IN Variant Domains AND SINGULARITIES

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ABSTRACT. Let \( U \) be an invariant component of the Fatou set of an entire transcendental function \( f \) such that the iterates of \( f \) tend to \( \infty \) in \( U \). Let \( P(f) \) be the closure of the set of the forward orbits of all critical and asymptotic values of \( f \). We show that there exists a sequence \( p_n \in P(f) \) such that \( \text{dist}(p_n, U) = \alpha(p_n) \), where \( \text{dist}(\cdot, \cdot) \) denotes Euclidean distance. On the other hand, we give an example where \( \text{dist}(P(f), U) > 0 \). In this example, \( U \) is bounded by a Jordan curve.

1. Introduction and Results

Let \( f \) be a transcendental entire (or a rational) function. The Fatou set \( F(f) \) is the subset of the complex plane \( \mathbb{C} \) (or the sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \)) where the iterates \( f^n \) of \( f \) form a normal family. The complement of \( F(f) \) is called the Julia set and denoted by \( J(f) \).

A connected component \( U \) of the Fatou set is called invariant if \( f(U) \subset U \). It is well-known that if \( f \) is a transcendental entire function and if \( U \) is an invariant component of \( F(f) \), then we have one of the following possibilities:

- There exists \( z_0 \in U \) such that \( f^n|_U \to z_0 \) as \( n \to \infty \). Then \( f(z_0) = z_0 \) and \( |f'(z_0)| < 1 \). Here \( U \) is called a Böttcher domain if \( f'(z_0) = 0 \) and a Schröder domain if \( 0 < |f'(z_0)| < 1 \).
- There exists \( z_0 \in \partial U \) such that \( f^n|_U \to z_0 \) as \( n \to \infty \). Then \( f(z_0) = z_0 \) and \( f'(z_0) = 1 \), and \( U \) is called a Leau domain.
- \( f^n|_U \to \infty \) as \( n \to \infty \). In this case, \( U \) is called a Baker domain.
- There exists an analytic homeomorphism \( \phi \) from \( U \) onto the unit disc such that \( \phi(f(\phi^{-1}(z))) = e^{2\pi i \alpha}z \) for some \( \alpha \in \mathbb{R}\backslash \mathbb{Q} \). Then \( U \) is called a Siegel disc.

There is a similar classification of invariant domains for rational functions. Here \( \infty \) does not play a special role so that Baker domains are not an independent possibility but turn out to be special cases of Böttcher, Schröder, or Leau domains (if we allow \( z_0 = \infty \) in their definition but replace \( f'(z_0) \) by \( f'(1/f(\frac{1}{z})) \) at \( z_0 = \infty \)). There is, however, also an additional type of invariant domain for rational functions, the Arnold–Herman ring. This is, by definition, a doubly-connected component \( U \) of \( F(f) \) such that \( \phi(f(\phi^{-1}(z))) = e^{2\pi i \alpha}z \) for some \( \alpha \in \mathbb{R}\backslash \mathbb{Q} \) and some analytic homeomorphism \( \phi \) from \( U \) onto an annulus \( \{z : 1 < |z| < r\} \), where \( 1 < r < \infty \).

A proof of the classification of invariant components for rational functions can be found in [7, §7.1], [9, §IV.2], and [17, §3.2]. These books are also recommended as introductions into the iteration theory of rational functions in general. The proof of the above classification of invariant components for transcendental entire
functions requires only minor modifications. (It is even easier than in the case of rational functions because a simple argument shows that the iterates tend to \( \infty \) in multiply-connected components. In particular, if \( \{f^n|U\} \) has non-constant limit functions, then \( U \) is simply-connected, and this implies that \( U \) is a Siegel disc in this case.) For an introduction into the iteration theory of transcendental entire functions we refer to [3], [8], [12, §4], and [14, §III].

An important role in iteration theory is played by the singularities of the inverse function \( f^{-1} \) of \( f \). Recall that the singularities of \( f^{-1} \) are the critical and asymptotic values of \( f \). (For rational functions, we have to consider only the critical values.) We denote the set of all singularities of \( f^{-1} \) by \( \text{sing}(f^{-1}) \) and define

\[
P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1})).
\]

Here and in the following \( \overline{E} \) denotes the closure of a set \( E \subset \mathbb{C} \).

The importance of the singularities of the inverse function in iteration theory is mainly due to the following results:

- If \( U \) is a Böttcher, Schröder, or Leau domain, then \( U \cap \text{sing}(f^{-1}) \neq \emptyset \).
- If \( U \) is a Siegel disc or Arnol’d–Herman ring, then \( \partial U \subset P(f) \).

Proofs of these results for rational functions can be found in [7, §§9.3], [9, §§III.2, V.1], and [17, §§3.4, 3.5, 3.8]. The arguments carry over to the transcendental case.

One may ask, whether there is also some relation between Baker domains and \( \text{sing}(f^{-1}) \) or \( P(f) \), compare [8, Question 4]. By a result of Eremenko and Lyubich [13, Theorem 1], there are no Baker domains if \( \text{sing}(f^{-1}) \) is bounded. On the other hand, Herman [14, p. 609] and Eremenko and Lyubich [11, Example 3] gave examples of Baker domains \( U \) satisfying \( U \cap \text{sing}(f^{-1}) = \emptyset \). Herman’s examples are related to analytic self-maps of the punctured plane having a Siegel disc or Arnol’d–Herman ring (see also [4, §5] and [6, §4]), and we have \( \partial U \subset P(f) \) in these examples. In the example of Eremenko and Lyubich, which is obtained with techniques from approximation theory, it is not clear whether there are relations between \( \partial U \) and \( P(f) \).

**Theorem 1.** The function \( f(z) = 2 - \log 2 + 2z - e^z \) has a Baker domain \( U \) such that \( \text{dist}(P(f), U) > 0 \).

Here and in the following \( \text{dist}(\cdot, \cdot) \) denotes the Euclidean distance between two sets (or points) in the plane.

**Theorem 2.** The Baker domain \( U \) in Theorem 1 is bounded by a Jordan curve in \( \mathbb{C} \).

Here we have added \( \infty \) to \( \partial U \). Another example of a Baker domain whose boundary is a Jordan curve in \( \mathbb{C} \) has been given by Baker and Weinreich [6, Theorem 3]. Their proof relied on deep results of Herman concerning circle diffeomorphisms. Our argument appears to be simpler.

We mention that Baker and Weinreich [6, Theorem 4] also proved that if \( \partial U \) is a Jordan curve in \( \mathbb{C} \) for some Baker domain \( U \) of an entire transcendental function \( f \), then \( f|U \) is univalent. The boundary of unbounded Böttcher domains, Schröder domains, Leau domains, and Siegel discs \( U \) is much more complicated [6, Theorem 1]: here \( \infty \) is in the impression of every prime end of \( U \).

The following result complements Theorem 1 in some sense,
Theorem 3. Let $f$ be a transcendental entire function with an invariant Baker domain $U$. If $U \cap \text{sing}(f^{-1}) = \emptyset$, then there exists a sequence $(p_n)$ such that $p_n \in P(f)$, $|p_n| \to \infty$, $|p_{n+1}/p_n| \to 1$, and $\text{dist}(p_n, U) = o(|p_n|)$ as $n \to \infty$.

The example of Theorem 1 shows that $\text{dist}(p_n, U) = o(|p_n|)$ and $|p_{n+1}/p_n| \to 1$ cannot be replaced by $\text{dist}(p_n, U) = o(1)$ and $|p_{n+1} - p_n| = o(1)$.

There is an analogous result for periodic domains. These are, by definition, components $U$ of $F(f)$ which satisfy $f^p(U) \subset U$ for some $p \in \mathbb{N}$, that is, components of the Fatou set that are invariant with respect to some iterate. To extend Theorem 3 to periodic domains we only have to note that $\text{sing}((f^p)^{-1}) = \bigcup_{n=0}^{p-1} f^n(\text{sing}(f^{-1}))$ and $P(f^p) = P(f)$, see [1, Lemma 2].

2. Proof of Theorem 1

First we determine $\text{sing}(f^{-1})$ and $P(f)$. To this end, we note that it is not difficult to see that $f$ does not have asymptotic values. Thus we have to consider only the critical values. The zeros of $f'(z) = 2 - e^z$ are given by

$$z_k = \log 2 + 2\pi k, \ k \in \mathbb{Z}.$$  

For $z \in \mathbb{C}$ and $k \in \mathbb{N}$ we have $f(z + 2k\pi i) = f(z) + 4k\pi i$ and induction shows that

$$f^n(z + 2k\pi i) = f^n(z) + 2^n k^{2\pi i}$$  

for $n \in \mathbb{N}$. In particular, since $f(\log 2) = \log 2$, we have $f(z_k) = z_{2k}$. This implies that

$$P(f) = \text{sing}(f^{-1}) = \{f(z_k) : k \in \mathbb{Z}\} = \{z_{2k} : k \in \mathbb{Z}\}.$$  

Next we note that if $\text{Re } z < -2$, then $\text{Re } f(z) < 2 - \log 2 + 2 \text{Re } z + e^{-2} < \text{Re } z$. This implies that there exists an invariant Baker domain $U$ containing the half-plane \{z : \text{Re } z < -2\}.

Induction shows that

$$f^n(z) = (2^n - 1)(2 - \log 2) + 2^n z - \sum_{j=0}^{n-1} 2^{n-j-1} \exp(f^j(z)).$$  

We define $h_n(z) = f^n(z)/2^n$ and deduce that $h_n(z)$ converges as $n \to \infty$ if $\text{Re } z < -2$. Moreover,

$$\lim_{n \to \infty} h_n(z) = 2 - \log 2 + z - \sum_{j=0}^{\infty} 2^{-j-1} \exp(f^j(z)) \sim 2 - \log 2 + z$$  

as $\text{Re } z \to -\infty$.

By a result of Baker [2, Theorem 1], $U$ is simply-connected. Using another result of Baker [5, Lemma 1] (see also [8, Lemma 7]) we deduce that if $K \subset U$ is compact, then there exists a constant $C$ such that $|f^n(z)| \leq C |f^n(z')|$ for sufficiently large $n$ and all $z, z' \in K$. It follows that $\{h_n\}$ is locally uniformly bounded in $U$ and hence normal in $U$. Thus $h_n(z)$ converges not only for $\text{Re } z < -2$ but in fact for all $z \in U$. By (2), the limit function is non-constant in $U$.

Recall now that $f(\log 2) = \log 2$ and $f'(\log 2) = 0$. This implies that there exists a component $U_0$ of $F(f)$ such that $f^n|_{U_0} \to \log 2$. (In the terminology introduced in the introduction, $U_0$ is a Bööttcher domain.) For $k \in \mathbb{Z}$, we define $U_k = \{z + 2\pi ki : z \in U_0\}$ and deduce from (1) that

$$f^n(z) = \log 2 + 2^n k^{2\pi i} + o(1)$$  

locally uniformly in $U_k$ as $n \to \infty$. We conclude that $U_k \in F(f)$. Moreover, $h_n \to 2\pi ki$ locally uniformly in $U_k$, that is, $h_n|_{U_k}$ tends to a constant limit as $n \to \infty$. Thus $U_k \cap U = \emptyset$. It follows that $\operatorname{dist}(P(f), U) \geq \operatorname{dist}(\log 2, \partial U_0) > 0$.

**Remark 1.** Our example is similar to that of Herman [14, p. 609f], who considered the function $g(z) = e^{2\pi i \alpha \log z^2}$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is chosen such that $g$ has a Siegel disc at zero and showed that the function $f(z) = \log g(e^z) = 2\pi i \alpha + z + e^z$ has a Baker domain $U$ satisfying $U \cap \operatorname{sing}(f^{-1}) = \emptyset$. The function $f$ of Theorem 1 is also of the form $\log g(e^z)$ for some entire function $g$, see §3 below.

**Remark 2.** The proof shows that $\operatorname{dist}(P(f), J(f)) > 0$ for the function $f$ of Theorem 1. Denote the class of all entire functions with this property by $C$. This class has been considered by Stallard [16], as one way to generalize the concept of hyperbolic (or expanding) rational functions (see [7, p. 225], [9, p. 89], or [17, p. 118]) to transcendental functions. She proves [16, Theorem A] that if $f \in C$ and $z \in J(f)$, then $|(f^n)'(z)| \to \infty$ as $n \to \infty$. We also mention that it follows from the main result of her paper [16, Theorem B] that the Julia set of the function of Theorem 1 has plane Lebesgue measure zero.

### 3. Proof of Theorem 2

We consider the function $g(z) = \frac{1}{2} e^{2z^2 - z}$. Clearly, 0 and 2 are fixed points of $g$ which are contained in Böttcher domains. We observe that $g(e^z) = \exp f(z)$ so that $g^n(e^z) = \exp(f^n(z))$ for all $n \in \mathbb{N}$. We shall show that $e^z \in J(g)$ if and only if $z \in J(f)$.

Suppose first that $z \in J(f)$. Then any neighborhood $N$ of $z$ contains points $a$ and $b$ such that $\Re f^k(a) < -2$ and $f^l(b) \in U_0$ for certain $k, l \in \mathbb{N}$, with $U_0$ as in the proof of Theorem 1. Hence $\Re f^m(a) \to -\infty$ and $f^m(b) \to \log 2$ as $n \to \infty$. We deduce that $g^n(e^a) \to 0$ and $g^n(e^b) \to 2$ as $n \to \infty$. This implies that $\exp(N) \cap J(g) \neq \emptyset$. Because $N$ can be chosen arbitrarily small we conclude that $e^z \in J(g)$.

Suppose now that $e^z \in J(g)$. Then any neighborhood $N$ of $z$ contains points $a$ and $b$ such that $g^k(e^a)$ and $g^l(e^b)$ are contained in the Böttcher domains of $g$ around 0 and 2 for certain $k, l \in \mathbb{N}$. Thus $g^n(e^a) \to 0$ and $g^n(e^b) \to 2$ as $n \to \infty$. We deduce that if $n$ is sufficiently large, then $f^n(a) \in U$ and $f^n(b) \in U_m$ for some $m$ (depending on $n$). Now $U$ and $U_m$ are distinct components of $P(f)$. We conclude that $f^n(N)$ intersects $J(f)$. Thus $N$ intersects $J(f)$ because $J(f)$ is completely invariant. Again, $N$ can be chosen arbitrarily small and we deduce that $z \in J(f)$. Altogether we see that $e^z \in J(g)$ if and only if $z \in J(f)$. In particular, $J(f)$ and hence $U$ are invariant under translation by $2\pi i$.

Denote by $V$ the Böttcher domain of $g$ that contains 0. Then $V = \exp U$ and $\partial V = \exp \partial U$. Thus it suffices to prove that $\partial V$ is a Jordan curve.

In order to do this we note that if $|z| = 1$, then $|g(z)| \geq \frac{5}{4} > 1$. This implies that $V \subset D$, where $D$ denotes the unit disc. Let $W$ be the component of $g^{-1}(D)$ that contains 0. Then $\overline{V} \subset W$ and $\overline{W} \subset D$. Moreover, $g$ is a proper mapping of degree 2 from $W$ onto $D$. Thus, in the terminology of [9, §VI.1] or [10], the triple $(g; W; D)$ is a polynomial-like mapping. By the basic result about polynomial-like mappings (see [9, Theorem VI.1.1] or [10, Theorem 1]), there exists a quasiconformal mapping $\varphi$ and a quadratic polynomial $p$ such that $g(z) = \varphi(p(\varphi^{-1}(z)))$ for $z \in W$. Since $g$ has a superattracting fixed point at 0 we deduce that $p$ has a superattracting fixed point so that $L^{-1}(p(L(z))) = z^2$ for some linear function $L$. This implies that
$V = \varphi(L(D))$ and $\partial V = \varphi(L(\partial D))$. Thus $\partial V$ is a Jordan curve (and in fact a quasicircle). This completes the proof of Theorem 2.

We describe a second, more elementary method to show $\partial V$ is a curve, although this method fails to yield that $\partial V$ is actually a Jordan curve. (Here more elementary is understood in the sense that quasiconformal mappings are not needed.)

To do this, we note that $P(g) = \text{sing}(g^{-1}) = \{0, 2\}$. Because 0 and 2 are contained in Böttcher domains of $g$ we have $\text{dist}(P(g), J(g)) > 0$. This, together with the result of Stallard [16, Theorem A] quoted in Remark 2, enables us to use the arguments of [9, §§V.4, VI.5] and [17, §5.5], where it is shown that simply-connected components of the Fatou set of hyperbolic rational functions are bounded by curves. In fact, let $\phi(z) = 2e^{-2z} + O(z^2)$ be the solution of Böttcher’s functional equation $\phi(z^2) = g(\phi(z))$ near 0. Then $\phi$ is a conformal mapping from the unit disc onto $V$. Let $R$ be a fixed number satisfying $0 < R < 1$ and define

$$\gamma_n(e^{i\theta}) = \phi \left( R^{-1/2} e^{i\theta} \right)$$

for $n \in \mathbb{N}$ and $0 \leq \theta \leq 2\pi$. As in [9, p. 94] and [17, p. 137] one can show that $\gamma_n(e^{i\theta})$ converges uniformly as $n \to \infty$, and the limit function maps the unit circle continuously onto $\partial V$.

4. Proof of Theorem 3

The following lemma summarizes Koebe’s one quarter theorem and a part of his distortion theorem, see [15, §1.2]. By $D(a, r)$ we denote the disc of radius $r$ around $a \in \mathbb{C}$.

**Lemma 1.** Let $f$ be analytic and univalent in $D(a, r)$. Then

$$f(D(a, r)) \supset D \left( f(a), \frac{1}{4} |f'(a)| r \right).$$

Moreover, we have

$$\frac{r^2 |(z - a)f'(a)|}{(r + |z - a|)^2} \leq |f(z) - f(a)| \leq \frac{r^2 |(z - a)f'(a)|}{(r - |z - a|)^2}$$

and

$$\frac{r - |z - a|}{r + |z - a|} \leq \frac{|(z - a)f'(z)|}{|f(z) - f(a)|} \leq \frac{r + |z - a|}{r - |z - a|}$$

for $z \in D(a, r)$.

By $\rho_U(z)$ we denote the density of the hyperbolic metric of a hyperbolic domain $U$ and by $\rho_U(z, z')$ the hyperbolic distance of $z, z' \in U$. The following estimate follows from Schwarz’s lemma and (3), that is, Koebe’s one quarter theorem. It can be found for example in [9, p. 13].

**Lemma 2.** If $U \subset \mathbb{C}$ is a simply-connected hyperbolic domain and $z \in U$, then

$$\frac{1}{2 \text{dist}(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{\text{dist}(z, \partial U)}$$

**Lemma 3.** Let $f$ be a transcendental entire function with an invariant Baker domain $U$. Suppose that $K \subset U$ is compact and that $\tau > 1$. Then there exists $n_0$ such that

$$D(f^n(z), \tau \text{dist}(f^n(z), \partial U)) \cap P(f) \neq \emptyset$$

for all $z \in K$ and $n \geq n_0$. 
Proof. Suppose that the conclusion does not hold. Then there exist sequences \((z_j)\) and \((n_j)\) such that \(z_j \in K\), \(n_j \to \infty\), and
\[
D(f^{n_j}(z_j), \tau \dist(f^{n_j}(z_j), \partial U)) \cap P(f) = \emptyset.
\]
Restricting to a subsequence if necessary we may assume that \(z_j \to z_0 \in K\) as \(j \to \infty\). This implies that \(\rho_U(z_j, z_0) \to 0\) and hence that \(\rho(f^{n_j}(z_j), f^{n_j}(z_0)) \to 0\) as \(j \to \infty\). From Lemma 2 we can deduce that \(\dist(f^{n_j}(z_j), \partial U) \sim \dist(f^{n_j}(z_0), \partial U)\) and \(|f^{n_j}(z_j) - f^{n_j}(z_0)|/ \dist(f^{n_j}(z_j), \partial U) \to 0\) as \(j \to \infty\). It follows that if \(1 < \sigma < \tau\), then
\[
D(f^{n_j}(z_0), \sigma \dist(f^{n_j}(z_0), \partial U)) \cap P(f) = \emptyset
\]
for all large \(j\). Of course, we may assume that (6) holds for all \(j\). We introduce the abbreviations \(\zeta_j = f^{n_j}(z_0)\) and \(\delta_j = \dist(\zeta_j, \partial U)\) so that (6) takes the form
\[
D(\zeta_j, \sigma \delta_j) \cap P(f) = \emptyset.
\]
Hence there exists branches \(\varphi_j\) of \((f^{n_j})^{-1}\) which are defined (and univalent) in \(D(\zeta_j, \sigma \delta_j)\) and satisfy \(\varphi_j(\zeta_j) = z_0\). We consider \(B_j = \{z : |z - \zeta_j| = \delta_j\}\) and \(C_j = \varphi_j(B_j)\). Then there exists \(b_j \in B_j \cap \partial U\) and \(c_j = \varphi_j(b_j) \in C_j \cap \partial U\).

By (4) we have
\[
|c_j - z_0| \leq \max_{z \in B_j} |\varphi_j(z) - z_0| \leq \left(\frac{\sigma + 1}{\sigma - 1}\right)^2 \min_{z \in B_j} |\varphi_j(z) - z_0| \leq \left(\frac{\sigma + 1}{\sigma - 1}\right)^2 \dist(z_0, \partial U)
\]
so that \(|c_j|\) is bounded. We may thus assume that \(c_j \to c \in \partial U\).

From (5) we deduce that
\[
|\varphi_j(b_j)| \geq \frac{\sigma - 1}{\sigma + 1} |\varphi_j(b_j) - z_0| = \frac{\sigma - 1}{\sigma + 1} |c_j - z_0| \geq \frac{\sigma - 1}{\sigma + 1} \frac{\dist(z_0, \partial U)}{\delta_j}.
\]
Combining this with (3) we find that
\[
\varphi_j(D(\zeta_j, \sigma \delta_j)) \supset D(c_j, \frac{1}{\delta_j} |\varphi_j(b_j)|(\sigma - 1)\delta_j)
\]
\[
\supset D(c_j, \frac{1}{4} (\sigma - 1)^2 \dist(z_0, \partial U))
\]
\[
\supset D(c, \delta)
\]
for some \(\delta > 0\) and all large \(j\). This implies that \(f^{n_j}\) is univalent in \(D(c, \delta)\). It is not difficult to see that this is impossible for \(c \in \partial U \subset J(f)\).

Proof of Theorem 3. Let \(f\) be an entire transcendental function with an invariant Baker domain \(U\) satisfying \(U \cap \text{sing}(f^{-1}) = \emptyset\). By a result of Baker [2, Theorem 1], \(U\) is simply-connected. Let \(h\) be a conformal mapping from \(U\) onto the upper half-plane \(H\). Then the function \(g(z) = h(f(h^{-1}(z)))\) is a linear transformation which leaves \(H\) invariant. By a suitable choice of \(h\) we can achieve that \(g(z) = z + 1\) or \(g(z) = az\) for some \(a > 1\).

In the first case we define
\[
K' = \{z = x + iy : 0 \leq x \leq 1, y \in \{e, e^2, \ldots, e^N\}\}
\]
and in the second case we define
\[
K' = \{z = re^{i\theta} : 1 \leq r \leq a, \theta \in \{\alpha_1, \alpha_2, \ldots, \alpha_N\}\}
\]
where $\alpha_N < \alpha_{N-1} < \cdots < \alpha_1 = \pi/2$ and $\int_{\alpha_{j+1}}^{\alpha_j} dt/\sin t = 1$ for $j = 1, 2, \ldots, N$. Here $N \in \mathbb{N}$ and $N \geq 2$. Then $\bigcup_{n=0}^{\infty} g^n(K')$ consists of $n$ straight lines tending to $\infty$. Because $\rho_H(z) = 1/|\text{Im} z|$ a simple computation shows that the hyperbolic distance (in $H$) between two such lines is at least $1$.

We now define $K = h^{-1}(K')$ and conclude that

$$\bigcup_{n=0}^{\infty} f^n(K) = h^{-1}\left(\bigcup_{n=0}^{\infty} g^n(K')\right)$$

consists of $N$ curves $\Gamma_1, \ldots, \Gamma_N$ tending to $\infty$ in $U$ such that the hyperbolic distance (in $U$) between two such curves is at least $1$.

For sufficiently large $r$, each of these curves intersects the circle $\{z : |z| = r\}$, that is, for all $j \in \{1, \ldots, N\}$ there exists $\xi_j \in \Gamma_j$ satisfying $|\xi_j| = r$. We claim that there exists $j \in \{1, \ldots, N\}$ such that

$$\text{dist}(\xi_j, \partial U) \leq \frac{5\pi r}{N}. \tag{7}$$

Suppose that this is not the case, that is, $\text{dist}(\xi_j, \partial U) > 5\pi r/N$ for all $j$. From Lemma 2 we deduce that if $|z - \xi_j| < \pi r/N$, then

$$\rho_U(z, \xi_j) \leq \int_{\xi_j}^{\xi_j} \frac{2|dw|}{\text{dist}(w, \partial U)} \leq \frac{N}{2\pi r} \int_{\xi_j}^{\xi_j} |dw| = \frac{N}{2\pi r} |z - \xi_j| < \frac{1}{2},$$

where the integral is along the straight line from $\xi_j$ to $z$. Since $\rho_U(\xi_j, \xi_k) \geq 1$ for $j \neq k$ this implies that the discs $D(\xi_j, \pi r/N), j = 1, 2, \ldots, N$, are disjoint. This is a contradiction because $|\xi_j| = r$ for all $j$. Hence there exists $j \in \{1, 2, \ldots, N\}$ satisfying (7).

We now apply Lemma 3 for $\tau = 2$ and deduce that if $r$ is large enough, then there exists $\xi = \xi_j$ such that $|\xi| = r$, $\text{dist}(\xi, \partial U) \leq 5\pi r/N$, and

$$D(\xi, 2 \text{dist}(\xi, \partial U)) \cap P(f) \neq \emptyset.$$ 

It follows that

$$D\left(\xi, \frac{10\pi r}{N}\right) \cap P(f) \neq \emptyset$$

so that there exists $p \in D(\xi, 10\pi r/N) \cap P(f)$. Then

$$\left(1 - \frac{10\pi}{N}\right)r \leq |p| \leq \left(1 + \frac{10\pi}{N}\right)r$$

and

$$\text{dist}(p, \partial U) \leq \frac{10\pi r}{N}.$$ 

The conclusion follows since $N$ can be chosen arbitrarily large.

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