem.

The Nevanlinna theory also gives information about the multiplicities with which the values are taken. Suppose that only finitely many of the $a_j$-points are simple for $j = 1, \ldots, q$. Then $\overline{N}(r,a_j) \leq \frac{1}{2}N(r,a_j) + O(\log r)$. We obtain

$$\begin{aligned}(q-2-o(1))T(r,w) &\leq \sum_{j=1}^{q} \overline{N}(r,a_j) \\ &\leq \frac{1}{2}\sum_{j=1}^{q} N(r,a_j) + O(\log r) \\ &\leq \left(\frac{q}{2} + o(1)\right) T(r,w), \quad r \notin E,\end{aligned}$$

if $w$ is transcendental, which implies that $q - 2 \leq \frac{q}{2}$ and hence that $q \leq 4$. We say that $a \in \widehat{\mathbb{C}}$ is *perfectly branched* if only finitely many of the $a$-points of $w$ are simple. Then the above consideration may be written as follows.

**Theorem 29** *A transcendental meromorphic function has at most four perfectly branched values.*

A similar consideration yields

**Theorem 30** *A transcendental entire function has at most two finite perfectly branched values.*

The Weierstraß $\wp$-function has the perfectly branched values $e_1, e_2, e_3, \infty$ and the sine function has the perfectly branched values $\pm 1$. Thus the numbers four and two in the preceding theorems are sharp.

In §8 we shall need the following

**Theorem 31** *A transcendental holomorphic self-map of $\mathbb{C}^*$ does not have perfectly branched values.*

*Proof.* Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be transcendental holomorphic and suppose that $b$ is a perfectly branched value of $f$. Then $f$ has infinitely many $b$-points $a_1, a_2, \ldots$ and there exists $p \in \mathbb{N}$ such that $f'(a_j) = 0$ for $j > p$. We apply the Nevanlinna theory to $w(z) = f(e^z)$. We have $N(r, 1/(\exp - a_j)) \sim r/\pi$ as $r \to \infty$ for each $j$ and thus

$$N\left(r, \frac{1}{w-b}\right) \geq \sum_{j=1}^{q} N\left(r, \frac{1}{\exp - a_j}\right) \geq \frac{(q - o(1))r}{\pi}$$

for each fixed $q$ so that

$$\lim_{r\to\infty} \frac{N\left(r, \frac{1}{w-b}\right)}{r} = \infty.$$

We also have

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{w-b}\right) &\leq \frac{1}{2} N\left(r, \frac{1}{w-b}\right) + \sum_{j=1}^{p} N\left(r, \frac{1}{\exp - a_j}\right) \\
&\leq \frac{1}{2} N\left(r, \frac{1}{w-b}\right) + \frac{(p + o(1))r}{\pi} \\
&\leq \left(\frac{1}{2} + o(1)\right) N\left(r, \frac{1}{w-b}\right)
\end{aligned} \tag{7}$$

Applying the two fundamental theorems we obtain

$$\begin{aligned}
N\left(r, \frac{1}{w-b}\right) &\leq T(r, w) + O(1) \\
&\leq (1 + o(1)) \left(\overline{N}(r, w) + \overline{N}\left(r, \frac{1}{w}\right) + \overline{N}\left(r, \frac{1}{w-b}\right)\right) \\
&= (1 + o(1))\overline{N}\left(r, \frac{1}{w-b}\right),
\end{aligned}$$

which contradicts (7). ■

We mention that there are connections between Nevanlinna's theory on the distribution of values developed in this chapter and Montel's theory of normal families given in §2. The interested reader is referred to [16] and [36, §§4.4-4.5].

# 7 Fixpoints of iterates and the proof that the Julia set is not empty

We shall use the Nevanlinna theory to prove Theorem 24 ($|J| = \infty$) for transcendental $f$. A preliminary result (of independent interest) is

**Lemma 32** *Let $f$ be entire transcendental. Then $f^2$ has infinitely many fixpoints.*

*Proof.* We assume that $f^2$ has only finitely many fixpoints $c_1, \ldots, c_p$. Then $f$ has only finitely many fixpoints. Thus

$$\overline{N}\left(r, \frac{1}{f^2(z) - z}\right) = O(\log r)$$

and

$$\overline{N}\left(r, \frac{1}{f(z) - z}\right) = O(\log r).$$

If $f^2(z) = f(z)$, then $f(z) = c_j$ for some $j$. Hence

$$\overline{N}\left(r, \frac{1}{f^2(z) - f(z)}\right) \le \sum_{j=1}^{p} \overline{N}\left(r, \frac{1}{f - c_j}\right) \le pT(r, f) + O(1)$$

by the First Fundamental Theorem. We define

$$h(z) = \frac{f^2(z) - z}{f(z) - z}$$

and deduce from (6) that

$$\begin{aligned}
& (1 - o(1))T(r, h) \\
\le\ & \overline{N}(r, h) + \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}\left(r, \frac{1}{h - 1}\right) \\
\le\ & \overline{N}\left(r, \frac{1}{f(z) - z}\right) + \overline{N}\left(r, \frac{1}{f^2(z) - z}\right) + \overline{N}\left(r, \frac{1}{f^2(z) - f(z)}\right) \\
\le\ & pT(r, f) + O(\log r), \quad r \notin E.
\end{aligned}$$

Here and in the following $E$ always denotes a set of finite measure, not necessarily the same at each occurence. (There should be no confusion with the set $E$ introduced in Definition 15.) From (2) and (3) we deduce that

$$\begin{aligned} T(r,f^2) &= T(r,f^2(z)-z)+O(\log r) \\ &= T(r,h(z)(f(z)-z))+O(\log r) \\ &\leq T(r,h)+T(r,f)+O(\log r) \end{aligned}$$

so that

$$T(r,f^2) \leq (p+1+o(1))T(r,f), \quad r \notin E.$$

Let now $b$ be a value taken infinitely often by $f$ and let $a_1, a_2, \ldots$ be the $b$-points of $f$. For fixed $q$ we have

$$\sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) \leq \overline{N}\left(r, \frac{1}{f^2-b}\right) \leq T(r,f^2)+O(1).$$

On the other hand, by (4) and since $\overline{N}(r,f) \equiv 0$,

$$\sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) \geq (q-1-o(1))T(r,f), \quad r \notin E.$$

The last three inequalities now yield that $p+1 \geq q-1$, which is a contradiction if we choose $q = p+3$. ■

A similar proof shows that $f^n$ has infinitely many fixpoints if $n \geq 2$, see [35]. It can also be shown that $f$ has even infinitely many periodic points of period $n$ if $n \geq 2$ [7].

If we apply Picard's Theorem instead of Nevanlinna theory to the auxiliary function $h$ used in the above proof, then we can only deduce that $f^2$ has at least one fixpoint. This does not suffice for the following application.

*Proof of Theorem* 24 *for transcendental* $f$. If $f$ is transcendental entire, then $f^2$ has infinitely many fixpoints by Lemma 32. Since $J(f^2) = J(f)$ by Lemma 13 (iii) we may assume that $f$ has infinitely many fixpoints. If $f$ is a transcendental holomorphic self-map of $\mathbb{C}^*$, then $h(z) = f(z)/z$ is also a transcendental holomorphic self-map of $\mathbb{C}^*$. Thus $h(z) \neq 0, \infty$ for $z \in \mathbb{C}^*$ and 0 or $\infty$ is an essential singularity of $h$. By the Great Picard Theorem $h$ takes the value 1 infinitely often which implies that $f$ has infinitely many fixpoints. Thus, in both cases, we may assume that $f$ has infinitely many fixpoints. If infinitely many of the fixpoints are in $J$, then we are done. If $F$ has more than one component, then we are also done because any path connecting two components of $F$ must meet $J$. There remains only the possibility that $F$ is connected and contains infinitely many fixpoints. Theorem 10 now implies that $f|_F$ is a conformal automorphism. But this is impossible for transcendental $f$ because $F$ is completely invariant. ■

# 8 The set of repelling periodic points

We have seen that repelling periodic points are in $J$. One of the basic results of the theory is

**Theorem 33** *$J$ is the closure of the set of all repelling periodic points.*

We remark that Julia [24] based his whole theory on the closure of the set of repelling periodic points, while Fatou [18] started with the set of non-normality as we did. Both obtained Theorem 33 which says that these two sets are actually equal, but their proofs were different. Both proofs do not apply to the cases $D = \mathbb{C}$ and $D = \mathbb{C}^*$. For the case $D = \mathbb{C}$ the result was proved first by Baker [3] and the case $D = \mathbb{C}^*$ was settled by Bhattacharyya [9], using Bakers's method which is based on Ahlfors's theory of covering surfaces [1]. Here we shall give a new proof due to Schwick [38].

We start with the following lemma due to Zalcman [45].

**Lemma 34** *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $U$ such that $f : U \to V \subset \widehat{\mathbb{C}}$ for all $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in U$, then there exist a sequence $(f_n)$ in $\mathcal{F}$, a sequence $(z_n)$ in $U$, a sequence $(\rho_n)$ of positive real numbers, and a non-constant meromorphic function $h : \mathbb{C} \to V$ such that $z_n \to z_0$, $\rho_n \to 0$, and $f_n(z_n + \rho_n z) \to h(z)$ locally uniformly in $\mathbb{C}$ as $n \to \infty$.*

*Proof.* By Marty's Theorem, there exist $f_n \in \mathcal{F}$ and $z_n^* \in U$ such $z_n^* \to z_0$ and $K_n = f_n^\sharp(z_n^*) \to \infty$ as $n \to \infty$. Define $r_n = \max\{2|z_n^* - z_0|, 1/\sqrt{K_n}\}$. Then $r_n \to 0$ as $n \to \infty$ and thus we may assume that $D(z_0, r_n) \subset U$. We choose $z_n \in D(z_0, r_n)$ such that

$$M_n = \max_{|z-z_0| \leq r_n} \left(1 - \frac{|z - z_0|^2}{r_n^2}\right) f_n^\sharp(z) = \left(1 - \frac{|z_n - z_0|^2}{r_n^2}\right) f_n^\sharp(z_n).$$

Then

$$M_n \geq \left(1 - \frac{|z_n^* - z_0|^2}{r_n^2}\right) f_n^\sharp(z_n^*) \geq \frac{3}{4} K_n$$

so that $M_n \to \infty$. We define

$$\rho_n = \frac{1}{f_n^\sharp(z_n)} = \frac{1}{M_n}\left(1 - \frac{|z_n - z_0|^2}{r_n^2}\right).$$

Then $\rho_n \to 0$ as $n \to \infty$ and

$$\begin{aligned}
\sqrt{K_n}\rho_n &= \frac{\sqrt{K_n}}{M_n} \frac{r_n + |z_n - z_0|}{r_n} \frac{r_n - |z_n - z_0|}{r_n} \\
&\leq \frac{2\sqrt{K_n}}{M_n r_n}(r_n - |z_n - z_0|) \\
&\leq \frac{8}{3\sqrt{K_n} r_n}(r_n - |z_n - z_0|) \\
&\leq \frac{8}{3}(r_n - |z_n - z_0|).
\end{aligned}$$

We deduce that if $|z| < \frac{3}{8}\sqrt{K_n}$, then

$$|z_n + \rho_n z - z_0| \leq |z_n - z_0| + \rho_n |z| < |z_n - z_0| + (r_n - |z_n - z_0|) \leq r_n$$

and hence $z_n + \rho_n z \in D(z_0, r_n) \subset U$. Thus $h_n(z) = f_n(z_n + \rho_n z)$ is defined for $|z| < \frac{3}{8}\sqrt{K_n}$. For $|z| < \frac{3}{16}\sqrt{K_n}$ we have

$$\begin{aligned}
h_n^\sharp(z) &= \rho_n f_n^\sharp(z_n + \rho_n z) \\
&\leq \frac{\rho_n M_n}{1 - \frac{|z_n - z_0|^2}{r_n^2}} \\
&= \frac{r_n^2 - |z_n - z_0|^2}{r_n^2 - |z_n + \rho_n z - z_0|^2} \\
&\leq \frac{r_n^2 - |z_n - z_0|^2}{r_n^2 - (|z_n - z_0| + \rho_n|z|)^2} \\
&= \frac{r_n + |z_n - z_0|}{r_n + |z_n - z_0| + \rho_n|z|} \; \frac{r_n - |z_n - z_0|}{r_n - (|z_n - z_0| + \rho_n|z|)} \\
&\leq \frac{r_n - |z_n - z_0|}{r_n - (|z_n - z_0| + \frac{1}{2}(r_n - |z_n - z_0|))} \\
&= 2.
\end{aligned}$$

Thus $\{h_n^\sharp\}_{n\in\mathbb{N}}$ is locally uniformly bounded in $\mathbb{C}$ and hence, by Marty's Theorem, $\{h_n\}_{n\in\mathbb{N}}$ is normal in $\mathbb{C}$. Restricting to a subsequence if necessary we may assume that $h_n \to h$ as $n \to \infty$ for some function $h$ meromorphic in $\mathbb{C}$. Because $h_n^\sharp(0) = 1$ by construction, we have $h^\sharp(0) = 1$ and thus $h$ is non-constant. Hurwitz's Theorem implies that $h : U \to V$. ■

**Remark** As already mentioned, Lemma 34 can be used to deduce Montel's Theorem from the Little Picard Theorem. In fact, if $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ are pairwise disjoint and if we suppose that the family of all meromorphic functions $f : U \to \widehat{\mathbb{C}}\backslash\{a_1, a_2, a_3\}$ is not normal for some domain $U$, then Lemma 34 yields a non-constant meromorphic function $h : \mathbb{C} \to \widehat{\mathbb{C}}\backslash\{a_1, a_2, a_3\}$, which is impossible by the Little Picard Theorem. Thus the above family is normal. Using the same argument we can deduce Theorem 6 from Liouville's Theorem.

*Proof of Theorem* 33. If $f$ is rational, we may assume that $\deg(f) \geq 5$ because otherwise we can consider $f^3$ instead of $f$. Let $A$ be the set of all $a \in D\backslash\{\infty\}$ for which the function $f(z) - a$ has less than 5 simple zeros. In the case that $f$ is rational let in addition $\infty$, $f(\infty) \in A$. Then $A$ is finite. This is clear if $f$ is rational and follows from Theorems 29–31 if $f$ is transcendental.

As $J$ is perfect it suffices to show that each point in $J\backslash A$ is limit point of repelling periodic points of $f$. So let $z_0 \in J\backslash A$. Applying Lemma 34 to the family $\{f^n\}_{n\in\mathbb{N}}$ we obtain sequences $(f^{n_j})$, $(z_j)$, $(\rho_j)$, and a non-constant meromorphic function $h : \mathbb{C} \to D$ such that $h_j(z) = f^{n_j}(z_j + \rho_j z) \to h(z)$ locally uniformly in $\mathbb{C}$ as $j \to \infty$. We define $g_j(z) = f^{n_j+1}(z_j + \rho_j z)$ and deduce that $g_j \to g = f(h)$. Since $z_0 \notin A$ there exist $u_1, \ldots, u_5$ such that $f(u_k) = z_0$ and $f'(u_k) \neq 0$. Theorem 29 now implies that if $h$ is transcendental, then there exist $k \in \{1, \ldots, 5\}$ and $w_0 \in \mathbb{C}$ such that $h(w_0) = u_k$ and $h'(w_0) \neq 0$. It is easy to see that such $k$ and $w_0$ also exist if $h$ is rational. It follows that $g(w_0) = f(h(w_0)) = f(u_k) = z_0$ and $g'(w_0) = f'(u_k)h'(w_0) \neq 0$.

Since $f^{n_j+1}(z_j + \rho_j z) - (z_j + \rho_j z) \to g(z) - z_0$ as $j \to \infty$ and $g(w_0) = z_0$ but $g \not\equiv z_0$, Hurwitz's Theorem implies that for $j$ large enough there exists $\zeta_j$ such that $f^{n_j+1}(z_j + \rho_j \zeta_j) - (z_j + \rho_j \zeta_j) = 0$ and $\zeta_j \to w_0$ as $j \to \infty$. It follows that $\xi_j = z_j + \rho_j \zeta_j$ is a fixpoint of $f^{n_j+1}$ and hence a periodic point of $f$. Moreover, $\xi_j \to z_0$ as $j \to \infty$.

From Weierstraß's Theorem we can deduce that $g_j'(\zeta_j) \to g'(w_0) \neq 0$. Since $g'(\zeta_j) = \rho_j (f^{n_j+1})'(\xi_j)$ and $\rho_j \to 0$ this implies that $|(f^{n_j+1})'(\xi_j)| \to \infty$ as $j \to \infty$. Hence $\xi_j$ is repelling if $j$ is large enough. ■

After Theorem 18 we remarked that if $K \subset D \backslash E$ is compact and $U \subset D$ is open and satisfies $U \cap J \neq \emptyset$, then there exists $N$ such that $K \subset \bigcup_{n=1}^{N} f^n(U)$. Theorem 33 can be used to show that there even exists $n$ such that $K \subset f^n(U)$. In fact, let $z_0 \in U$ be a repelling periodic point of period $p$. Then there exists a neighborhood $V$ of $z_0$ such that $V \subset f^p(V)$ and $V \subset U$. We thus have $\bigcup_{n=1}^{N} f^{pn}(V) = f^{pN}(V)$ for all $N$ and hence $K \subset \bigcup_{n=1}^{N} f^{pn}(V) = f^{pN}(V) \subset f^{pN}(U)$ if $N$ is large.

One can even show that $K \subset f^n(U)$ for all large $n$. This can also be done without using Theorem 33, see [6, Theorem 4.2.5].

# 9 Local theory near fixpoints

We have seen already that the periodic points play an important role in complex dynamics. In this section we study the behavior of the iterates near such a point. Clearly it suffices to consider fixpoints.

**Theorem 35** *Let $z_0 \in \mathbb{C}$ be a fixpoint of multiplier $\lambda = f'(z_0)$. Suppose that $\lambda \neq 0$ and $|\lambda| \neq 1$. Then there exists a function $S$ holomorphic in a neighborhood of* 0 *such that $S(0) = z_0$, $S'(0) \neq 0$, and*

$$f(S(z)) = S(\lambda z) \tag{8}$$

*for all $z$ sufficiently close to* 0. *The function $S$ is unique if normalized by* $S'(0) = 1$.

Because $S'(0) \neq 0$, the inverse function $\phi = S^{-1}$ exists in a neighborhood of $z_0$ and (8) may be written as $f(z) = S(\lambda S^{-1}(z))$ near $z_0$ so that (in an obvious sense) $f$ is *locally conjugate* to the map $g(z) = \lambda z$ near $z_0$. In terms of $\phi$, (8) takes the form

$$\phi(f(z)) = \lambda \phi(z). \tag{9}$$

The functional equations (8) and (9) are named after Schröder [37], who studied them first, but the theorem is due to Kœnigs [25]. For the validity of the theorem it is not necessary that $f$ is meromorphic in $D$, but only that $f$ is holomorphic in a neighborhood of the fixpoint $z_0$. The assumption that $f$ is meromorphic in $D$ is needed, however, for the following corollary.

**Corollary 36** *Let $f$, $z_0$, and $\lambda$ be as in Theorem* 35. *If $|\lambda| > 1$, then there exists a meromorphic function $S : \mathbb{C} \to D$ satisfying* (8).

To prove the corollary we only have to observe that (8) permits analytic continuation to the whole plane if $|\lambda| > 1$. ■

It follows that $S$ is entire if $f$ is entire and $S$ is an entire function without zeros if $f$ is a holomorphic self-map of $\mathbb{C}^*$. For example, if $f(z) = z^2$ and $z_0 = 1$, then $\lambda = 2$ and $S(z) = e^z$. If $f(z) = 2z^2 - 1$ and $z_0 = 1$, then $\lambda = 4$ and $S(z) = \cos\sqrt{z}$. These examples are somewhat misleading, however, because the function $S$ can be expressed in terms of elementary functions only in exceptional cases, see [34].

*Proof of Theorem* 35. First we consider the case that $|\lambda| < 1$. We may assume without loss of generality that $z_0 = 0$ so that $f(z) = \lambda z + O(z^2)$ as $z \to 0$, say $|f(z) - \lambda z| \le K|z|^2$ for $|z| \le r$. Given $\delta > 0$, we may also assume that $r$ is chosen such that $|f(z)| \le (|\lambda| + \delta)|z|$ and therefore $|f^n(z)| \le (|\lambda| + \delta)^n |z|$ for $|z| \le r$. Hence

$$|f^{n+1}(z) - \lambda f^n(z)| = |f(f^n(z)) - \lambda f^n(z)| \le K|f^n(z)|^2 \le Kr^2 \left(|\lambda| + \delta\right)^{2n}$$

for $|z| \le r$. It follows that

$$\left| \frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \right| \le \frac{Kr^2 \left(|\lambda| + \delta\right)^{2n}}{|\lambda|^{n+1}} = \frac{Kr^2}{|\lambda|} \left( \frac{\left(|\lambda| + \delta\right)^2}{|\lambda|} \right)^n$$

for $|z| \le r$. Choosing $\delta$ such that $(|\lambda| + \delta)^2 \le |\lambda|$ we find that $f^n(z)/\lambda^n$ converges locally uniformly for $|z| \le r$, say $f^n/\lambda^n \to \phi$ in $D(0, r)$. Then

$$\phi(f(z)) = \lim_{n\to\infty} \frac{f^n(f(z))}{\lambda^n} = \lambda \lim_{n\to\infty} \frac{f^{n+1}(z)}{\lambda^{n+1}} = \lambda\phi(z).$$

Thus $\phi$ satisfies (9). Since $\frac{d}{dz}\frac{f^n(z)}{\lambda^n}|_{z=0} = 1$ we have $\phi'(0) = 1$. Hence $S = \phi^{-1}$ exists in a neighborhood of 0 and $S'(0) = 1$. We have already noted that (8) is equivalent to (9).

Now we consider the case that $|\lambda| > 1$. Then $g = f^{-1}$ exists in a neighborhood of $z_0 = 0$ and $g'(0) = 1/\lambda$. We obtain $S$ satisfying $g(S(z)) = S(\frac{1}{\lambda}z)$. It follows that $S(\lambda z) = f(g(S(\lambda z)) = f(S(\lambda \frac{1}{\lambda} z)) = f(S(z))$ and we have (8).

It is clear that if $S(z)$ satisfies (8), then so does $S(cz)$ for all $c \in \mathbb{C}^*$. Suppose now that $S_1$ and $S_2$ satisfy (8) and $S_1'(0) = S_2'(0) = 1$. Then

$g = S_2^{-1} \circ S_1$ satisfies $g(\lambda z) = \lambda g(z)$ and $g'(0) = 1$. A look at the power series of $g$ now shows that $g(z) \equiv z$ so that $S_1 = S_2$. ■

We shall use Theorem 35 to connect attracting periodic points to the singularities of the inverse function $f^{-1}$ of $f$.

Let $\varphi$ be a single-valued branch of $f^{-1}$ defined in some domain $U$. Let $\gamma : [0,1] \to D$ be a path with $\gamma(0) \in U$ such that $\varphi$ can be continued analytically along $\gamma$ into the point $\gamma(t)$ for each $t \in (0,1)$, but not for $t = 1$. Then $a = \gamma(1)$ is called a *singularity of* $f^{-1}$. It turns out that this can occur in two cases:

(i) $\varphi(\gamma(t)) \to z_0 \in D$ as $t \to 1$. Then $f(z_0) = a$ and $f$ is not univalent in any neighborhood of $z_0$. We call this an *algebraic singularity* and say that $a$ is a *critical value* and $z_0$ is a *critical point* of $f$. If $z_0, a \neq \infty$, then $z_0$ is a critical point if and only if $f'(z_0) = 0$.

(ii) $\varphi(\gamma(t)) \to z_0 \in \widehat{\mathbb{C}} \backslash D$ as $t \to 1$. We call this a *transcendental singularity* and say that $a$ is an *asymptotic value* of $f$. Equivalently, with $\Gamma(t) = \varphi(\gamma(t))$, we have $\Gamma(t) \to z_0$ and $f(\Gamma(t)) \to a$.

Clearly case (ii) cannot occur if $D = \widehat{\mathbb{C}}$.

Earlier we had defined the basin of attraction $A(z_0)$ of an attracting periodic point $z_0$. The *immediate basin of attraction* of an attracting periodic point $z_0$ is defined as the component of $F$ that contains $z_0$ and it is denoted by $A^*(z_0)$. Clearly, we always have $A^*(z_0) \subset A(z_0)$ and we have $A^*(z_0) = A(z_0)$ if and only if $A(z_0)$ is connected.

**Theorem 37** *Let $z_0$ be an attracting periodic point of period $p$. Then*

$$\bigcup_{j=0}^{p-1} A^*(f^j(z_0))$$

*contains a singularity of $f^{-1}$.*

*Proof.* For simplicity we consider only the case that $p = 1$ so that $z_0$ is a fixpoint. The general case can be reduced to this case. Without loss of generality we may assume that $z_0 = 0$ and also that $\infty \in J$ and hence $\infty \notin A(0)$ if $f$ is rational. If $f'(0) = 0$, then 0 is a critical value (as well as a critical point) and we are done. Thus we may assume that $\lambda = f'(0)$ satisfies $0 < |\lambda| < 1$ so that Theorem 35 is applicable. From (8) we deduce that if $S(\lambda z) \in A(0)$, then $S(z) \in A(0)$. Hence $S(D(0,r)) \subset A(0)$ if $S$ is defined and holomorphic in $D(0,r)$. As $S(D(0,r))$ is connected, we even have $S(D(0,r)) \subset A^*(0)$.

If $S$ were entire we would obtain $S(\mathbb{C}) \subset A^*(0) \subset D\backslash J$, contradicting Picard's Theorem because $|J| = \infty$. Hence $S$ is not entire and there exists $r > 0$ such that $S$ is holomorphic in $D(0,r)$, but not in $D(0,R)$ for any $R > r$. Hence there exists a singularity $\zeta$ of $S$ satisfying $|\zeta| = r$. Since $S(z) = f^{-1}(S(\lambda z))$ and since $S(\lambda z)$ is holomorphic at $z = \zeta$ we see that $a = S(\lambda\zeta)$ is a singularity of $f^{-1}$. As noted above, $a = S(\lambda\zeta) \in S(D(0,r)) \subset A^*(0)$. ■

As an example we consider the iteration of quadratic polynomials. Any quadratic polynomial is conjugate to a polynomial of the form $f_c(z) = z^2 + c$ and thus it suffices to consider the dynamics of such polynomials. The function $f_c^{-1}$ has two critical points: 0 and $\infty$. As $\infty$ is a superattracting fixpoint, we deduce from Theorem 37 that if $f_c$ has a cycle of finite attracting periodic points, then the corresponding cycle of immediate basins of attraction contains the critical value $c$. (The proof of Theorem 37 actually shows that it also contains the critical point 0.) In particular, there is at most one cycle of attracting periodic points. We define

$$H = \{c : f_c \text{ has a finite attracting periodic point}\}$$

and

$$M = \{c : |f_c^n(0)| \not\to \infty \text{ as } r \to \infty\}.$$

The set $M$ is called the *Mandelbrot set*. The above considerations yield $H \subset M$. It is a major open question whether $H$ equals the interior of $M$.

Similarly one may analyze the dynamics of $f_\lambda(z) = \lambda e^z$. Here $f'_\lambda(z) \neq 0$ so that there are no critical values. Clearly 0 is an asymptotic value because $f_\lambda(x) \to 0$ as $x \to -\infty$, $x \in \mathbb{R}$. It is easy to see that there are no other asymptotic values so that 0 is the only singularity of $f^{-1}$. (Of course, this follows also immediately from the explicit form $f_\lambda^{-1}(w) = \log(w/\lambda)$.) As above we deduce that $f_\lambda$ has at most one cycle of attracting periodic points. It is conjectured that the set of all $\lambda$ for which this is the case is dense in $\mathbb{C}$.

We now consider fixpoints with multiplier $\lambda$ which do not satisfy the hypotheses of Theorem 35; that is, $\lambda = 0$ or $|\lambda| = 1$. We will only briefly sketch the results without going into the details of the proofs.

First, if $z_0$ is irrationally indifferent, so that $|\lambda| = 1$, but $\lambda$ is not a root of unity, then there still exists a formal power series $S(z) = z_0 + z + b_2 z^2 + b_3 z^3 + \dots$ which satisfies (8). If this series converges in a neighborhood of 0, then $z_0 \in F$ and the component of $F$ containing $z_0$ is called a *Siegel disk*. The name is chosen in honor of Siegel [39] who first established that this case can actually occur. If $z_0$ is contained in a Siegel disk $U$ and $D(0,r)$ is the largest disk around 0 where $S$ is holomorphic, then $S : D(0,r) \to U$ is a bijection and (8) holds for all $z \in D(0,r)$. Before Siegel's work, Pfeifer [32] and Cremer [13, 14] had shown that the series for $S$ does not always converge. Given a function $f$ with an irrationally indifferent fixpoint $z_0$, say of multiplier $\lambda = e^{2\pi i\alpha}$, $0 < \alpha < 1$, the question whether the formal power series solution $S$ of (8) converges or not is rather delicate and depends on number theoretic properties of $\alpha$. We refer the interested reader to the papers cited above as well as to the recent work of Yoccoz [43, 44] and the surveys [22, 31].

Next we consider the case that $\lambda$ is a root of unity. Replacing $f$ by a suitable iterate we may assume that $\lambda = 1$.

**Theorem 38** *Suppose that*

$$f(z) = z + a(z - z_0)^{m+1} + O\left((z - z_0)^{m+2}\right)$$

*as $z \to z_0$, where $m \geq 1$ and $a \neq 0$. Define $\theta_j = -\frac{\arg a + 2\pi j}{m}$, $j = 0, \dots, m$. Then there exist $m$ domains $U_j \subset \{z : \theta_j < \arg(z - z_0) < \theta_{j+1}\}$, $j = 0, \dots, m-1$, with $z_0 \in \partial U_j$ and with piecewise smooth boundaries tangent to the rays $\arg(z - z_0) = \theta_j$ and $\arg(z - z_0) = \theta_{j+1}$ such that $f(U_j) \subset U_j$ and $f^n|_{U_j} \to z_0$ as $n \to \infty$ for each $j \in \{0, \dots, m-1\}$.*

*Sketch of the proof.* We may assume that $z_0 = 0$ and consider for fixed $j \in \{0, \dots, m-1\}$ the function $h : \{z : \theta_j < \arg z < \theta_{j+1}\} \to \mathbb{C} \backslash (-\infty, 0]$ defined by $h(z) = -1/maz^m$. We define $g = h \circ f \circ h^{-1}$ and find that

$$g(z) = z + 1 + O\left(\frac{1}{|z|^{1/m}}\right)$$

as $|z| \to \infty$. From this we can deduce that for a suitable domain $G$ of the form $G = \{z = x + iy : x > \alpha - \beta|z|^\gamma\}$ where $\alpha, \beta > 0$, $\gamma > 1$, we have $g(G) \subset G$ and $g^n|_G \to \infty$ as $n \to \infty$. It follows that $U_j = h^{-1}(G)$ has the desired properties. ■

The domains $U_j$ in Theorem 38 are called *Leau petals* of $f$ at $z_0$. Clearly they are contained in $F$. A component of $F$ containing a Leau petal is called a *Leau domain.* (If $z_0$ is a rationally indifferent periodic point of $f$, then there exists $n$ such that $z_0$ is a fixpoint of $f^n$ with multiplier 1. Thus, more generally, we call a domain a Leau petal or a Leau domain of $f$ if it has this property with respect to some iterate of $f$. Analogously we define Siegel disks to periodic points of period greater than one.) Again there is a functional equation related to the behavior of $f$ in a Leau petal. This is usually expressed in terms of the function $g$ introduced in the above proof. The result is that, for a suitable domain $G$ as above, there exists a function $A$ satisfying

$$A(g(z)) = A(z) + 1 \tag{10}$$

for $z \in G$. The equation (10) is known as Abel's equation. Similarly as we deduced Theorem 37 from (8) we can deduce the following result from (10).

**Theorem 39** *A cycle of Leau domains contains a singularity of* $f^{-1}$.

Finally we consider the case that $\lambda = 0$, say

$$f(z) = z_0 + a_m(z - z_0)^m + O\left((z - z_0)^{m+1}\right)$$

as $z \to z_0$, where $a_m \neq 0$ and $m \geq 2$. Clearly $f^n \to z_0$ in some neighborhood of $z_0$. Again there is a functional equation related to this case, namely Böttcher's functional equation

$$\phi(f(z)) = \phi(z)^m. \tag{11}$$

If $b_1^{m-1} = a_m$, then (11) has a solution $\phi(z) = b_1(z - z_0) + b_2(z - z_0)^2 + \dots$ holomorphic in a neighborhood of $z_0$. In fact, if we restrict (without loss of generality) to the case $z_0 = 0$, then we have $\phi(z) = \lim_{n\to\infty} \sqrt[m^n]{f^n(z)}$ for a suitable branch of the root.

Because of the functional equations (11) and (8) an immediate attractive basin is also called *Böttcher domain* or *Schröder domain*, depending on whether the attracting periodic point contained in it is superattracting or not.

# 10 The classification of the periodic components of the Fatou set

**Definition 40** We say that a component $U$ of $F$ is *periodic* if $f^p(U) \subset U$ for some $p \in \mathbb{N}$ and the smallest $p$ with this property is called the *period* of $U$. If $p = 1$ then $U$ is called *invariant*. If $f^n(U)$ is contained in a periodic component for some $n$, then $U$ is called *preperiodic*. Otherwise $U$ is called *wandering*.

So far we have met four types of invariant (or periodic) components, namely Böttcher domains, Schröder domains, Leau domains, and Siegel disks, although we have not proved that Siegel disks do actually exist. A further possibility is that of a *Herman ring*. This is, by definition, an invariant doubly connected component of $F$ such that (8) holds for some conformal map $S$ from an annulus $\{z : 1 < |z| < R\}$ onto $U$ and some $\lambda$ satisfying $|\lambda| = 1$. That these do actually occur was shown by Herman [21, p. 138], extending earlier work of Arnol'd [2]. Another possibility of an invariant component is that of a *Baker domain*. This is, by definition, an invariant component $U$ of $F$ with the property that $f^n|_U \to z_0$ as $n \to \infty$ for some $z_0 \in \widehat{\mathbb{C}} \backslash D$. An example was already given by Fatou [19] who showed that $f(z) = z + 1 + e^{-z}$ has a Baker domain $U$ containing the right half-plane. The (simple) proof is left as an exercise. So far we have defined Herman rings and Baker domains only in the invariant case. Analogously as before, a periodic component of period $p$ is called a Herman ring or a Baker domain of $f$ if it has this property with respect to $f^p$.

In this chapter we shall classify the periodic components of $F$. Of course, it suffices to consider invariant components. Most of the work has already been done in §3. In fact, let $U$ be an invariant component of $F$ and apply Theorem 10 to $D = U$. If we have case (i) of Theorem 10, then $U$ is a Böttcher or Schröder domain. It can be deduced from Remark 3 (which we did not prove) after Theorem 10 that if we have case (iii) of that theorem, then $U$ is a Siegel disk or Herman ring. Using this and considering case (ii) in more detail we obtain

**Theorem 41** *A periodic component of $F$ is a Böttcher domain, Schröder domain, Leau domain, Siegel disk, Herman ring, or Baker domain.*

*Proof.* As mentioned above, it suffices to consider case (ii) of Theorem 10, applied to $D = U$, where $U$ is an invariant component of $F$.

Without loss of generality we assume that $\infty \notin U$. Choose $w_0 \in U$ and let $V \subset U$ be an open connected set whose closure is in $U$ and that contains $w_0$ and $f(w_0)$. Any limit function of $\{f^n|_U\}$ is constant and the set of limit functions of $\{f^n|_U\}$ is equal to $\bigcap_{N=1}^{\infty} \overline{\bigcup_{n=N}^{\infty} f^n(V)}$ and thus connected. We deduce that if no limit function is in $D$, then $U$ is a Baker domain. If there exists a limit function in $D$, say $f^{n_k}|_U \to a \in D$ as $k \to \infty$, then we can deduce as in the proof of Theorem 10 that $a$ is a fixpoint of $f$. Because the fixpoints of $f$ are isolated we conclude that there are no other limit functions of $\{f^n|_U\}$. Thus $f^n|_U \to a$ as $n \to \infty$, and $a \in \partial U$. We may assume that $a \neq \infty$.

We will show that $U$ is a Leau domain; that is, $f'(a) = 1$. It is clear that $|f'(a)| \geq 1$ because otherwise $a$ is an attracting fixpoint and we are in case (i) and $U$ is a Schröder or Böttcher domain. Also, it is not difficult to show that we have $|f'(a)| \leq 1$ because $f^n|_U \to a$ as $n \to \infty$. Hence $|f'(a)| = 1$.

There exists $\delta > 0$ such that $f$ is univalent in $D(a, \delta)$. For $N$ sufficiently large, $W = \bigcup_{n=N}^{\infty} f^n(V)$ satisfies $f(W) \subset W \subset D(a, \delta)$. Hence $f$ is univalent in $W$. For fixed $z_0 \in V$ we consider $\phi_n(z) = (f^n(z) - a)/(f^n(z_0) - a)$. Because $a, \infty \notin U$ and hence $a, \infty \notin W$ we have $f^n(z) \neq a, \infty$ and $\phi_n(z) \neq 0, \infty$ for $z \in W$. Since the $\phi_n$ are univalent in $W$ and $\phi_n(z_0) = 1$ we have $\phi_n(z) \neq 0, 1, \infty$ for $z \in W_0 = W \backslash \{z_0\}$. Thus $\{\phi_n\}_{n \in \mathbb{N}}$ is normal in $W_0$. Suppose that $\phi_{n_j} \to \phi$ in $W_0$. By the maximum principle, any fixed circle around $z_0$ contains points $u_j$ and $v_j$ such that $|\phi_{n_j}(u_j)| < |\phi_{n_j}(z_0)| = 1 < |\phi_{n_j}(v_j)|$ and thus points $u$ and $v$ such that $|\phi(u)| \leq 1 \leq |\phi(v)|$. In particular, $\phi \not\equiv 0, \infty$. For $z_1 \in W_0$ we consider $\psi_j(z) = \phi_{n_j}(z)/\phi_{n_j}(z_1) = (f^{n_j}(z) - a)/(f^{n_j}(z_1) - a)$. As above we see that $\psi_j \neq 0, 1, \infty$ in $W_1 = W \backslash \{z_1\}$ and hence $\{\psi_j\}_{j \in \mathbb{N}}$ is normal in $W_1$. We deduce that $\{\phi_{n_j}\}_{j \in \mathbb{N}}$ is normal in $W_1$ and hence normal in $W = W_0 \cup W_1$. It follows that $\phi_{n_j} \to \phi$ in $W$, with $\phi(z_0) = 1$. By Hurwitz's Theorem, $\phi$ is either constant (and hence $\phi \equiv 1$) or univalent.

Since $f(a) = a$ and $f^n|_U \to a$ as $n \to \infty$,

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{n \to \infty} \frac{f(f^n(z)) - a}{f^n(z) - a} = \lim_{n \to \infty} \frac{f^{n+1}(z) - a}{f^n(z) - a}$$

and hence

$$\phi(f(z)) = \lim_{j \to \infty} \phi_{n_j}(f(z))$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{f^{n_j}(f(z)) - a}{f^{n_j}(z) - a} \frac{f^{n_j}(z) - a}{f^{n_j}(z_0) - a} \qquad (12)\\
&= f'(a)\phi(z).
\end{aligned}
$$

If $\phi \equiv 1$, then $f'(a) = 1$ as claimed. If $\phi$ is univalent in $W$, then we obtain from (12) that $\phi(f^n(z)) = f'(a)^n\phi(z)$ and in particular that $\phi(f^n(z_0)) = f'(a)^n$. Since $|f'(a)| = 1$ there exists a sequence $(m_j)$ such that $f'(a)^{m_j} \to 1 = \phi(z_0)$. Thus $f'(a)^{m_j} \in \phi(W)$ for large $j$. We deduce that $\phi^{-1}(f'(a)^{m_j}) \to \phi^{-1}(1) = z_0$. But $\phi^{-1}(f'(a)^{m_j}) = f^{m_j}(z_0) \to a \neq z_0$, a contradiction. ■

Theorem 41 is of particular importance in view of the following result of Sullivan [41, 42].

**Theorem 42** *Rational functions do not have wandering domains.*

In other words, if $f$ is rational, then each component of $F$ is preperiodic. The proof of Theorem 42 is, unfortunately, too complicated to be included in this introductory course.

The last two theorems give a complete description of the possible limiting behavior of the sequence $(f^n)$ in components of $F$ if $f$ is rational: every component of $F$ is mapped by some iterate onto a Böttcher domain, Schröder domain, Leau domain, Siegel disk, or Herman ring.

Using Theorems 37 and 39 one can, in principle, determine the cycles of Böttcher, Schröder, and Leau domains by looking at the forward orbit of the singularities of $f^{-1}$ (which in the case of rational functions are just the critical values). There is also a relation between Siegel disks and Herman rings and the singularities of $f^{-1}$: the boundary of a Siegel disk or Herman ring is contained in the closure of the forward orbit of the singularities of $f^{-1}$.

On the other hand, there do not seem to be simple relations between Baker domains and the singularities of $f^{-1}$.

The major difference between the dynamics of rational and transcendental functions, however, is that for transcendental entire functions and holomorphic self-maps of $\mathbb{C}^*$ the conclusion of Theorem 42 does not hold; that is, wandering domains can occur. The first example of an entire function with a wandering domain is due to Baker [5]. It is not well understood yet which types of limiting behavior of $(f^n)$ can occur in wandering domains. For example, it is unknown whether for a wandering domain $U$ there always exists a sequence $(f^{n_k})$ and $z_0 \in \widehat{\mathbb{C}}\backslash D$ such that $f^n|_U \to z_0$ as $k \to \infty$.

# References


[1] L. V. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* 65 (1935), 157-194.

[2] V. I. Arnol'd, Small denominators I: On the mappings of the circumference onto itself, *Amer. Math. Soc. Transl.* (2) 46 (1965), 213-284; translation from *Izv. Akad Nauk. SSSR Ser. Mat.* 25 (1961), 21-86.

[3] I. N. Baker, Repulsive fixpoints of entire functions, *Math. Z.* 104 (1968), 252-256.

[4] I. N. Baker, Limit functions and sets of non-normality in iteration theory, *Ann. Acad. Sci. Fenn. (Ser. A, I. Math.)* 467 (1970).

[5] I. N. Baker, An entire function which has wandering domains, *J. Australian Math. Soc. (Ser. A)* 22 (1976), 173-176.

[6] A. F. Beardon, *Iteration of rational functions*, Springer, New York, Berlin, Heidelberg 1991.

[7] W. Bergweiler, Periodic points of entire functions: proof of a conjecture of Baker, *Complex Variables Theory Appl.* 17 (1991), 57-72.

[8] W. Bergweiler, Iteration of meromorphic functions, *Bull. Amer. Math. Soc. (N. S.)* 29 (1993), 151-188.

[9] P. Bhattacharyya, *Iteration of analytic functions*, PhD Thesis, University of London, 1969.

[10] P. Blanchard, Complex analytic dynamics on the Riemann sphere, *Bull. Amer. Math. Soc. (N. S.)* 11 (1984), 85-141.

[11] L. Carleson and T. W. Gamelin, *Complex dynamics*, Springer, New York, Berlin, Heidelberg 1993.

[12] J. B. Conway, *Functions of one complex variable*, Second edition, Springer, New York, Berlin, Heidelberg 1978.

[13] H. Cremer, Zum Zentrumsproblem, *Math. Ann.* 98 (1926), 151-163.

[14] H. Cremer, Über die Häufigkeit der Nichtzentren, *Math. Ann.* 115 (1938), 573-580.

[15] A. Douady and J. H. Hubbard, Itération des polynômes quadratiques complexes, *C. R. Acad. Sci. Paris (Série I)* 294 (1982), 123-126.

[16] D. Drasin, Normal families and the Nevanlinna theory, *Acta Math.* 122 (1969), 231-263.

[17] A. E. Eremenko and M. Yu. Lyubich, The dynamics of analytic transformations, *Leningrad Math. J.* 1 (1990), 563-634; translation from *Algebra i Analiz* 1 (1989).

[18] P. Fatou, Sur les équations fonctionelles, *Bull. Soc. Math. France* 47 (1919), 161-271; 48 (1920), 33-94, 208-314.

[19] P. Fatou, Sur l'itération des fonctions transcendantes entières, *Acta Math.* 47 (1926), 337-360.

[20] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford 1964.

[21] M. R. Herman, Exemples de fractions rationelles ayant une orbite dense sur la sphère de Riemann, *Bull. Soc. Math. France* 112 (1984), 93-142.

[22] M. R. Herman, Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point, *Proceedings VIIIth Int. Congress Math. Phys.*, edited by M. Mebkhout and R. Sénéor, World Scientific, Singapore, New Jersey, London, Hong Kong, 1987, 138-184.

[23] G. Jank and L. Volkmann, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel, Boston, Stuttgart, 1985.

[24] G. Julia, Sur l'itération des fonctions rationelles, *J. Math. Pures Appl.* (7) 4 (1918), 47-245.

[25] G. Kœnigs, Recherches sur les integrales de certaines équations fonctionelles, *Ann. Sci. École Norm. Sup.* (3) 1 (1884), 1-41.

[26] M. Yu. Lyubich, The dynamics of rational transforms: the topological picture, *Russian Math. Surveys* 41:4 (1986), 43-117; translation from *Uspekhi Mat. Nauk* 41:4 (1986), 35-95.

[27] J. Milnor, Dynamics in one complex variable: introductory lectures, Stony Brook Institute for Mathematical Sciences, Preprint 1990/5.

[28] M. Misiurewicz, On iterates of $e^z$, *Ergodic Theory Dynamical Systems* 1 (1981), 103-106.

[29] R. Nevanlinna, *Analytic functions*, Springer, Berlin, Heidelberg, New York 1970.

[30] H.-O. Peitgen and P. H. Richter, *The beauty of fractals*, Springer, Berlin, Heidelberg, New York, Tokyo, 1986.

[31] R. Pérez-Marco, Solution complète au problème de Siegel de linéarisation d'une application holomorphe au voisinage d'un point fixe (d'aprés J.-C. Yoccoz), *Astérisque* 206 (1992), 273-310 (Séminaire Bourbaki, Exp. 753).

[32] G. Pfeifer On the conformal mapping of curvilinear angles. The functional equation $\phi[f(x)] = a_1\phi(x)$, *Trans. Amer. Math. Soc.* 18 (1917), 185-198.

[33] H. Radström, On the iteration of analytic functions, *Math. Scand.* 1 (1953), 85-92.

[34] J. F. Ritt, Transcendental transcendency of certain functions of Poincaré, *Math. Ann.* 95 (1925/26), 671-682.

[35] P. C. Rosenbloom, L'itération des fonctions entières, *C. R. Acad. Sci. Paris* 227 (1948), 382-383.

[36] J. L. Schiff, *Normal Families*, Springer, New York, Berlin, Heidelberg 1993.

[37] E. Schröder, Ueber iterirte Functionen, *Math. Ann.* 3 (1871), 296-322.

[38] W. Schwick, Repelling periodic points in the Julia set, *Bull. London Math. Soc.*, to appear.

[39] C. L. Siegel, Iteration of analytic functions, *Ann. Math.* 43 (1942), 607-612.

[40] N. Steinmetz, *Rational iteration*, Walter de Gruyter, Berlin 1993.

[41] D. Sullivan, Itération des fonctions analytiques complexes, *C. R. Acad. Sci. Paris (Série I)* 294 (1982), 301-303.

[42] D. Sullivan, Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains, *Ann. Math.* 122 (1985), 401-418.

[43] J.-C. Yoccoz, Théorème de Siegel, nombres de Brjuno et polynômes quadratiques, preprint, 1987.

[44] J.-C. Yoccoz, Linéarisation des germes de difféomorphismes holomorphes de (C,0), *C. R. Acad. Sci. Paris (Série I)* 306 (1988), 55-58.

[45] L. Zalcman, A heuristic principle in complex function theory, *Amer. Math. Monthly* 82 (1975), 813-817.

Technische Universität Berlin, Fachbereich Mathematik, Sekr. MA 8-2,
Straße des 17. Juni 136, D-10623 Berlin

Email: bergweil@math.tu-berlin.de