On the zeros of certain homogeneous differential polynomials

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1 Introduction and main result

E. Mues [10] proved in 1978 that if $a \in \mathbf{C}\backslash\{1\}$ and if f is a transcendental entire function which is not of the form $f(z) = \exp(\alpha z + \beta)$ where $\alpha, \beta \in \mathbf{C}$, then $f(z)f''(z) - af'(z)^2$ has at least one zero. The case $a = 0$ is due to W. K. Hayman [5, Theorem 5]. As shown by examples like $f(z) = \cos z$ the conclusion need not hold if $a = 1$, see [10] for further examples.

If we allow f to be meromorphic, then we have further exceptional values for a. In fact, an easy computation shows that if $a = (n+1)/n$ for some $n \in \mathbf{N}$ and if $f(z) = F(z)^{-n}$ for an entire function F with the the property that F'' has no zeros, then $f(z)f''(z) - af'(z)^2 = -nF(z)^{-2n-1}F''(z)$ has no zeros. It seems reasonable to conjecture if f is a transcendental meromorphic function not of the form $f(z) = \exp(\alpha z + \beta)$ and if $a \neq 1$ and $a \neq (n+1)/n$, then $f(z)f''(z) - af'(z)^2$ has at least one zero. This has recently been proved by J. K. Langley [8] in the case that $a = 0$ and had been obtained earlier by Mues [9] in this case for functions of finite lower order.

Theorem *Let f be a meromorphic function of finite order and $a \in \mathbf{C}$. If $a \neq 1$ and $a \neq (n+1)/n$ for all $n \in \mathbf{N}$ and if $f(z)f''(z) - af'(z)^2$ has only finitely many zeros, then f has the form $f(z) = R(z)e^{P(z)}$ for a rational function R and a polynomial P.*

Corollary *Let f be a transcendental meromorphic function of finite order and let a be as above. If $f(z)f''(z) - af'(z)^2$ has no zero, then f is of the form $f(z) = \exp(\alpha z + \beta)$ where $\alpha, \beta \in \mathbf{C}$.*

Mues [10] also characterized the polynomials with the property that $f(z)f''(z) - af'(z)^2$ has no zero for some $a \neq 1$. The proof of the corollary will show that one can also determine all rational functions satisfying the hypothesis of the corollary.

2 Lemmas

The following result was proved in [3, Corollary 3].

Lemma 1 *Let g be a meromorphic function of finite order. If g has only finitely many critical values, then g has only finitely many asymptotic values.*

The next result can be proved using a logarithmic change of variable. This device was used by A. E. Eremenko and M. Yu. Lyubich [4, §2] in iteration theory, see also [2, p. 173]. According to Eremenko and Lyubich, O. Teichmüller had used this change of variable before in value distribution theory.

Lemma 2 *Let g be a meromorphic function and suppose that $g(0) \neq \infty$ and that the set of finite critical and asymptotic values of g is bounded. Then there exists $R > 0$ such that*

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R}$$

for all $z \in \mathbf{C}\backslash\{0\}$ which are not poles of g.

For the convenience of the reader we sketch the proof of Lemma 2. We choose $R > |g(0)|$ such that all finite critical and asymptotic values of g lie in $|w| < R$. As shown in [2, Lemma 8] we may, increasing R if necessary, also assume that $|g(z)| < R$ on some curve Γ connecting 0 and ∞. The conclusion is trivial if $|g(z)| \leq R$. Suppose that $|g(z_0)| > R$ and put $u_0 = \log z_0$ and $v_0 = \log g(z_0)$ for some branch of the logarithm. Then we can define $\Phi(z) = \log(g^{-1}(e^z))$ as a single-valued function in $H = \{z : \text{Re } z > \log R\}$ such that $\Phi(v_0) = u_0$. Now $\Phi(H) \cap \log \Gamma = \emptyset$ for all branches of the logarithm so that $\Phi(H)$ does not contain any disk of radius greater than π. Thus

$$|\Phi'(v)| < \frac{\pi}{L(\text{Re } v - \log R)}$$

for $v \in H$, where L is Landau's constant. If we express this inequality in terms of g and use the inequality $L \geq \frac{1}{2}$ [1, p. 364] we obtain the conclusion, see [2, p. 173] for details.

3 Proof of the theorem and the corollary

Suppose that f has finite order and $f(z)f''(z) - af'(z)^2$ has only finitely many zeros. We consider the function

$$g(z) = z - h\frac{f(z)}{f'(z)} \tag{1}$$

where $h = 1/(1-a)$. Then

$$g'(z) = 1 - h + h\frac{f(z)f''(z)}{f'(z)^2} = h\left(\frac{f(z)f''(z)}{f'(z)^2} - a\right)$$

has only finitely many zeros. Note that we have used here the hypothesis that $a \neq (n+1)/n$ because $g'(z_0) = h\,((n+1)/n - a)$ if z_0 is a pole of multiplicity n of f.

Moreover, g has finite order and thus, by Lemma 1, g has only finitely many asymptotic values. We may assume without loss of generality that $f'(0) \neq 0$ so that $g(0) \neq \infty$. Then g satisfies the hypotheses of Lemma 2. We deduce that if R is as in this lemma and if ζ is a fixed point of g, then

$$|g'(\zeta)| \geq \frac{1}{2\pi} \log \frac{|\zeta|}{R}. \tag{2}$$

Now we suppose that f is not of the form $f(z) = R(z)e^{Q(z)}$ so that its logarithmic derivative f'/f and hence g are transcendental. We shall use standard notations and results from Nevanlinna theory, see [6, 7]. Because

$$\frac{1}{g(z) - z} = -\frac{1}{h}\frac{f'(z)}{f(z)}$$

we have

$$m\left(r, \frac{1}{g(z) - z}\right) = m\left(r, \frac{f'}{f}\right) + O(1)$$

as $r \to \infty$. Thus

$$m\left(r, \frac{1}{g(z) - z}\right) = O(\log r)$$

as $r \to \infty$ by the lemma on the logarithmic derivative. Nevanlinna's first fundamental theorem now implies that $g(z) - z$ has infinitely many zeros; that is, g has infinitely many fixed points. Let ζ be a fixed point of g. Clearly, ζ is either a zero or a pole of f. A simple computation shows that $g'(\zeta) = 1 - h/m$ if ζ is a zero of multiplicity m of f and $g'(\zeta) = 1 + h/m$ if ζ is a pole of multiplicity m of f. We deduce that $|g'(\zeta)| \le 1 + |h|$ for all fixed points ζ of f. Clearly, this contradicts (2) if $|\zeta|$ is large enough. This contradiction completes the proof of the theorem.

To prove the corollary, we suppose that $f(z)f''(z) - af'(z)^2$ has no zero, proceed as above, and find that f has the form $f(z) = R(z)e^{Q(z)}$ and that g is a rational function such that g' does not have zeros. Moreover, since $f(z)f''(z) - af'(z)^2 \ne 0$, the zeros of f' are all simple. Hence g does not have multiple poles. It is not difficult to see that these restrictions on g imply that g is either constant or a linear transformation. If f'/f is constant, then f has the required form. If f'/f is not constant, then

$$\frac{f'(z)}{f(z)} = \frac{h}{z - g(z)} \to 0$$

as $z \to \infty$. On the other hand,

$$\frac{f'(z)}{f(z)} = \frac{R'(z)}{R(z)} + Q'(z).$$

We conclude that $Q'(z) \to 0$ as $z \to \infty$. Hence $Q' = 0$ so that Q is constant. This contradicts the hypothesis that f is transcendental.

4 Remarks

1. The function g defined by (1) is the function iterated in the relaxed Newton method for finding the zeros of f. Note, however, that we do not suppose here that $|h-1| < 1$ which would be necessary to ensure that the iterates of g converge to zeros of f.

2. Mues [10, p. 333] considered essentially the same auxiliary function. More precisely, he worked with g/h instead of g.

3. Lemma 1 and results from iteration theory concerning fixed points of multiplier 1 were used in [3] to prove that the equation $f'(z)f(z) = c$ has infinitely many solutions if f

is a transcendental meromorphic function of finite order and $c \in \mathbf{C}\backslash\{0\}$. We remark that instead of the results from iteration theory, one could also use Lemma 2. On the other hand, although Lemma 2 and its proof do not involve iteration, this lemma has turned out to be very useful in the context of iteration, compare [2, 4].

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