On the product of a meromorphic function and its derivative *

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Abstract

It is shown that if f is a transcendental meromorphic function of finite order and if c is a polynomial which does not vanish identically, then $f'f - c$ has infinitely many zeros.

1 Introduction and main result

In 1959, W. K. Hayman [5, p. 36] proved that if f is a transcendental meromorphic function and n an integer satisfying $n \geq 3$, then $f'f^n$ takes every non-zero complex value infinitely often. He conjectured [7, Problem 1.19] that this remains valid for $n = 1$ and $n = 2$. The case $n = 2$ was settled by E. Mues [9] in 1979.

Recently, A. E. Eremenko and the author [3] proved Hayman's conjecture. The proof consists of two parts. The first step is to prove the result for the special case that the function has finite order and the second step is to reduce the general case to this special case. (This second step in the proof was obtained independently by H. H. Chen and M. L. Fang as well as by L. Zalcman.)

When I visited the Hong Kong University of Science and Technology in June 1994, Yik-Man Chiang asked me whether $f'f - c$ has infinitely many zeros if f is a transcendental meromorphic function and c is a meromorphic function which does not vanish identically and satisfies $T(r, c) = o(T(r, f))$ as $r \to \infty$. Here $T(r, f)$ denotes the Nevanlinna characteristic of f, see [6, 8, 10] for the notations and results of Nevanlinna theory used in this paper.

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It follows from recent results of Qingde Zhang [11] that the answer to Chiang's question is "yes" if the deficiency $\delta(\infty, f)$ of the poles of f satisfies $\delta(\infty, f) > 7/9$.

I have been unable to answer Chiang's question in general, but the purpose of this note is to give an affirmative answer in the case that c is a polynomial and that f has finite order.

Theorem *Let f be a transcendental meromorphic function of finite order and let c be a polynomial which does not vanish identically. Then $f'f - c$ has infinitely many zeros.*

The proof uses results from [3], but also requires some new ideas.

2 Lemmas

The following result was proved in [3, Corollary 3].

Lemma 1 *Let g be a meromorphic function of finite order. If g has only finitely many critical values, then g has only finitely many asymptotic values.*

We shall also need the following lemma whose proof can be found in [2, Lemma 2]. A closely related result was given by A. E. Eremenko and M. Yu. Lyubich [4, §2], see also [1, p. 173]. The main idea in the proof of [2, Lemma 2] is the logarithmic change of variable that was also used in [4, §2].

Lemma 2 *Let g be a meromorphic function and suppose that $g(0) \neq \infty$ and that the set of finite critical and asymptotic values of g is bounded. Then there exists $R > 0$ such that*

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R}$$

for all $z \in \mathbf{C} \backslash \{0\}$ which are not poles of g.

3 The case that f has finitely many zeros

In this section, we prove the theorem under the additional hypothesis that f has only finitely many zeros so that $N(r, 1/f) = O(\log r)$ as $r \to \infty$.

We define $g = f'f/c$. Then all poles of g which are not zeros of c have multiplicity at least 3 so that $\overline{N}(r,g) \le \frac{1}{3}N(r,g) + O(\log r)$.

From Nevanlinna's second fundamental theorem we can deduce that

$$\begin{aligned} 0 &\le N(r,f) + N\left(r,\frac{1}{f}\right) - N_1(r) + O(\log r) \\ &= N(r,f) - \left(N\left(r,\frac{1}{f'}\right) + 2N(r,f) - N(r,f')\right) + O(\log r) \\ &= N(r,f') - N(r,f) - N\left(r,\frac{1}{f'}\right) + O(\log r) \\ &= \overline{N}(r,f) - N\left(r,\frac{1}{f'}\right) + O(\log r) \end{aligned}$$

as $r \to \infty$. (Here we have used the second fundamental theorem for $q = 2$. Usually, it is only stated for $q \ge 3$, but is is easy to see that it also holds for $q = 2$.) Thus

$$N\left(r,\frac{1}{f'}\right) \le \overline{N}(r,f) + O(\log r) = \overline{N}(r,g) + O(\log r).$$

It follows that

$$N\left(r,\frac{1}{g}\right) \le N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f'}\right) \le \overline{N}(r,g) + O(\log r).$$

The second fundamental theorem (with $q = 3$) now implies that

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + O(\log r) \\ &\le 2\overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g-1}\right) + O(\log r) \\ &\le \frac{2}{3}N(r,g) + \overline{N}\left(r,\frac{1}{g-1}\right) + O(\log r) \\ &\le \frac{2}{3}T(r,g) + \overline{N}\left(r,\frac{1}{g-1}\right) + O(\log r) \end{aligned}$$

so that

$$\overline{N}\left(r,\frac{1}{g-1}\right) \ge \frac{1}{3}T(r,g) - O(\log r).$$

In particular, $g-1$ and hence $f'f - c$ have infinitely many zeros.

4 The case that f has infinitely many zeros

In this section, we assume that f has infinitely many zeros $z_1, z_2, \ldots$ and define $g = \frac{1}{2}f^2 - C$ where $C(z) = \int_0^z c(t)dt$. Then $g' = f'f - c$. We note that $C(z_j) \to \infty$ as $j \to \infty$. In particular, we may assume that $C(z_j) \neq 0$ for all j.

We have to show that g' has infinitely many zeros and thus suppose that this is not the case. We may assume without loss of generality that 0 is not a pole of f. Using Lemma 1 we see that g satisfies the hypotheses of Lemma 2.

It follows that

$$\left| \frac{z_j g'(z_j)}{g(z_j)} \right| \geq \frac{1}{2\pi} \log \frac{|g(z_j)|}{R} = \frac{1}{2\pi} \log \frac{|C(z_j)|}{R}.$$

In particular, $|z_j g'(z_j)/g(z_j)| \to \infty$ as $j \to \infty$. On the other hand,

$$\frac{g'(z_j)}{g(z_j)} = \frac{c(z_j)}{C(z_j)}$$

which implies that $z_j g'(z_j)/g(z_j)$ tends to a finite limit as $j \to \infty$. This contradiction completes the proof of the theorem.

5 Remarks

1. The method of §3 also works if f has infinite order. We only have to replace the error terms $O(\log r)$ by $S(r, f)$ and $S(r, g)$ at the appropriate places. Similarly, the argument in §3 does not require that c is a polynomial but only that $T(r, c) = o(T(r, f))$ as $r \to \infty$. The hypotheses that f have finite order and that c be a polynomial are used only in §4.

2. The proof shows that the theorem holds more generally for all rational functions c which are of the form $c = C'$ for a rational function C and satisfy $\lim_{z\to\infty} c(z) \neq 0$.

3. In [3] it was shown not only that $f'f$ takes every non-zero value infinitely often, if f is a transcendental meromorphic function, but more general differential polynomials like $P(f)f'$ with a non-constant polynomial P or $(f^m)^{(l)}$ with $m > l > 0$ were also considered. The result of this paper has similar extensions.

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References

[1] W. Bergweiler, Iteration of meromorphic functions, Bull. Amer. Math. Soc. (N. S.) **29** (1993), 151–188.

[2] W. Bergweiler, On the zeros of certain homogeneous differential polynomials, Arch. Math. (Basel) **64** (1995), 199–202.

[3] W. Bergweiler and A. Eremenko, On the singularities of the inverse of a meromorphic function of finite order, Rev. Mat. Iberoamericana **11** (1995), 355–373.

[4] A. E. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier (Grenoble) **42** (1992), 989–1020.

[5] W. K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. Math. **70** (1959), 9–42.

[6] W. K. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.

[7] W. K. Hayman, Research problems in function theory, The Athlone Press, London, 1967.

[8] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser, Basel, Boston, Stuttgart, 1985.

[9] E. Mues, Über ein Problem von Hayman, Math. Z. **164** (1979), 239–259.

[10] R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, Göttingen, Heidelberg, 1953.

[11] Q. Zhang, On the value distribution of $\varphi(z)f(z)f'(z)$ (Chinese), Acta Math. Sinica **37** (1994), 91-97.

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