INVARIANT DOMAINS AND SINGULARITIES

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ABSTRACT. Let U be an invariant component of the Fatou set of an entire transcendental function f such that the iterates of f tend to ∞ in U. Let $P(f)$ be the closure of the set of the forward orbits of all critical and asymptotic values of f. We show that there exists a sequence $p_n \in P(f)$ such that $\operatorname{dist}(p_n, U) = o(|p_n|)$, where $\operatorname{dist}(\cdot,\cdot)$ denotes Euclidean distance. On the other hand, we give an example where $\operatorname{dist}(P(f), U) > 0$. In this example, U is bounded by a Jordan curve.

1. INTRODUCTION AND RESULTS

Let f be a transcendental entire (or a rational) function. The *Fatou set* $F(f)$ is the subset of the complex plane $\mathbb{C}$ (or the sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) where the iterates f^n of f form a normal family. The complement of $F(f)$ is called the *Julia set* and denoted by $J(f)$.

A connected component U of the Fatou set is called *invariant* if $f(U) \subset U$. It is well-known that if f is a transcendental entire function and if U is an invariant component of $F(f)$, then we have one of the following possibilities:

- There exists $z_0 \in U$ such that $f^n|_U \to z_0$ as $n \to \infty$. Then $f(z_0) = z_0$ and $|f'(z_0)| < 1$. Here U is called a *Böttcher domain* if $f'(z_0) = 0$ and a *Schröder domain* if $0 < |f'(z_0)| < 1$.
- There exists $z_0 \in \partial U$ such that $f^n|_U \to z_0$ as $n \to \infty$. Then $f(z_0) = z_0$ and $f'(z_0) = 1$, and U is called a *Leau domain.*
- $f^n|_U \to \infty$ as $n \to \infty$. In this case, U is called a *Baker domain.*
- There exists an analytic homeomorphism ϕ from U onto the unit disc such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\alpha} z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then U is called a *Siegel disc.*

There is a similar classification of invariant domains for rational functions. Here ∞ does not play a special role so that Baker domains are not an independent possibility but turn out to be special cases of Böttcher, Schröder, or Leau domains (if we allow $z_0 = \infty$ in their definition but replace $f'(z_0)$ by $\frac{d}{dz}(1/f(\frac{1}{z}))|_{z=0}$ if $z_0 = \infty$). There is, however, also an additional type of invariant domain for rational functions, the *Arnol'd–Herman ring.* This is, by definition, a doubly-connected component U of $F(f)$ such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\alpha} z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and some analytic homeomorphism ϕ from U onto an annulus $\{z : 1 < |z| < r\}$, where $1 < r < \infty$.

A proof of the classification of invariant components for rational functions can be found in [7, §7.1], [9, §IV.2], and [17, §3.2]. These books are also recommended as introductions into the iteration theory of rational functions in general. The proof of the above classification of invariant components for transcendental entire

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functions requires only minor modifications. (It is even easier than in the case of rational functions because a simple argument shows that the iterates tend to ∞ in multiply-connected components. In particular, if $\{f^n|_U\}$ has non-constant limit functions, then U is simply-connected, and this implies that U is a Siegel disc in this case.) For an introduction into the iteration theory of transcendental entire functions we refer to [3], [8], [12, §4], and [14, §III].

An important role in iteration theory is played by the singularities of the inverse function f^{-1} of f. Recall that the singularities of f^{-1} are the critical and asymptotic values of f. (For rational functions, we have to consider only the critical values.) We denote the set of all singularities of f^{-1} by $\operatorname{sing}(f^{-1})$ and define

$$P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\operatorname{sing}(f^{-1}))}.$$

Here and in the following $\overline{E}$ denotes the closure of a set $E \subset \mathbb{C}$.

The importance of the singularities of the inverse function in iteration theory is mainly due to the following results:

- If U is a Böttcher, Schröder, or Leau domain, then $U \cap \operatorname{sing}(f^{-1}) \neq \emptyset$.
- If U is a Siegel disc or Arnol'd–Herman ring, then $\partial U \subset P(f)$.

Proofs of these results for rational functions can be found in [7, §9.3], [9, §§III.2, V.1], and [17, §§3.4, 3.5, 3.8]. The arguments carry over to the transcendental case.

One may ask, whether there is also some relation between Baker domains and $\operatorname{sing}(f^{-1})$ or $P(f)$, compare [8, Question 4]. By a result of Eremenko and Lyubich [13, Theorem 1], there are no Baker domains if $\operatorname{sing}(f^{-1})$ is bounded. On the other hand, Herman [14, p. 609] and Eremenko and Lyubich [11, Example 3] gave examples of Baker domains U satisfying $U \cap \operatorname{sing}(f^{-1}) = \emptyset$. Herman's examples are related to analytic self-maps of the punctured plane having a Siegel disc or Arnol'd–Herman ring (see also [4, §5] and [6, §4]), and we have $\partial U \subset P(f)$ in these examples. In the example of Eremenko and Lyubich, which is obtained with techniques from approximation theory, it is not clear whether there are relations between ∂U and $P(f)$.

Theorem 1. *The function* $f(z) = 2 - \log 2 + 2z - e^z$ *has a Baker domain* U *such that* $\operatorname{dist}(P(f), U) > 0$.

Here and in the following $\operatorname{dist}(\cdot, \cdot)$ denotes the Euclidean distance between two sets (or points) in the plane.

Theorem 2. *The Baker domain* U *in Theorem 1 is bounded by a Jordan curve in* $\hat{\mathbb{C}}$.

Here we have added ∞ to ∂U. Another example of a Baker domain whose boundary is a Jordan curve in $\hat{\mathbb{C}}$ has been given by Baker and Weinreich [6, Theorem 3]. Their proof relied on deep results of Herman concerning circle diffeomorphisms. Our argument appears to be simpler.

We mention that Baker and Weinreich [6, Theorem 4] also proved that if ∂U is a Jordan curve in $\hat{\mathbb{C}}$ for some Baker domain U of an entire transcendental function f, then $f|_U$ is univalent. The boundary of unbounded Böttcher domains, Schröder domains, Leau domains, and Siegel discs U is much more complicated [6, Theorem 1]: here ∞ is in the impression of every prime end of U.

The following result complements Theorem 1 in some sense.

Theorem 3. *Let f be a transcendental entire function with an invariant Baker domain U. If $U \cap \operatorname{sing}(f^{-1}) = \emptyset$, then there exists a sequence (p_n) such that $p_n \in P(f)$, $|p_n| \to \infty$, $|p_{n+1}/p_n| \to 1$, and $\operatorname{dist}(p_n, U) = o(|p_n|)$ as $n \to \infty$.*

The example of Theorem 1 shows that $\operatorname{dist}(p_n, U) = o(|p_n|)$ and $|p_{n+1}/p_n| \to 1$ cannot be replaced by $\operatorname{dist}(p_n, U) = o(1)$ and $|p_{n+1} - p_n| = o(1)$.

There is an analogous result for *periodic* domains. These are, by definition, components U of $F(f)$ which satisfy $f^p(U) \subset U$ for some $p \in \mathbb{N}$, that is, components of the Fatou set that are invariant with respect to some iterate. To extend Theorem 3 to periodic domains we only have to note that $\operatorname{sing}((f^p)^{-1}) = \bigcup_{n=0}^{p-1} f^n(\operatorname{sing}(f^{-1}))$ and $P(f^p) = P(f)$, see [1, Lemma 2].

2. Proof of Theorem 1

First we determine $\operatorname{sing}(f^{-1})$ and $P(f)$. To this end, we note that it is not difficult to see that f does not have asymptotic values. Thus we have to consider only the critical values. The zeros of $f'(z) = 2 - e^z$ are given by

$$z_k = \log 2 + 2\pi i k, \;\; k \in \mathbb{Z}.$$

For $z \in \mathbb{C}$ and $k \in \mathbb{N}$ we have $f(z + 2k\pi i) = f(z) + 4k\pi i$ and induction shows that

$$f^n(z + 2k\pi i) = f^n(z) + 2^n k 2\pi i \tag{1}$$

for $n \in \mathbb{N}$. In particular, since $f(\log 2) = \log 2$, we have $f(z_k) = z_{2k}$. This implies that

$$P(f) = \operatorname{sing}(f^{-1}) = \{f(z_k) : k \in \mathbb{Z}\} = \{z_{2k} : k \in \mathbb{Z}\}.$$

Next we note that if $\operatorname{Re} z < -2$, then $\operatorname{Re} f(z) < 2 - \log 2 + 2 \operatorname{Re} z + e^{-2} < \operatorname{Re} z$. This implies that there exists an invariant Baker domain U containing the half-plane $\{z : \operatorname{Re} z < -2\}$.

Induction shows that

$$f^n(z) = (2^n - 1)(2 - \log 2) + 2^n z - \sum_{j=0}^{n-1} 2^{n-j-1} \exp\left(f^j(z)\right).$$

We define $h_n(z) = f^n(z)/2^n$ and deduce that $h_n(z)$ converges as $n \to \infty$ if $\operatorname{Re} z < -2$. Moreover,

$$\lim_{n\to\infty} h_n(z) = 2 - \log 2 + z - \sum_{j=0}^{\infty} 2^{-j-1} \exp\left(f^j(z)\right) \sim 2 - \log 2 + z \tag{2}$$

as $\operatorname{Re} z \to -\infty$.

By a result of Baker [2, Theorem 1], U is simply-connected. Using another result of Baker [5, Lemma 1] (see also [8, Lemma 7]) we deduce that if $K \subset U$ is compact, then there exists a constant C such that $|f^n(z)| \leq C|f^n(z')|$ for sufficiently large n and all $z, z' \in K$. It follows that $\{h_n\}$ is locally uniformly bounded in U and hence normal in U. Thus $h_n(z)$ converges not only for $\operatorname{Re} z < -2$ but in fact for all $z \in U$. By (2), the limit function is non-constant in U.

Recall now that $f(\log 2) = \log 2$ and $f'(\log 2) = 0$. This implies that there exists a component U_0 of $F(f)$ such that $f^n|_{U_0} \to \log 2$. (In the terminology introduced in the introduction, U_0 is a Böttcher domain.) For $k \in \mathbb{Z}$, we define $U_k = \{z + 2\pi k i : z \in U_0\}$ and deduce from (1) that

$$f^n(z) = \log 2 + 2^n k 2\pi i + o(1)$$

locally uniformly in U_k as $n \to \infty$. We conclude that $U_k \in F(f)$. Moreover, $h_n \to 2\pi ki$ locally uniformly in U_k, that is, $h_n|_{U_k}$ tends to a constant limit as $n \to \infty$. Thus $U_k \cap U = \emptyset$. It follows that $\operatorname{dist}(P(f), U) \geq \operatorname{dist}(\log 2, \partial U_0) > 0$.

Remark 1. Our example is similar to that of Herman [14, p. 609f], who considered the function $g(z) = e^{2\pi i\alpha} z e^z$ where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is chosen such that g has a Siegel disc at zero and showed that the function $f(z) = \log g(e^z) = 2\pi i\alpha + z + e^z$ has a Baker domain U satisfying $U \cap \operatorname{sing}(f^{-1}) = \emptyset$. The function f of Theorem 1 is also of the form $\log g(e^z)$ for some entire function g, see §3 below.

Remark 2. The proof shows that $\operatorname{dist}(P(f), J(f)) > 0$ for the function f of Theorem 1. Denote the class of all entire functions with this property by C. This class has been considered by Stallard [16], as one way to generalize the concept of *hyperbolic* (or *expanding*) rational functions (see [7, p. 225], [9, p. 89], or [17, p. 118]) to transcendental functions. She proves [16, Theorem A] that if $f \in C$ and $z \in J(f)$, then $|(f^n)'(z)| \to \infty$ as $n \to \infty$. We also mention that it follows from the main result of her paper [16, Theorem B] that the Julia set of the function of Theorem 1 has plane Lebesgue measure zero.

3. Proof of Theorem 2

We consider the function $g(z) = \frac{1}{2} z^2 e^{2-z}$. Clearly, 0 and 2 are fixed points of g which are contained in Böttcher domains. We observe that $g(e^z) = \exp f(z)$ so that $g^n(e^z) = \exp(f^n(z))$ for all $n \in \mathbb{N}$. We shall show that $e^z \in J(g)$ if and only if $z \in J(f)$.

Suppose first that $z \in J(f)$. Then any neighborhood N of z contains points a and b such that $\operatorname{Re} f^k(a) < -2$ and $f^l(b) \in U_0$ for certain $k, l \in \mathbb{N}$, with U_0 as in the proof of Theorem 1. Hence $\operatorname{Re} f^n(a) \to -\infty$ and $f^n(b) \to \log 2$ as $n \to \infty$. We deduce that $g^n(e^a) \to 0$ and $g^n(e^b) \to 2$ as $n \to \infty$. This implies that $\exp(N) \cap J(g) \neq \emptyset$. Because N can be chosen arbitrarily small we conclude that $e^z \in J(g)$.

Suppose now that $e^z \in J(g)$. Then any neighborhood N of z contains points a and b such that $g^k(e^a)$ and $g^l(e^b)$ are contained in the Böttcher domains of g around 0 and 2 for certain $k, l \in \mathbb{N}$. Thus $g^n(e^a) \to 0$ and $g^n(e^b) \to 2$ as $n \to \infty$. We deduce that if n is sufficiently large, then $f^n(a) \in U$ and $f^n(b) \in U_m$ for some m (depending on n). Now U and U_m are distinct components of $F(f)$. We conclude that $f^n(N)$ intersects $J(f)$. Thus N intersects $J(f)$ because $J(f)$ is completely invariant. Again, N can be chosen arbitrarily small and we deduce that $z \in J(f)$. Altogether we see that $e^z \in J(g)$ if and only if $z \in J(f)$. In particular, $J(f)$ and hence U are invariant under translation by $2\pi i$.

Denote by V the Böttcher domain of g that contains 0. Then $V = \exp U$ and $\partial V = \exp \partial U$. Thus it suffices to prove that ∂V is a Jordan curve.

In order to do this we note that if $|z| = 1$, then $|g(z)| \geq \frac{e}{2} > 1$. This implies that $V \subset D$, where D denotes the unit disc. Let W be the component of $g^{-1}(D)$ that contains 0. Then $\overline{V} \subset W$ and $\overline{W} \subset D$. Moreover, g is a proper mapping of degree 2 from W onto D. Thus, in the terminology of [9, §VI.1] or [10], the triple $(g; W, D)$ is a polynomial-like mapping. By the basic result about polynomial-like mappings (see [9, Theorem VI.1.1] or [10, Theorem 1]), there exists a quasiconformal mapping φ and a quadratic polynomial p such that $g(z) = \varphi(p(\varphi^{-1}(z)))$ for $z \in W$. Since g has a superattracting fixed point at 0 we deduce that p has a superattracting fixed point so that $L^{-1}(p(L(z))) = z^2$ for some linear function L. This implies that

$V = \varphi(L(D))$ and $\partial V = \varphi(L(\partial D))$. Thus ∂V is a Jordan curve (and in fact a quasicircle). This completes the proof of Theorem 2.

We describe a second, more elementary method to show ∂V is a curve, although this method fails to yield that ∂V is actually a Jordan curve. (Here more elementary is understood in the sense that quasiconformal mappings are not needed.)

To do this, we note that $P(g) = \operatorname{sing}(g^{-1}) = \{0, 2\}$. Because 0 and 2 are contained in Böttcher domains of g we have $\operatorname{dist}(P(g), J(g)) > 0$. This, together with the result of Stallard [16, Theorem A] quoted in Remark 2, enables us to use the arguments of [9, §§V.4, VI.5] and [17, §5.5], where it is shown that simply-connected components of the Fatou set of hyperbolic rational functions are bounded by curves. In fact, let $\phi(z) = 2e^{-2}z + O(z^2)$ be the solution of Böttcher's functional equation $\phi(z^2) = g(\phi(z))$ near 0. Then ϕ is a conformal mapping from the unit disc onto V. Let R be a fixed number satisfying $0 < R < 1$ and define

$$\gamma_n(e^{i\theta}) = \phi\left(R^{1/2^n} e^{i\theta}\right)$$

for $n \in \mathbb{N}$ and $0 \leq \theta \leq 2\pi$. As in [9, p. 94] and [17, p. 137] one can show that $\gamma_n(e^{i\theta})$ converges uniformly as $n \to \infty$, and the limit function maps the unit circle continuously onto ∂V.

4. Proof of Theorem 3

The following lemma summarizes Koebe's one quarter theorem and a part of his distortion theorem, see [15, §1.2]. By $D(a, r)$ we denote the disc of radius r around $a \in \mathbb{C}$.

Lemma 1. *Let f be analytic and univalent in $D(a, r)$. Then*

$$f(D(a,r)) \supset D\left(f(a), \frac{1}{4}|f'(a)|r\right). \tag{3}$$

Moreover, we have

$$\frac{r^2|(z-a)f'(a)|}{(r+|z-a|)^2} \leq |f(z) - f(a)| \leq \frac{r^2|(z-a)f'(a)|}{(r-|z-a|)^2} \tag{4}$$

and

$$\frac{r-|z-a|}{r+|z-a|} \leq \frac{|(z-a)f'(z)|}{|f(z)-f(a)|} \leq \frac{r+|z-a|}{r-|z-a|} \tag{5}$$

for $z \in D(a, r)$.

By $\rho_U(z)$ we denote the density of the hyperbolic metric of a hyperbolic domain U and by $\rho_U(z, z')$ the hyperbolic distance of $z, z' \in U$. The following estimate follows from Schwarz's lemma and (3), that is, Koebe's one quarter theorem. It can be found for example in [9, p. 13].

Lemma 2. *If $U \subset \mathbb{C}$ is a simply-connected hyperbolic domain and $z \in U$, then*

$$\frac{1}{2\operatorname{dist}(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{\operatorname{dist}(z, \partial U)}$$

Lemma 3. *Let f be a transcendental entire function with an invariant Baker domain U. Suppose that $K \subset U$ is compact and that $\tau > 1$. Then there exists n_0 such that*

$$D(f^n(z), \tau \operatorname{dist}(f^n(z), \partial U)) \cap P(f) \neq \emptyset$$

for all $z \in K$ and $n \geq n_0$.

Proof. Suppose that the conclusion does not hold. Then there exist sequences (z_j) and (n_j) such that $z_j \in K$, $n_j \to \infty$, and

$$D(f^{n_j}(z_j), \tau \operatorname{dist}(f^{n_j}(z_j), \partial U)) \cap P(f) = \emptyset.$$

Restricting to a subsequence if necessary we may assume that $z_j \to z_0 \in K$ as $j \to \infty$. This implies that $\rho_U(z_j, z_0) \to 0$ and hence that $\rho(f^{n_j}(z_j), f^{n_j}(z_0)) \to 0$ as $j \to \infty$. From Lemma 2 we can deduce that $\operatorname{dist}(f^{n_j}(z_j), \partial U) \sim \operatorname{dist}(f^{n_j}(z_0), \partial U)$ and $|f^{n_j}(z_j) - f^{n_j}(z_0)| / \operatorname{dist}(f^{n_j}(z_j), \partial U) \to 0$ as $j \to \infty$. It follows that if $1 < \sigma < \tau$, then

$$D(f^{n_j}(z_0), \sigma \operatorname{dist}(f^{n_j}(z_0), \partial U)) \cap P(f) = \emptyset \tag{6}$$

for all large j. Of course, we may assume that (6) holds for all j. We introduce the abbreviations $\zeta_j = f^{n_j}(z_0)$ and $\delta_j = \operatorname{dist}(\zeta_j, \partial U)$ so that (6) takes the form

$$D(\zeta_j, \sigma\delta_j) \cap P(f) = \emptyset.$$

Hence there exists branches φ_j of $(f^{n_j})^{-1}$ which are defined (and univalent) in $D(\zeta_j, \sigma\delta_j)$ and satisfy $\varphi_j(\zeta_j) = z_0$. We consider $B_j = \{z : |z - \zeta_j| = \delta_j\}$ and $C_j = \varphi_j(B_j)$. Then there exists $b_j \in B_j \cap \partial U$ and $c_j = \varphi_j(b_j) \in C_j \cap \partial U$.

By (4) we have

$$|c_j - z_0| \leq \max_{z \in B_j} |\varphi_j(z) - z_0| \leq \left(\frac{\sigma+1}{\sigma-1}\right)^2 \min_{z \in B_j} |\varphi_j(z) - z_0| \leq \left(\frac{\sigma+1}{\sigma-1}\right)^2 \operatorname{dist}(z_0, \partial U)$$

so that $|c_j|$ is bounded. We may thus assume that $c_j \to c \in \partial U$.

From (5) we deduce that

$$|\varphi_j'(b_j)| \geq \frac{\sigma-1}{\sigma+1} \frac{|\varphi_j(b_j) - z_0|}{|b_j - \zeta_j|} = \frac{\sigma-1}{\sigma+1} \frac{|c_j - z_0|}{\delta_j} \geq \frac{\sigma-1}{\sigma+1} \frac{\operatorname{dist}(z_0, \partial U)}{\delta_j}.$$

Combining this with (3) we find that

$$\begin{aligned}
\varphi_j\left(D\left(\zeta_j, \sigma\delta_j\right)\right) &\supset \varphi_j\left(D\left(b_j, (\sigma-1)\delta_j\right)\right) \\
&\supset D\left(c_j, \frac{1}{4}|\varphi_j'(b_j)|(\sigma-1)\delta_j\right) \\
&\supset D\left(c_j, \frac{1}{4}\frac{(\sigma-1)^2}{\sigma+1} \operatorname{dist}(z_0, \partial U)\right) \\
&\supset D(c, \delta)
\end{aligned}$$

for some $\delta > 0$ and all large j. This implies that f^{n_j} is univalent in $D(c, \delta)$. It is not difficult to see that this is impossible for $c \in \partial U \subset J(f)$.

Proof of Theorem 3. Let f be an entire transcendental function with an invariant Baker domain U satisfying $U \cap \operatorname{sing}(f^{-1}) = \emptyset$. By a result of Baker [2, Theorem 1], U is simply-connected. Let h be a conformal mapping from U onto the upper half-plane H. Then the function $g(z) = h(f(h^{-1}(z)))$ is a linear transformation which leaves H invariant. By a suitable choice of h we can achieve that $g(z) = z + 1$ or $g(z) = az$ for some $a > 1$.

In the first case we define

$$K' = \{z = x + iy : 0 \leq x \leq 1, y \in \{e, e^2, \ldots, e^N\}\}$$

and in the second case we define

$$K' = \{z = re^{i\theta} : 1 \leq r \leq a, \theta \in \{\alpha_1, \alpha_2, \ldots, \alpha_N\}\}$$

where $\alpha_N < \alpha_{N-1} < \cdots < \alpha_1 = \pi/2$ and $\int_{\alpha_{j+1}}^{\alpha_j} dt/\sin t = 1$ for $j = 1, 2, \ldots, N$. Here $N \in \mathbb{N}$ and $N \geq 2$. Then $\bigcup_{n=0}^{\infty} g^n(K')$ consists of n straight lines tending to ∞. Because $\rho_H(z) = 1/\operatorname{Im} z$ a simple computation shows that the hyperbolic distance (in H) between two such lines is at least 1.

We now define $K = h^{-1}(K')$ and conclude that

$$\bigcup_{n=0}^{\infty} f^n(K) = h^{-1}\left(\bigcup_{n=0}^{\infty} g^n(K')\right)$$

consists of N curves $\Gamma_1, \ldots, \Gamma_N$ tending to ∞ in U such that the hyperbolic distance (in U) between two such curves is at least 1.

For sufficiently large r, each of these curves intersects the circle $\{z : |z| = r\}$, that is, for all $j \in \{1, \ldots, N\}$ there exists $\xi_j \in \Gamma_j$ satisfying $|\xi_j| = r$. We claim that there exists $j \in \{1, \ldots, N\}$ such that

$$\operatorname{dist}(\xi_j, \partial U) \leq \frac{5\pi r}{N}. \tag{7}$$

Suppose that this is not the case, that is, $\operatorname{dist}(\xi_j, \partial U) > 5\pi r/N$ for all j. From Lemma 2 we deduce that if $|z - \xi_j| < \pi r/N$, then

$$\rho_U(z, \xi_j) \leq \int_{\xi_j}^{z} \rho_U(w)|dw| \leq \int_{\xi_j}^{z} \frac{2|dw|}{\operatorname{dist}(w, \partial U)} \leq \frac{N}{2\pi r}\int_{\xi_j}^{z} |dw| = \frac{N}{2\pi r}|z - \xi_j| < \frac{1}{2},$$

where the integral is along the straight line from ξ_j to z. Since $\rho_U(\xi_j, \xi_k) \geq 1$ for $j \neq k$ this implies that the discs $D(\xi_j, \pi r/N)$, $j = 1, 2, \ldots, N$, are disjoint. This is a contradiction because $|\xi_j| = r$ for all j. Hence there exists $j \in \{1, 2, \ldots, N\}$ satisfying (7).

We now apply Lemma 3 for $\tau = 2$ and deduce that if r is large enough, then there exists $\xi = \xi_j$ such that $|\xi| = r$, $\operatorname{dist}(\xi, \partial U) \leq 5\pi r/N$, and

$$D\left(\xi, 2\operatorname{dist}(\xi, \partial U)\right) \cap P(f) \neq \emptyset.$$

It follows that

$$D\left(\xi, \frac{10\pi r}{N}\right) \cap P(f) \neq \emptyset$$

so that there exists $p \in D\left(\xi, 10\pi r/N\right) \cap P(f)$. Then

$$\left(1 - \frac{10\pi}{N}\right) r \leq |p| \leq \left(1 + \frac{10\pi}{N}\right) r$$

and

$$\operatorname{dist}(p, \partial U) \leq \frac{10\pi r}{N}.$$

The conclusion follows since N can be chosen arbitrarily large.

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