On the growth of meromorphic functions of infinite order *

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Abstract

Let f be a meromorphic function of infinite order, $T(r,f)$ its Nevanlinna (or Ahlfors-Shimizu) characteristic, and $M(r,f)$ its maximum modulus. It is proved that

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{rT'(r,f)}\leq\pi$$

and

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)\psi(\log T(r,f))}=0$$

if $\psi(x)/x$ is non-decreasing, $\psi'(x)\leq\sqrt{\psi(x)}$, and $\int^{\infty}dx/\psi(x)<\infty$.

1 Introduction and results

Let f be a meromorphic function. We shall use the standard notation of Nevanlinna theory [6, 7, 9]. In particular, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $M(r,f)$ the maximum modulus of f.

In 1969, Govorov [5] proved an old conjecture of Paley which says that if f is entire and the order ρ of f satisfies $\frac{1}{2}\leq\rho<\infty$, then

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}\leq\pi\rho. \tag{1}$$

Soon afterwards, Petrenko [10] proved that (1) remains valid for meromorphic functions, even if the order is replaced by the lower order.

The relative growth of $T(r,f)$ and $\log M(r,f)$ for entire functions of infinite order has been considered by Chuang [2], Marchenko and Shcherba [8], and Dai, Drasin, and Li [3]. It is shown in these papers that if $\psi(x)$ is increasing and positive for $x\geq x_0>0$ and if

$$\int_{x_0}^{\infty}\frac{dx}{\psi(x)}<\infty, \tag{2}$$

then

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)\psi(\log T(r,f))}=0. \tag{3}$$

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In [3], it is even proved that

$$\log M(r,f) = o(T(r,f)\psi(\log T(r,f))) \tag{4}$$

as $r \to \infty$ through a set of logarithmic density one. In [8] and [3], is also shown that the results are best possible in some sense.

The case that f is meromorphic has also been considered in [3] where it was shown that

$$\log M(r,f) = o(T(r,f)\psi(\log T(r,f)) \log \psi(\log T(r,f))) \tag{5}$$

as $r \to \infty$ through a set of logarithmic density one.

A different approach has been taken in [1] where $\log M(r,f)$ has been compared with the derivative of $T(r,f)$. More generally, $\log M(r,f)$ has been compared with $\gamma'(r)$ for an increasing and differentiable function $\gamma(r)$ satisfying $T(r,f) \leq \gamma(r)$ for all large r. It was shown in [1] that under these hypotheses

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{r\gamma'(r)} \leq \pi, \tag{6}$$

if f is an entire function of infinite order. Here the constant π is best possible.

Our first result is that this is true for meromorphic functions as well.

Theorem 1 *Let f be a meromorphic function of infinite order and let γ be an increasing and differentiable function such that $T(r,f) \leq \gamma(r)$ for all large r. Then* (6) *holds.*

In particular, we have

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{rT'(r,f)} \leq \pi.$$

This also holds with the Nevanlinna characteristic replaced by the Ahlfors-Shimizu characteristic.

Using similar methods as in the proof of Theorem 1 we obtain the following result.

Theorem 2 *Let f be a meromorphic function of infinite order and let $\psi(x)$ be positive and continuously differentiable for $x \geq x_0 > 0$ such that $\psi(x)/x$ is non-decreasing, $\psi'(x) \leq \sqrt{\psi(x)}$, and* (2) *is satisfied. Then* (3) *holds.*

We conjecture that, under the hypotheses of Theorem 2, (4) holds on a set of logarithmic density one so that the extra factor $\log\psi(\log T(r,f))$ occuring in (5) is not necessary.

We do not know whether the hypotheses made about ψ besides (2) are necessary. On the other hand, we note that these hypotheses are similar to those made in [8] and [3] in order to show that (2) is best possible.

Our proofs are based on the method of Petrenko as developed by Fuchs [4] and a lemma for real functions.

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2 A growth lemma for real functions

An important part in our proofs is played by the following lemma.

Lemma 1 *Let $\Phi(x)$ be increasing and differentiable for $x \geq x_0 > 0$ and assume that*

$$\limsup_{x\to\infty} \frac{\Phi(x)}{x} = \infty.$$

Then there exist sequences (x_j), (M_j), and (ε_j) satisfying $x_j \to \infty$, $M_j \to \infty$, $\varepsilon_j \to 0$, and $\Phi'(x_j) \to \infty$ as $j \to \infty$ such that

$$\Phi(x_j + h) \leq \Phi(x_j) + \Phi'(x_j)h + \varepsilon_j$$

for $|h| \leq \frac{M_j}{\Phi'(x_j)}$.

If, in addition, ψ is given as in Theorem 2, then (x_j) can be chosen such that

$$\Phi'(x_j) = o(\psi(\Phi(x_j))$$

as $j \to \infty$.

Without the claim about ψ, this lemma was proved in [1, Lemma 1]. The following proof uses a similar method. We remark that this additional claim about ψ is only needed for the proof of Theorem 2 while [1, Lemma 1] suffices for the proof of Theorem 1.

Proof of Lemma 1. We define $p(t)$ by the differential equation $p'(t) = \psi(p(t))$ with initial condition $p(0) = x_0$. Then $p(t)$ is increasing and standard lemmas of Borel type (compare [9, p. 253]) show that there exists $\beta > 0$ such that $\lim_{t\to\beta} p(t) = \infty$.

As in [1] we find for any given $c > 0$ arbitrarily large u such that $\Phi(5u) \geq 2\Phi(2u) \geq 2cu$. We choose $u > \max\{2x_0, \beta x_0/2\Phi(x_0)\}$ with this property and define

$$F_a(x) = \frac{\Phi(2u)}{x_0} p\left(\frac{ax}{u}\right)$$

for $0 < a \leq \frac{\beta}{2}$ and $x_0 \leq x < \frac{u\beta}{a}$. Then

$$F_{\beta/2}(x) = \frac{\Phi(2u)}{x_0} p\left(\frac{\beta x}{2u}\right) \geq \frac{\Phi(2u)}{x_0} p\,(0) = \Phi(2u) \geq \Phi(x)$$

for $x_0 \leq x \leq 2u$. Hence the set

$$E = \left\{a : 0 < a \leq \frac{\beta}{2}, F_a(x) \geq \Phi(x) \text{ for } x_0 \leq x < \frac{u\beta}{a}\right\}$$

is not empty. We define $b = \inf E$. To find a lower bound for b we note that there exists α satisfying $0 < \alpha < \beta$ such that $p(\alpha) = 2x_0$. We deduce that if $0 < a < \frac{\alpha}{5}$, then

$$F_a(5u) = \frac{\Phi(2u)}{x_0} p(5a) \leq 2\Phi(2u) \leq \Phi(5u).$$

Hence $\frac{\alpha}{5} \leq b \leq \frac{\beta}{2}$.

As in [1] we define $F = F_b$ and deduce that there exists $v \in (2u, \frac{u\beta}{b})$ such that $F(v) = \Phi(v)$, $F'(v) = \Phi'(v)$, and $F(x) \geq \Phi(x)$ for $x \in (2u, \frac{u\beta}{b})$. We note that

$$F'(v) = \frac{\Phi(2u)}{x_0} p'\left(\frac{bv}{u}\right)\frac{b}{u} \geq \frac{\Phi(2u)b}{x_0 u} p'(0) = \frac{\Phi(2u)b\psi(x_0)}{x_0 u} \geq \frac{cb\psi(x_0)}{x_0} \geq \frac{c\alpha\psi(x_0)}{5x_0}$$

so that $F'(v)$ can be made arbitrarily large by choosing c large.

Using $\psi'(x) \leq \sqrt{\psi(x)}$ and $u > \beta x_0/2\Phi(x_0) \geq bx_0/\Phi(x_0)$ one can show that $F''(x) \leq F'(x)^{3/2}$. Let M be a positive constant. We deduce that if $F'(v) > M^2$ and $0 \leq h \leq \frac{M}{F'(v)}$, then

$$1 - \sqrt{\frac{F'(v)}{F'(v+h)}} = \frac{\sqrt{F'(v)}}{2} \int_v^{v+h} \frac{F''(x)}{F'(x)^{3/2}} dx \leq \frac{\sqrt{F'(v)}}{2} \int_v^{v+h} dx \leq \frac{M}{2\sqrt{F'(v)}}$$

so that $F'(v+h) \leq (1+\frac{\varepsilon}{M})F'(v)$ for any given $\varepsilon > 0$, provided c is large enough. We deduce that

$$\begin{aligned} \Phi(v+h) &\leq F(v+h) \\ &= F(v) + \int_v^{v+h} F'(x)dx \\ &\leq \Phi(v) + F'(v+h)h \\ &\leq \Phi(v) + \left(1 + \frac{\varepsilon}{M}\right) F'(v)h \\ &= \Phi(v) + \Phi'(v)h + \varepsilon\frac{F'(v)h}{M} \\ &\leq \Phi(v) + \Phi'(v)h + \varepsilon \end{aligned}$$

for $0 \leq h \leq \frac{M}{F'(v)}$. The case $-\frac{M}{F'(v)} \leq h < 0$ is similar so that

$$\Phi(v+h) \leq \Phi(v) + \Phi'(v)h + \varepsilon$$

holds for $|h| \leq \frac{M}{F'(v)} = \frac{M}{\Phi'(v)}$. Since ε and M were arbitrary, the conclusion follows.

3 Proofs of Theorems 1 and 2

Proof of Theorem 1. We define $\Phi(x) = \log \gamma(e^x)$. Since f has infinite order, the hypotheses of Lemma 1 are satisfied. Choose $(x_j), (M_j)$ and (ε_j) according to Lemma 1 and define $\rho_j = e^{x_j}$ and $\mu_j = \Phi'(x_j)$. Then

$$\gamma(r) \leq (1+\varepsilon_j)\gamma(\rho_j)\left(\frac{r}{\rho_j}\right)^{\mu_j} \tag{7}$$

for $\left|\log \frac{r}{\rho_j}\right| \leq \frac{M_j}{\mu_j}$. Lemma 1 says that $M_j \to \infty$. Replacing, if necessary, (M_j) by a sequence of smaller numbers, we may achieve that $M_j \to \infty$ as slowly as we please. Also, $\mu_j \to \infty$ so that we may assume that $\mu_j \geq \frac{1}{2}$ for all j. We define (p_j) and (P_j) by

$$\log \frac{\rho_j}{p_j} = \log \frac{P_j}{\rho_j} = \frac{M_j}{\mu_j}$$

so that (7) holds for $p_j \le r \le P_j$. We consider the set

$$A_j = \left\{ r; \rho_j \le r \le P_j, \gamma(r) \le \frac{1}{\sqrt{M_j}} \gamma(\rho_j) \left(\frac{r}{\rho_j} \right)^{\mu_j} \right\}$$

and define $R_j = P_j$ if $A_j = \emptyset$ and $R_j = \min A_j$ otherwise. Similarly, we consider

$$B_j = \left\{ r; p_j \le r \le \rho_j, \gamma(r) \le \frac{1}{\sqrt{M_j}} \gamma(\rho_j) \left(\frac{r}{\rho_j} \right)^{\mu_j} \right\}$$

and define $r_j = p_j$ if $B_j = \emptyset$ and $r_j = \max B_j$ otherwise. We also define $S_j = e^{-\frac{1}{\mu_j}} R_j, T_j = e^{-\frac{2}{\mu_j}} R_j, t_j = r_j$, and $s_j = e^{-\frac{1}{\mu_j}} t_j$. Then $s_j < t_j < \rho_j < T_j < S_j < R_j$.

Following Fuchs [4, equation (5.7)] we obtain from Petrenko's formula:

$$\begin{aligned} & \int_{t_j}^{T_j} u^{-\mu_j - 1} \log M(u, f) du \\ < \quad & \pi \mu_j \int_{s_j}^{S_j} r^{-\mu_j - 1} m(r, f) dr \\ + \quad & \frac{\pi}{\mu_j} \sum_{s_j \le |b| \le S_j} |b|^{-\mu_j} \\ + \quad & A \mu_j \left(s_j^{2\mu_j} \int_{s_j}^{\infty} u^{-3\mu_j - 1} du \, T(t_j, f) + S_j^{-2\mu_j} \int_0^{S_j} u^{\mu_j - 1} du \, T(S_j, f) \right) \end{aligned} \tag{8}$$

where the sum is taken over all poles of f in the annulus $s_j \le |z| \le S_j$ and where A is an absolute constant.

Here we have taken $\mu = \mu_j$ and $\gamma = 2\mu_j$ which is permissible since $\mu_j \ge \frac{1}{2}$. Fuchs proves (8) for the case that $t_j = 2s_j$ and $S_j = 2T_j$, but the general case $s_j < t_j < T_j < S_j$ can be proved by the same method. Fuchs also requires $\gamma > 2\mu$, but the result remains valid if $\gamma = 2\mu$.

Following Fuchs we have

$$\begin{aligned} & \sum_{s_j \le |b| \le S_j} |b|^{-\mu_j} \\ \le \quad & S_j^{-\mu_j} n(S_j, f) + \mu_j \int_{s_j}^{S_j} t^{-\mu_j - 1} n(t, f) dt \\ \le \quad & S_j^{-\mu_j} n(S_j, f) + \mu_j S_j^{-\mu_j} N(S_j, f) + \mu_j^2 \int_{s_j}^{S_j} t^{-\mu_j - 1} N(t, f) dt. \end{aligned}$$

Since

$$\begin{aligned} N(R_j, f) \quad & \ge \quad \int_{S_j}^{R_j} \frac{n(t, f)}{t} dt \ge n(S_j, f) \int_{S_j}^{R_j} \frac{dt}{t} \\ & = \quad n(S_j, f) \log \frac{R_j}{S_j} = \frac{1}{\mu_j} n(S_j, f) \end{aligned}$$

and $S_j^{-\mu_j} = eR_j^{-\mu_j}$ we obtain

$$\sum_{s_j \le |b| \le S_j} |b|^{-\mu_j} \le 2e\mu_j R_j^{-\mu_j} N(R_j, f) + \mu_j^2 \int_{s_j}^{S_j} t^{-\mu_j - 1} N(t, f) dt.$$

Substituting this in (8) and computing the last two integrals in (8) we deduce that

$$\int_{t_j}^{T_j} u^{-\mu_j - 1} \log M(u, f) du < \pi \mu_j \int_{s_j}^{S_j} r^{-\mu_j - 1} T(r, f) dr + B \left(t_j^{-\mu_j} T(t_j, f) + R_j^{-\mu_j} T(R_j, f) \right)$$

for some absolute constant B. We wish to replace the integral on the right side by an integral from t_j to T_j. Therefore we note that

$$\mu_j \int_{s_j}^{t_j} r^{-\mu_j - 1} T(r, f) dr \le \mu_j T(t_j, f) \int_{s_j}^{t_j} r^{-\mu_j - 1} dr \le T(t_j, f) s_j^{-\mu_j} = e t_j^{-\mu_j} T(t_j, f)$$

Similarly,

$$\mu_j \int_{T_j}^{S_j} r^{-\mu_j - 1} T(r, f) dr \le 2eR_j^{-\mu_j} T(R_j, f).$$

Hence

$$\begin{aligned} & \int_{t_j}^{T_j} r^{-\mu_j - 1} \log M(r, f) dr \\ \le \quad & \pi \mu_j \int_{t_j}^{T_j} r^{-\mu_j - 1} T(r, f) dr + C \left(t_j^{-\mu_j} T(t_j, f) + R_j^{-\mu_j} T(R_j, f) \right) \end{aligned}$$

where C is an absolute constant. Of course, this implies that

$$\begin{aligned} & \int_{t_j}^{T_j} r^{-\mu_j - 1} \log M(r, f) dr \\ \le \quad & \pi \mu_j \int_{t_j}^{T_j} r^{-\mu_j - 1} \gamma(r) dr + C \left(t_j^{-\mu_j} \gamma(t_j) + R_j^{-\mu_j} \gamma(R_j) \right). \end{aligned} \tag{9}$$

We want to show that the second term on the right hand side of (9) is small compared with the first one. To this end, we define

$$I_j = \mu_j \int_{t_j}^{T_j} r^{-\mu_j - 1} \gamma(r) dr$$

and we note that

$$
\begin{aligned}
I_j &\geq \mu_j \int_{\rho_j}^{T_j} r^{-\mu_j - 1}\gamma(r)dr \\
&\geq \mu_j \gamma(\rho_j)\left(\rho_j^{-\mu_j} - T_j^{-\mu_j}\right).
\end{aligned} \tag{10}
$$

If $A_j \neq \emptyset$, then

$$
\frac{1}{\sqrt{M_j}}\gamma(\rho_j)\left(\frac{R_j}{\rho_j}\right)^{\mu_j} = \gamma(R_j) \geq \gamma(\rho_j)
$$

so that

$$
\left(\frac{T_j}{\rho_j}\right)^{\mu_j} = e^{-2}\left(\frac{R_j}{\rho_j}\right)^{\mu_j} \geq e^{-2}\sqrt{M_j} \to \infty.
$$

Hence (10) implies that

$$
I_j \geq (1 - o(1))\gamma(\rho_j)\rho_j^{-\mu_j}. \tag{11}
$$

But if $A_j = \emptyset$, then

$$
\left(\frac{T_j}{\rho_j}\right)^{\mu_j} = e^{-2}\left(\frac{R_j}{\rho_j}\right)^{\mu_j} = e^{M_j - 2}
$$

and (11) follows again from (10). We now show that

$$
\gamma(R_j)R_j^{-\mu_j} = o(I_j). \tag{12}
$$

This follows immediately from the definition of R_j if $A_j \neq \emptyset$. But if $A_j = \emptyset$, then

$$
\begin{aligned}
I_j &\geq \mu_j \int_{\rho_j}^{T_j} \frac{1}{\sqrt{M_j}}\gamma(\rho_j)\left(\frac{r}{\rho_j}\right)^{\mu_j} r^{-\mu_j - 1}dr \\
&= \mu_j \frac{1}{\sqrt{M_j}}\gamma(\rho_j)\rho_j^{-\mu_j}\log\frac{T_j}{\rho_j} \\
&= \frac{1}{\sqrt{M_j}}\gamma(\rho_j)\rho_j^{-\mu_j}(M_j - 2) \\
&\geq \frac{1}{1 + \varepsilon_j}\frac{M_j - 2}{\sqrt{M_j}}\gamma(R_j)R_j^{-\mu_j}
\end{aligned} \tag{13}
$$

and (12) follows.

Next we show that

$$
\gamma(t_j)t_j^{-\mu_j} = o(I_j). \tag{14}
$$

If $B_j \neq \emptyset$, this follows immediately from (11). But if $B_j = \emptyset$, then we obtain similarly as in (13)

$$\begin{aligned} I_j &\geq \mu_j \int_{t_j}^{\rho_j} \frac{1}{\sqrt{M_j}} \gamma(\rho_j) \left(\frac{r}{\rho_j}\right)^{\mu_j} r^{-\mu_j - 1} dr \\ &= \mu_j \frac{1}{\sqrt{M_j}} \gamma(\rho_j) \rho_j^{-\mu_j} \log \frac{\rho_j}{t_j} \\ &= \sqrt{M_j} \gamma(\rho_j) \rho_j^{-\mu_j} \\ &\geq \frac{\sqrt{M_j}}{1+\varepsilon_j} \gamma(t_j) t_j^{-\mu_j} \end{aligned}$$

and (14) follows.

Combining (9), (12), and (14) we obtain

$$\int_{t_j}^{T_j} r^{-\mu_j - 1} \log M(r,f) dr \leq (1+o(1)) \pi I_j. \tag{15}$$

Integration by parts shows that

$$I_j = \gamma(t_j) t_j^{-\mu_j} - \gamma(T_j) T_j^{-\mu_j} + \int_{t_j}^{T_j} r^{-\mu_j - 1} r \gamma'(r) dr. \tag{16}$$

(Note that γ is absolutely continuous because it is increasing and differentiable.) Combining (14), (15), and (16) we obtain

$$\int_{t_j}^{T_j} r^{-\mu_j - 1} \log M(r,f) dr \leq (1+o(1)) \pi \int_{t_j}^{T_j} r^{-\mu_j - 1} r \gamma'(r) dr$$

It follows that there exist $\xi_j \in [t_j, T_j]$ such that

$$\log M(\xi_j, f) \leq (1+o(1)) \pi \xi_j \gamma'(\xi_j) \tag{17}$$

We may assume that M_j tends to ∞ so slowly that $p_j \to \infty$. Because $\xi_j \geq t_j \geq p_j$ this implies that $\xi_j \to \infty$. Hence Theorem 1 follows from (17).

Proof of Theorem 2. We proceed as in the proof of Theorem 1 to obtain (15), choosing $\gamma(r) = T(r,f)$ in the definition of I_j. It follows that there exists $\zeta_j \in [t_j, T_j]$ such that

$$\log M(\zeta_j, f) \leq (1+o(1)) \pi \mu_j T(\zeta_j, f)$$

By Lemma 1 we have

$$\mu_j = \Phi'(x_j) = o(\psi(\Phi(x_j))) = o(\psi(\log T(\rho_j, f)))$$

Hence

$$\log M(\zeta_j, f) = o(\psi(\log T(\rho_j, f)) T(\zeta_j, f)) \tag{18}$$

We shall prove that

$$\psi(\log T(\rho_j, f)) \le 2\psi(\log T(t_j, f)) \tag{19}$$

for sufficiently large j. Then Theorem 2 follows immediately from (18) and (19) because $t_j \le \zeta_j$.

It remains to prove (19). We have

$$\begin{aligned} &\psi(\log T(\rho_j, f)) - \psi(\log T(t_j, f)) \\ = &\int_{\log T(t_j,f)}^{\log T(\rho_j,f)} \psi'(x)dx \\ \le &\int_{\log T(t_j,f)}^{\log T(\rho_j,f)} \sqrt{\psi(x)}dx \\ \le &\sqrt{\psi(\log T(\rho_j, f))} \log \frac{T(\rho_j, f)}{T(t_j, f)} \end{aligned}$$

and

$$\frac{T(\rho_j, f)}{T(t_j, f)} \le \sqrt{M_j} \left(\frac{\rho_j}{t_j}\right)^{\mu_j} \le \sqrt{M_j} \left(\frac{\rho_j}{p_j}\right)^{\mu_j} = \sqrt{M_j} e^{M_j}$$

Hence

$$\psi(\log T(\rho_j, f)) - \psi(\log T(t_j, f)) \le \sqrt{\psi(\log T(\rho_j, f))} \log(\sqrt{M_j} e^{M_j}) \tag{20}$$

By choosing M_j slowly increasing, we can achieve that

$$\log(\sqrt{M_j} e^{M_j}) \le \frac{1}{2}\sqrt{\psi(\log T(\rho_j, f))}. \tag{21}$$

Combining (20) and (21) we deduce (19). This completes the proof of Theorem 2.

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