Hypertranscendency of Conjugacies in Complex Dynamics

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The dynamics of rational functions near fixed points are closely connected to the functional equations of Schröder, Böttcher, and Abel. We show that the solutions of these equations are hypertranscendental except in some cases, where they are given in terms of exponential, trigonometric, and elliptic functions. This generalizes earlier work of Ritt [Math. Ann. **95**, 671-682 (1926); F.d.M. 52:321], Carroll [Michigan Math. J. **32**, 47-57 (1985); Zbl. 574:30028], and Borwein [Proc. Amer. Math. Soc. **107**, 215-221 (1989); Zbl. 674:39005].

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0 Introduction

The dynamics of rational functions near fixed points are closely connected to the functional equations of Schröder, Böttcher, and Abel. We determine in which cases the solutions of these functional equations satisfy algebraic differential equations. This generalizes earlier work of Ritt, Carroll, and Borwein.

1 Local Conjugacies

In the iteration theory of rational functions [1, 5, 14], the fixed points of a function (and of its iterates) play an important role. In order to understand the dynamics near the fixed point, the rational function is usually conjugated to a simpler one in a neighborhood or in a part of a neighborhood of the fixed point. In fact this local theory preceeded the global theory developed by Fatou [7] and Julia [8].

We now introduce the various conjugacies. For proofs and more details see [1, Chap. 6], [5, Chap. II], or [14, Chap. 3]. Let R be a rational function of degree at least 2 and let $z_0 \in \mathbf{C} \cup \{\infty\}$ be a fixed point of R, that is, $R(z_0) = z_0$. We define $L(z) = z + z_0$ if $z_0 \in \mathbf{C}$ and $L(z) = 1/z$ if $z_0 = \infty$. With $R_0 = L^{-1} \circ R \circ L$ we have $R_0(0) = 0$. We call $\lambda = R_0'(0)$ the *multiplier* of z_0. Clearly, $\lambda = R'(z_0)$ if $z_0 \neq \infty$.

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The fixed point z_0 is called

- superattracting if $\lambda = 0$,
- attracting if $0 < |\lambda| < 1$,
- rationally indifferent if λ is a root of unity,
- irrationally indifferent if $|\lambda| = 1$, but λ is not a root of unity,
- repelling if $|\lambda| > 1$.

Suppose first that z_0 is attracting, irrationally indifferent, or repelling. Then Schröder's functional equation

$$f_0(\lambda z) = R_0(f_0(z))$$

has a formal power series solution $f_0(z) = \sum_{n=1}^{\infty} a_n z^n$, which is unique if we normalize it by $a_1 = 1$. It converges in some neighborhood of 0 if z_0 is attracting or repelling and may or may not converge in the irrationally indifferent case. The function $f = L \circ f_0$ satisfies the equation

$$f(\lambda z) = R(f(z)). \tag{1}$$

In the case of convergence, we call f the *Schröder function* of R at z_0.

Suppose next that z_0 is a superattracting fixed point so that $R_0(z) = \sum_{n=p}^{\infty} b_n z^n$, where $p \geq 2$ and $b_p \neq 0$. Then, for any a_1 satisfying ${a_1}^{p-1} = b_p$, Böttcher's functional equation

$$f_0(z^p) = R_0(f_0(z))$$

has a unique solution $f_0(z) = \sum_{n=1}^{\infty} a_n z^n$, which converges in some neighborhood of 0. The function $f = L \circ f_0$ satisfies

$$f(z^p) = R(f(z))$$

and is called a *Böttcher function* of R at z_0.

Finally, suppose that z_0 is a rationally indifferent fixed point, in which case the description of the local conjugacies is slightly more complicated, compare e. g. [5, Chap. II.5]. It suffices to consider the case $\lambda = 1$, because otherwise we can replace R by a suitable iterate. Then $R_0(z) = z + \sum_{n=p}^{\infty} b_n z^n$, where $p \geq 2$ and $b_p \neq 0$. We may assume that $b_p = 1$, because this can be achieved by replacing $L(z)$ by $L(cz)$ with a suitable value of c. For a fixed j, $0 \leq j \leq p-2$, we consider the function

$$h : \left\{ z \;\middle|\; \frac{2\pi j}{p-1} < \arg z < \frac{2\pi (j+1)}{p-1} \right\} \to \mathbf{C} \setminus (-\infty, 0]$$

given by $h(z) = -z^{-p+1}$ and define $S(z) = h(R_0(h^{-1}(z)))$. Then Abel's functional equation

$$g(S(z)) = g(z) + 1$$

has a solution g in certain domains

$$\Omega_\delta = \{z \mid |\text{Im } z| > -\delta \text{ Re } z + C_\delta \text{ or } \text{Re } z > C_0\},$$

which are invariant under S. The solution g is unique up to an additive constant. The image of such a domain Ω_δ under $L \circ h^{-1}$ is called a *petal* at z_0, and it is invariant under R.

The function $f = g \circ h \circ L^{-1}$ is defined in such a petal and satisfies

$$f(R(z)) = f(z) + 1.$$

It is called an *Abel function* of R at z_0.

For short, we shall refer to Schröder, Böttcher, and Abel functions as *local conjugacies.*

If R and S are rational functions such that $R = L \circ S \circ L^{-1}$ for some Möbius transformation L, then we say that R and S are *linearly conjugate.*

2 Hypertranscendency and Statement of Results

In this paper we are interested in the question, whether local conjugacies can satisfy algebraic differential equations. A function which satisfies an algebraic differential equation is called *differentially algebraic*, otherwise it is called *hypertranscendental* (or *transcendentally transcendental*). Since Hölder proved in 1887 that the Gamma function is hypertranscendental, there have been many results concerning the hypertranscendency of other (classes of) functions. For more information on this subject we refer to the papers of Rubel [11, 12, 13].

Ritt [10] was the first to consider the question, whether certain local conjugacies are hypertranscendental. It is an observation of Poincaré [9, p. 318] that Schröder functions to repelling fixed points extend to functions meromorphic in the plane. On the other hand, Schröder functions to repelling fixed points are essentially the only functions meromorphic in the plane that satisfy (1). Ritt proved that if a differentially algebraic function f is meromorphic in the plane and satisfies (1) for a rational function R of degree at least 2, then f is a Möbius transformation of $\exp(\alpha z^r)$, $\cos(\alpha z^r + \beta)$, $\wp(z^r + \beta)$, $\wp^2(z^r + \beta)$, $\wp'(z^r + \beta)$, or $\wp^3(z^r + \beta)$. Here r is a rational number with the property that the resulting function is meromorphic in the plane, α is a non-zero constant, and β is an appropriate fraction of the period. The function $\wp^2(z^r + \beta)$ occurs only if $g_3 = 0$ and $\wp'(z^r + \beta)$ and $\wp^3(z^r + \beta)$ occur only if $g_2 = 0$.

It is easy to see that if f is a Möbius transformation of $\exp(\alpha z^r)$, then the corresponding rational function R is linearly conjugate to the monomial $M_d(z) = z^d$ or to z^{-d}. Here and in the following d denotes the degree of R. Similarly, if f is a Möbius transformation of

$\cos(\alpha z^r+\beta)$, then R is linearly conjugate to $\pm T_d(z)$, where T_d denotes the d-th Chebychev polynomial, which is defined by $\cos dz = T_d(\cos z)$.

For example the Schröder function to $T_2(z) = 2z^2 - 1$ at the fixed point 1 is given by $\cosh\sqrt{2z} = \cos i\sqrt{2z}$, while the Schröder function to T_2 at $-1/2$ is given by $\cos(2z/\sqrt{3} - 2\pi/3)$.

For the explicit form of a rational function having $\wp$ as a solution of the Schröder functional equation we refer to [1, Chap. 4.3]. Rational functions corresponding to $\wp'$, $\wp^2$, or $\wp^3$ can be obtained similarly.

The monomials M_d and the Chebychev polynomials T_d give also rise to rational (and hence differentially algebraic) Böttcher functions. Clearly, the Böttcher functions to M_d at 0 are given by $f(z) = \rho z$, where $\rho^{d-1} = 1$.

The Böttcher functions of $\pm T_d$ at ∞ are given by $f(z) = \frac{1}{2}(\rho z + 1/\rho z)$, where $\rho^{d-1} = \pm 1$. It was shown in [2] that M_d and $\pm T_d$ are, up to linear conjugations, the only polynomials which have an algebraic Böttcher function at ∞. (Note that the definition of a Böttcher function in [2] is slightly different from the one given here.)

Carroll [6] proved that Schröder functions to finite Blaschke products with (attracting) fixed point at 0 are hypertranscendental.

Borwein's result in [3] says that the Böttcher function of z^2+c at ∞ is hypertranscendental if c is a positive real number.

The purpose of this paper is to show that Ritt's result, which seemingly dealt only with Schröder functions at repelling fixed points, may be used to determine all differentially algebraic local conjugacies.

Theorem *(i) The only differentially algebraic Schröder functions are those given by Ritt. In particular, Schröder functions to attracting fixed points are always hypertranscendental.*

(ii) Differentially algebraic Böttcher functions are either Möbius transformations or they are Möbius transformations of $\rho z + 1/\rho z$, where $\rho^{d-1} = \pm 1$ and $d \geq 2$ is an integer. The corresponding rational functions are linearly conjugate to M_d and $\pm T_d$, respectively.

(iii) Abel functions are always hypertranscendental.

Besides Ritt's theorem, the proof is based on results of Boshernitzan and Rubel [4] on coherent families of functions. By definition, a family of functions is *coherent* if there exists an algebraic differential equation which is satisfied by all functions of the family. We shall also use some results from the Fatou-Julia iteration theory, compare [1, 5, 14].

3 Proof of the Theorem

Let R be a rational function and let f be a differentially algebraic local conjugacy at the fixed point z_0 of R. We may assume without loss of generality that $z_0 = 0$, because compositions of f with Möbius transformations are also differentially algebraic.

Our result is contained in Ritt's theorem if 0 is a repelling fixed point. Therefore we shall assume that 0 is not repelling but superattracting, attracting, irrationally indifferent, or rationally indifferent with multiplier 1. Note that in these cases there are certain invariant components of the Fatou set $F(R)$ of R associated with the fixed point 0. (Recall that we assumed in the irrationally indifferent case that the series for f converges in a neighborhood of 0.) Here, as usual, the Fatou set $F(R)$ of R is the set of points, where the iterates of R form a normal family. In particular, $F(R) \neq \emptyset$ in our case. The complement of $F(R)$ is called the Julia set of R and denoted by $J(R)$. Next we note that the iterates $\{R^n\}$ of R form a coherent family. This follows immediately from [4, Thm. 6.1] if f is a Schröder function and from [4, Thm. 6.3] if f is a Böttcher function. The case of an Abel function is similar. In fact, we have

$$R^n(z) = f^{-1}(f(z) + n)$$

in a petal, and the right hand side is coherent if f is differentially algebraic, cf. [4, Thm. 2.2].

Now there exists $k \in \mathbf{N}$ such that R^k has a repelling fixed point z_1. For example, this follows from the fact that the repelling fixed points of the iterates of R are dense in $J(R)$, which is a non-empty perfect set. We denote by Φ the Schröder function of R^k at z_1. Because $\{R^n\}$ is coherent, Φ is differentially algebraic by [4, Thm. 6.1]. Thus Φ is one of the functions occurring in Ritt's list.

If Φ is related to the $\wp$-function, then $F(R) = \emptyset$, cf. [1, Chap. 4.3]. As explained above this is impossible in our case. Thus Φ is a Möbius transformation of $\exp(\alpha z^r)$ or $\cos(\alpha z^r + \beta)$. We deduce that R^k is linearly conjugate to a monomial, a Chebychev polynomial, or the negative of a Chebychev polynomial. The non-repelling fixed points of these polynomials are superattracting. Hence f is a Böttcher function and from the remarks in Chapter 1 we conclude that f has the required form. This in turn implies that R itself is linearly conjugate to M_d or $\pm T_d$. Here $R(z) = z^{-d}$ does not occur, because z^{-d} does not have a superattracting fixed point. This shows that the assertion of the Theorem holds.

Remarks. 1. Rubel [12, p. 663] credits Eremenko with the result that monomials and Chebychev polynomials are the only (suitably normalized) polynomials whose iterates form a coherent family.

2. It was shown in [2] that if p and q are polynomials of the same degree $d \geq 2$ and if f is conformal at ∞ and satisfies $f(p(z)) = q(f(z))$, then f is transcendental unless both p and q are linearly conjugate to M_d or $\pm T_d$. We conjecture that, apart from these exceptional cases, f is even hypertranscendental under the above hypotheses.

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