Proof of a conjecture of Baker concerning the distribution of fixpoints

Walter Bergweiler, Jim Clunie and Jim Langley

1 Introduction and main result

The main result of this paper is the following theorem.

Theorem 1 *Let $f(z)$ be an entire transcendental function and l a straight line in the complex plane. Then $f(f(z))$ has infinitely many fixpoints that do not lie on l.*

This result was conjectured by Baker (see [3, p. 494] and [12, Problem 2.23]). Baker [3, Theorem 1] proved this for the case where the order $\rho(f)$ of f satisfies $\rho(f) < \frac{1}{2}$. Our proof works for the case $\rho(f) > 0$ so that the combination of both results yields the desired conclusion.

We recall some results in iteration theory that led Baker to his conjecture. The Fatou set of an entire (or rational) function f is the subset of the plane (or sphere) where the iterates of f form a normal family, and the Julia set is its complement. While Julia sets are usually quite complicated, there are examples of rational functions whose Julia set is an interval. For example, the Julia set of the Chebychev polynomials consists of the interval $[-1, 1]$. Baker [2] proved that the Julia set of a transcendental entire function cannot be contained in a straight line l. Since, by a classical theorem of Fatou [8, p. 354], every point in the Julia set is a limit point of fixpoints of the iterates of f, this implies that, for any given line l, some iterate of f has fixpoints that are not contained in l. Baker's conjecture was that this is already the case for the second iterate. (Clearly, the fixpoints of f itself may all lie on l.)

One may generalize Theorem 1 as follows.

Theorem 2 *Let $f(z)$ be an entire transcendental function, l a straight line in the complex plane and $n \geq 2$. Then the n-th iterate of $f(z)$ has infinitely many fixpoints that do not lie on l.*

This was proved by Baker [3, Theorem 4] if the order of the n-th iterate is finite. Our method works for the case where it is greater than two.

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2 Preliminaries

The proof of the theorem requires the following lemma concerning the Fourier coefficients $c_m = c_m(r, h)$ of $\log|h(re^{i\theta})|$, where h is an analytic function. By definition,

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\log|h(re^{i\theta})|d\theta$$

for each integer m.

Lemma A *Let h be analytic in $|z| < r$ and continuous in $|z| \le r$ with finitely many zeros $z_1, z_2, \ldots, z_n$ all lying in $0 < |z| < r$. If*

$$\log h(z) = \sum_{m=0}^{\infty} a_m z^m \tag{1}$$

in a neighbourhood of 0, then

$$c_m = \frac{1}{2}a_m r^m + \frac{1}{2m}\sum_{j=1}^{n}\left(\left(\frac{r}{z_j}\right)^m - \left(\frac{\overline{z_j}}{r}\right)^m\right) \tag{2}$$

for each positive integer m.

Proofs of this lemma, together with applications, can be found in the papers of Edrei and Fuchs [6, p. 312], Miles and Shea [14, p. 379] and Rubel [17, Lemma 1]. In these papers it is assumed that h is analytic in $|z| \le r$, but the version stated above can be deduced from this case. For further applications of the lemma and further references we refer to the survey paper by Rubel [18].

We require the following theorem on entire functions with real zeros, which is a combination of a result of Edrei, Fuchs and Hellerstein [7, Corollary 1.2] for functions of finite order and part of a result of Miles [13, Theorem 1] for infinite order. Here the notation is that of [11].

Theorem B *Suppose that f is an entire function of order ρ, and that all but finitely many zeros of f are real. If $2 < \rho < \infty$ then*

$$\delta(0, f) = \liminf_{r\to\infty}\frac{m\left(r, \frac{1}{f}\right)}{T(r, f)} > 0.$$

If $\rho = \infty$ then

$$N\left(r, \frac{1}{f}\right) = o(m_2(r, f)),$$

where

$$m_2(r, f)^2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\log^+|f(re^{i\theta})|\right)^2 d\theta,$$

so that in particular

$$N\left(r, \frac{1}{f}\right) = o(\log M(r, f)).$$

In [7, Corollary 1.2] the result is formulated only for the case where all zeros of f are real, but it is proved in [7, p. 138] that the conclusion remains valid if f has finitely many nonreal zeros (and in fact if the counting function of the nonreal zeros is sufficiently small). Theorem B implies that there cannot exist arbitrarily large r such that $\log|f(z)| \sim \log M(r,f)$ on $|z| = r$. For the existence of such r would imply that $T(r,f) \sim \log M(r,f)$ and that $m(r,1/f) = 0$, leading to $N(r,1/f) \sim T(r,f)$ using Nevanlinna's first fundamental theorem, which gives a contradiction.

3 Proof of the theorems

We shall prove only Theorem 1 and leave the minor changes that are necessary to handle the more general Theorem 2 to the reader.

Without loss of generality we may assume that l is the real axis, because otherwise we may consider $L(f(L^{-1}(z)))$ instead of $f(z)$, where $L(z)$ is a linear function that maps l onto the real axis. Similarly, we may assume that $f(f(0)) \neq 0$.

We now assume that the conclusion of the theorem is false, that is, we assume that $f(f(z))$ has only finitely many non-real fixpoints. Of course, this implies that $f(z)$ has only finitely many non-real fixpoints.

Denote by $M(r,f)$ and $\nu(r,f)$ the maximum modulus and central index of f, respectively, and suppose that $K > 1$ and $\varepsilon > 0$. We shall show that

$$\frac{1}{1+\varepsilon}\log|f(w_1)| \leq \log|f(w_2)| \leq (1+\varepsilon)\log|f(w_1)| \tag{3}$$

for all complex numbers w_1 and w_2 satisfying

$$\frac{1}{K}M(r,f) \leq |w_j| \leq KM(r,f), \tag{4}$$

provided $r \notin E$ where E is an exceptional set of finite logarithmic measure arising from Wiman-Valiron theory.

We proceed as in [4, Proof of Theorem 2] with, in the notation there, $g(z) = h(z) = f(z)$. We choose a small $\delta > 0$ and an integer $N > 5/\delta$. Now it is shown in [4] that if $r \notin E$ and there exist w_1 and w_2 such that (4) holds but (3) does not, then there is a point u_0 with $|u_0| \sim r$ such that with $u_j = u_0(1 + 2\pi ijN/\nu(r,f))$, at least one of the three disks $D(u_j, 5|u_0|/\delta\nu(r,f))$, $j \in \{1,2,3\}$, contains a fixpoint of f. The same argument shows that at least two of the four disks $D(u_j, 5|u_0|/\delta\nu(r,f))$, $j \in \{1,2,3,4\}$, contain a fixpoint of f. For sufficiently large r, at most one of these four disks can intersect the real axis. This is a contradiction. Hence (4) implies (3).

Letting ε tend to zero and choosing $r \notin E$ and $t = M(r,f)$ we see that there exists a sequence (t_ν) tending to ∞ such that $\log|f(z)| \sim \log M(t_\nu, f)$ for $|z| = t_\nu$. Of course we also have

$$\log|f(z) - z| \sim \log M(t_\nu, f(z) - z) \tag{5}$$

for $|z| = t_\nu$. Since all but finitely many zeros of $f(z) - z$ are real, Theorem B and the remark following it imply that $\rho(f) = \rho(f(z) - z) \leq 2$. (The proof in [13] is based on (2), and it will be apparent from the rest of our proof how (2) can be used to obtain the inequality $\rho(f(z) - z) \leq 2$. In fact we shall adapt some of the ideas used there.)

In view of the result of Baker already mentioned in the introduction, we may now assume that $1/2 \le \rho(f) \le 2$, but our argument in fact works for $0 < \rho(f) < \infty$. Since $\rho(f) > 0$ we have $\rho(f(f)) = \infty$ by a result of Pólya ([16], see also [11, Theorem 2.9]). Defining F by $F(z) = f(f(z)) - z$, we also have $\rho(F) = \infty$ and since all but finitely many zeros of F are real, a theorem of Gol'dberg ([9, Corollary 1], see also [10, p. 342] and [13, p. 137]) implies that F has infinite lower order and so does $f(f)$.

We choose small positive σ and ε. As in [4, Proof of Theorem 1], we deduce from (5) and the convexity of $\log M(t, f)$ as a function of $\log t$ that there exists $s_\nu \notin E$ satisfying $(t_\nu)^{1-2\sigma} \le s_\nu \le (t_\nu)^{1-\sigma}$ such that $\log |f(z)| > \log M(s_\nu, f)$ on $|z| = t_\nu$. Thus there must exist a closed curve Γ lying entirely in $s_\nu \le |z| < t_\nu$ on which $|f(z)| = M(s_\nu, f)$. Here we are using an idea from [1, p. 129]. Because $s_\nu \notin E$ we have, from (3),

$$(1+\varepsilon) \log |f(f(z))| \ge \log M(M(s_\nu, f), f) \ge \log M(s_\nu, f(f))$$

for z on Γ. Now, provided ε is small enough, the convexity of $\log M(t, f(f))$ enables us to choose (r_ν) with $(s_\nu)^{1-2\sigma} \le r_\nu \le (s_\nu)^{1-\sigma}$ such that on Γ we have $|f(f(z))| > M(r_\nu, f(f))$. Here r_ν may be chosen so that the circle $|w| = M(r_\nu, f(f))$ contains no critical value of f or of $f(f)$. Moreover, we have $(t_\nu)^{(1-2\sigma)^2} \le r_\nu$. Now the annulus $r_\nu \le |z| \le t_\nu$ must contain a simple closed curve Γ_0 surrounding the origin once with the property that $|f(f(z))| = M(r_\nu, f(f))$ for z on Γ_0.

Following [4, p. 66] we define G_0 to be the interior of Γ_0, $G_1 = f(G_0)$ and $G_2 = f(G_1)$. Clearly G_2 is the disk of radius $M(r_\nu, f(f))$ about the origin. We further define Γ_j to be the boundary of G_j for $j = 1, 2$. Since f has finite order and $f(f)$ has infinite lower order it is easy to see that for t large we have $M(t, f(f)) > M(t^2, f)$, which implies that Γ_{j-1} lies in the interior of G_j for $j = 1$, 2. Further, Γ_1 must be a simple curve, by the choice of r_ν, and we assume that for $j = 1, 2$, $f(z)$ describes p_j times the curve Γ_j as z describes the curve Γ_{j-1} once. Then f is a p_j-fold covering from G_{j-1} to G_j for $j = 1$, 2. We denote by P_2 the number of zeros of F in G_0, counting multiplicities, and by $\overline{P_2}$ the corresponding number where multiplicities are ignored. Then $P_2 = p_1 p_2$ for sufficiently large ν by Rouché's theorem and we proceed, using the argument from [4, p. 67], to show that

$$\overline{P_2} = (1 + o(1))P_2. \tag{6}$$

To prove (6), suppose that $z_0 \in G_0$ is a zero of $F(z) = f(f(z)) - z$ of multiplicity $m + 1 \ge 2$. Then z_0 is an indifferent fixpoint of $f(f)$ with multiplier 1, and by standard results in iteration theory (as summarized, for example, in Lemma 6 of [4]) there exist m disjoint components D_j, called Leau domains, of the Fatou set of $f(f)$, each having z_0 as a boundary point and such that the iterates of $f(f)$ converge to z_0 locally uniformly in D_j. Moreover, each D_j is mapped into itself by $f(f)$ and contains a singularity of the inverse function of $f(f)$.

We next show that each D_j is contained in G_0, using the argument from Lemma 8 of [4]. Now each D_j contains a sub-domain L_j, a "Leau petal", such that z_0 lies on the boundary of L_j and $f(f)$ maps L_j into itself. Moreover, L_j may be chosen so as to lie in an arbitrarily small neighbourhood of z_0. If D_j meets Γ_0 we join L_j to Γ_0 by a path γ through D_j, on which the iterates of $f(f)$ converge uniformly to z_0. However, since γ meets L_j and $f(f(\Gamma_0)) = \Gamma_2$, while Γ_0 lies inside Γ_2, it is easy to prove by induction that the image of γ under each iterate of $f(f)$ must meet Γ_0. This contradiction proves the assertion that each D_j is contained in G_0.

Thus for each j a singularity of the inverse function of $f(f))$ in D_j must correspond to a critical point of $f(f))$ in G_0 and so to a zero of f' in D_j or in $f(D_j)$. In either case this zero of f' must lie in G_1. Since $f(f)$ maps each D_j into itself, and since Leau domains corresponding to distinct multiple zeros of F must be disjoint, by the dynamics of $f(f)$ there, the images of distinct Leau domains of $f(f)$ under f must also be disjoint. Thus we have shown that $P_2 - \overline{P_2}$ is at most the number of zeros of f' in G_1, and since Γ_1 is a level curve of f we obtain $p_1 p_2 = P_2 \leq \overline{P_2} + p_2 - 1$. But p_1 must be large if r_ν is large enough, and thus (6) is proved.

We use (6) to show that $\overline{P_2}$ is large. If a is a fixed complex number and ν is sufficiently large, then P_2 equals the number of zeros of $f(f(z)) - a$ in the interior of Γ_0. Hence

$$P_2 \geq n\left(r_\nu, \frac{1}{f(f(z)) - a}\right), \tag{7}$$

where, as usual, $n(r,h)$ denotes the number of poles of a meromorphic function h in $|z| \leq r$, counted according to multiplicity. By $\overline{n}(r,h)$ we denote the corresponding number where multiplicities are ignored. We deduce from (6) and (7) that

$$\overline{n}\left(t_\nu, \frac{1}{F(z)}\right) \geq (1 - o(1))n\left(r_\nu, \frac{1}{f(f(z)) - a}\right). \tag{8}$$

Since $f(f)$ has infinite lower order we have

$$\lim_{\nu\to\infty} \frac{\log n\left(r_\nu, \frac{1}{f(f(z))-a}\right)}{\log r_\nu} = \infty \tag{9}$$

for a suitable value of a. This follows from [15, p. 280]. It can also be deduced from Nevanlinna's second fundamental theorem if we are prepared to replace (r_ν) by a suitable subsequence. Since $t_\nu \leq (r_\nu)^{1+\varepsilon}$ we deduce from (8) and (9) that

$$\lim_{\nu\to\infty} \frac{\log \overline{n}\left(t_\nu, \frac{1}{F(z)}\right)}{\log t_\nu} = \infty,$$

that is, the sequence of distinct zeros of F does not have finite exponent of convergence.

Now $f(z)$ maps zeros of $F(z)$ to zeros of $F(z)$. In fact, $f(z)$ is a bijection of the set of zeros of $F(z)$ onto itself. Hence $f(z)$ takes real values at the zeros of $F(z)$, with at most finitely many exceptions. We define $g(z) = f(z) - \overline{f(\overline{z})}$ and deduce that $g(z)$ vanishes at the zeros of $F(z)$, with at most finitely many exceptions. Hence the zeros of $g(z)$ do not have finite exponent of convergence. On the other hand, $\rho(g) \leq \rho(f) < \infty$. This implies that g vanishes identically, that is, $f(z) = \overline{f(\overline{z})}$, so that $f(z)$ is real on the real axis.

Therefore we may assume now that the curves Γ_0 are symmetric with respect to the real axis. For a fixed ν, we consider the conformal map $\varphi(z)$ from the unit disk D onto the interior of Γ_0, normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$. Then $\varphi(z)$ and $\varphi^{-1}(z)$ are real on the real axis. Also, $\varphi(z)$ has a continuous extension to the closure of D. We denote the extension again by $\varphi(z)$ and define $h(z) = F(\varphi(z))$ for $|z| \leq 1$. We also define $M_\nu = M(r_\nu, F)$ and note that

$$|h(e^{it})| \sim M_\nu \tag{10}$$

as $\nu \to \infty$. A simple computation shows that the coefficient a_2 in the expansion (1) is given by

$$a_2 = \frac{1}{2}\left(\frac{F''(0)F(0) - F'(0)^2}{F(0)^2}\varphi'(0)^2 + \frac{F'(0)}{F(0)}\varphi''(0)\right).$$

(Note that $F(0) \neq 0$ because we assumed that $f(f(0)) \neq 0$.) Since $\frac{1}{2}|\varphi''(0)| \leq 2|\varphi'(0)|$ by Bieberbach's theorem [5, Theorem 2.2] and $|\varphi'(0)| \leq t_\nu$ by Schwarz's lemma, we have

$$|a_2| = O\left((t_\nu)^2\right) \tag{11}$$

as $\nu \to \infty$. Next we have

$$\begin{aligned} |c_2(1,h)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-i2t}\log\left|h(e^{it})\right| dt\right| \\ &= \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-i2t}\left(\log|h(e^{it})| - \log M_\nu\right) dt\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log|h(e^{it})| - \log M_\nu\right| dt \\ &= o(1) \end{aligned} \tag{12}$$

by (10). Denote by $z_1, z_2, \ldots$ the zeros of $F(z)$ and by Z_ν the the set of all z_j that are contained in the interior of Γ_0. For $z_j \in Z_\nu$, we define $w_j = \varphi^{-1}(z_j)$. Combining (2), (11) and (12) we deduce that

$$\sum_{z_j \in Z_\nu}\left(\left(\frac{1}{w_j}\right)^2 - \overline{w_j}^2\right) = O\left((t_\nu)^2\right).$$

We have $|z_j| = |\varphi(w_j)| \leq t_\nu |w_j|$ by Schwarz's lemma so that

$$\left|\frac{1}{w_j}\right|^2 = O\left((t_\nu)^2\right).$$

for those j such that z_j is non-real. Since there are at most finitely many j for which z_j is non-real and since w_j is real if and only if z_j is real, we deduce that

$$\sum_{z_j \in Z_\nu}\left(\left|\frac{1}{w_j}\right|^2 - |w_j|^2\right) = O\left((t_\nu)^2\right). \tag{13}$$

Integration by parts yields

$$\begin{aligned} \sum_{z_j \in Z_\nu}\left(\left|\frac{1}{w_j}\right|^2 - |w_j|^2\right) &= \int_0^1 \left(\frac{1}{t^2} - t^2\right) dn(t, 1/h) \\ &= 2\int_0^1 \frac{n(t,1/h)}{t}\left(\frac{1}{t^2} + t^2\right) dt \\ &\geq 2\int_0^1 \frac{n(t,1/h)}{t} dt. \end{aligned} \tag{14}$$

By Jensen's formula,

$$\int_0^1 \frac{n(t,1/h)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |h(e^{it})| dt - \log |h(0)| = (1+o(1)) \log M_\nu. \tag{15}$$

Combining (13), (14) and (15), we obtain

$$\log M_\nu = O\left((t_\nu)^2\right) = O\left((r_\nu)^{2+2\varepsilon}\right)$$

which implies that $F(z)$ has lower order at most $2+2\varepsilon$. This contradicts our previous finding that $F(z)$ has infinite lower order and thus completes the proof of the theorem.

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Lehrstuhl II für Mathematik, RWTH Aachen, D-52056 Aachen, Germany;
E–mail: sf010be@dacth11.bitnet

Department of Mathematics, University of York, YO1 5DD;

Department of Mathematics, University of Nottingham, NG7 2RD;
E–mail: jkl@maths.nott.ac.uk