WEAKLY REPELLING FIXPOINTS AND THE CONNECTIVITY OF WANDERING DOMAINS

WALTER BERGWEILER AND NORBERT TERGLANE

ABSTRACT. It is proved that if a transcendental meromorphic function f has a multiply-connected wandering domain, then f has a fixpoint z_0 such that $|f'(z_0)| > 1$ or $f'(z_0) = 1$. Entire functions with a multiply-connected wandering domain have infinitely many such fixpoints. These results are used to show that solutions of certain differential equations do not have wandering domains at all.

1. INTRODUCTION AND RESULTS

Let f be a meromorphic function defined in the complex plane $\mathbb{C}$ or on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In the latter case, f is rational, and we shall always assume that the degree of f is at least two. In the first case, we shall always assume that f is transcendental.

The Fatou set F is the subset of $\hat{\mathbb{C}}$ where the iterates f^n of f are defined and form a normal family. The complement of F is called the Julia set and denoted by J. If U is a component of F, then $f^n(U)$ is contained in some component of F which we denote by U_n. If $U_n \cap U_m = \emptyset$ for all $n \neq m$, then U is called wandering.

Sullivan [38, 39] proved that rational functions do not have wandering domains. Transcendental entire or meromorphic functions, however, may have wandering domains, see [2, 3, 4, 8, 18, 26, 39]. Some examples of wandering domains are simply-connected, like those in [18, 26, 39], while others are multiply-connected, compare [2, 4, 8]. On the other hand, several classes of transcendental entire and meromorphic functions which do not have wandering domains are known [3, 10, 13, 14, 17, 20, 23, 35].

Following Shishikura [34] we call a fixpoint z_0 of a meromorphic function f *weakly repelling* if $|f'(z_0)| > 1$ or $f'(z_0) = 1$, with a slight modification if $z_0 = \infty$ (which can happen only for rational f). It is classical, see [21, I, p. 168] and [29, p. 85, p. 243], that a rational function of degree greater than one has at least one weakly repelling fixpoint. Shishikura [34], sharpening earlier results of Przytycki [32], proved that if a rational function has only one weakly repelling fixpoint, then its Julia set is connected, that is, all components of the Fatou set are simply-connected. Because of Sullivan's theorem [38, 39], Shishikura needed not to consider wandering domains but proved his result by considering the various types of preperiodic components of the Fatou set.

In this paper, we shall show that Shishikura's method may also be used to obtain results for wandering domains.

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Theorem 1. *Let f be a transcendental meromorphic function and suppose that f has a multiply-connected wandering domain. Then f has at least one weakly repelling fixpoint.*

For entire functions, we have a stronger result.

Theorem 2. *Let f be a transcendental entire function and suppose that f has a multiply-connected wandering domain. Then f has infinitely many weakly repelling fixpoints.*

Our results can be used to show that certain classes of meromorphic functions do not have wandering domains at all. Let R be the class of all meromorphic functions f which satisfy one of the following differential equations:

$$f'(z) = q(z)(f(z) - z)^2, \tag{1}$$

$$f'(z) = q(z)(f(z) - z)(f(z) - \sigma), \tag{2}$$

$$f'(z)^2 = q(z)(f(z) - z)^2(f(z) - \sigma), \tag{3}$$

$$f'(z)^2 = q(z)(f(z) - z)^2(f(z) - \sigma)(f(z) - \tau). \tag{4}$$

Here q is a rational function and $\sigma, \tau \in \mathbb{C}$.

Theorem 3. *Suppose $f \in R$. Then f does not have wandering domains.*

We note that the differential equations of the form $f'(z)^n = Q(z, f(z))$, where Q is rational in z and $f(z)$, which admit transcendental meromorphic solutions have been classified by Steinmetz [36, p. 25], see also Bank and Kaufman [11] and Jank and Volkmann ([27], [28, §18]). If q is a polynomial, then every solution of (1) and (2) is meromorphic, see for example [28, Satz 20.2]. Of course, the solutions of (2) can be given explicitly. In fact, the substitution $g(z) = 1/(f(z) - \sigma)$ transforms (2) into a linear differential equation for g. If q has the form $q(z) = p(z)^2$ for some polynomial p, then every solution of (3) and (4) is meromorphic, compare [28, Satz 20.3]. The same substitution as above transforms (4) into a differential equation for g which is similar to (3) in the sense that the right hand side is a cubic polynomial in g.

We remark that the class R is of some interest in connection with Newton's method of finding the zeros of an entire or meromorphic function. In fact, if g satisfies the differential equation $g''(z) = q(z)g(z)$, then the Newton iteration function $f(z) = z - g(z)/g'(z)$ satisfies (1). Thus Newton's method for solutions of such differential equations does not lead to wandering domains. For example, by choosing $g(z)$ equal to $\sin z$ or $\cos z$, we find that $z - \tan z$ and $z + \cot z$ do not have wandering domains. Similarly, if g satisfies the differential equation $g''(z) + q(z)(z-\sigma)g'(z) - q(z)g(z) = 0$, then $f(z) = z - g(z)/g'(z)$ satisfies (2). Functions satisfying (3) and (4) also occur as Newton iteration functions for solutions of certain differential equations.

Finally, we mention that rational functions are clearly in R so that Theorem 3 may be viewed as a generalization of Sullivan's theorem. In the proof, however, we shall restrict ourselves to the case that f is transcendental.

As an introduction to iteration theory, we recommend Beardon's [12] and Steinmetz's [37] book and Milnor's [30] lecture notes for rational functions and the survey articles of Baker [5] and Eremenko and Lyubich [19] for transcendental entire (as well as rational) functions. The iteration theory of transcendental meromorphic functions is surveyed in [15]. The classical references are Fatou [21] and Julia [29] for rational and Fatou [22] for transcendental entire functions.

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2. Lemmas

Lemma 1. (Shishikura [34, Theorem 2.1 and Corollary 2.2]) *Let V_0, V_1 be simply connected open sets in $\hat{\mathbb{C}}$ with $V_0 \neq \hat{\mathbb{C}}$, and f an analytic mapping from a neighborhood N of $\hat{\mathbb{C}} \backslash V_0$ to $\hat{\mathbb{C}}$ such that $f(\partial V_0) = \partial V_1$ and $f(V_0 \cap N) \subset V_1$. Suppose that for some $k \geq 1$, $f^j(V_1)$ are defined and*

$$f^j(V_1) \cap V_0 = \emptyset \ \ (0 \leq j < k-1),$$

and one of the following (a) or (b) holds:

(a) *$k = \infty$, i. e. for any $j \geq 0$, $f^j(V_1) \cap V_0 = \emptyset$;*
(b) *$f^{k-1}(\overline{V_1}) \subset V_0$.*

Then f has a weakly repelling fixpoint in $\hat{\mathbb{C}} \backslash \overline{V_0}$.

Lemma 2. *Let γ be a simple closed curve in $\mathbb{C}$ and let f be a function satisfying*

(i) *f is meromorphic in $\overline{\operatorname{int}(\gamma)}$,*
(ii) *$\operatorname{int}(\gamma)$ contains a pole of f,*
(iii) *$f(\gamma) \subset \operatorname{ext}(\gamma)$,*
(iv) *∞ and γ are in the same component of $\hat{\mathbb{C}} \backslash f(\gamma)$.*

Then $\operatorname{int}(\gamma)$ contains a weakly repelling fixpoint of f.

Here $\operatorname{int}(\gamma)$ and $\operatorname{ext}(\gamma)$ denote the interior and exterior of the curve γ.

Lemma 3. *Let γ be a simple closed curve in $\mathbb{C}$ and let f be a function satisfying conditions (i) - (iii) of Lemma 2. Instead of (iv), suppose that ∞ and γ are in different components of $\hat{\mathbb{C}} \backslash f(\gamma)$. Furthermore, suppose that $f(\gamma)$ contains a simple closed curve σ with the following properties:*

(i) *$\gamma \subset \operatorname{int}(\sigma)$,*
(ii) *f is meromorphic in $\overline{\operatorname{int}(\sigma)}$,*
(iii) *$f(\sigma) \subset \operatorname{ext}(\sigma)$.*

Then $\operatorname{int}(\sigma)$ contains a weakly repelling fixpoint of f.

Lemma 4. *Suppose that there are simply-connected domains V_0, V_1 and that there is a function f meromorphic in a neighborhood N of $\hat{\mathbb{C}} \backslash V_0$ such that*

(i) *$f(\partial V_0) = \partial V_1$,*
(ii) *$f(N \cap V_0) \subset V_1$,*
(iii) *$\infty \in V_0$,*
(iv) *$\overline{V_1} \subset \hat{\mathbb{C}} \backslash \overline{V_0}$,*
(v) *f has a pole in $\hat{\mathbb{C}} \backslash \overline{V_0}$,*
(vi) *there exists a neighborhood N' of ∂V_0 such that $f^n|_{N'}$ is defined for all n and that $f^m(N') \cap f^n(N') = \emptyset$ for $m \neq n$.*

Suppose also f does not have a weakly repelling fixpoint in $\hat{\mathbb{C}}\backslash\overline{V_0}$. Then there exist V_0', V_1' satisfying $\overline{V_0} \subset V_0'$ such that the above hypotheses are satisfied with V_0, V_1 replaced by V_0', V_1'. Moreover, there exists $m \geq 1$ such that if $\varepsilon > 0$ is given, then $\partial V_0' \subset N_\varepsilon(f^m(\partial V_0))$ can be achieved by a suitable choice of V_0'.

Here, for $\varepsilon > 0$ and $S \subset \hat{\mathbb{C}}$, $N_\varepsilon(S)$ denotes the ε-neighborhood of S, that is, the set of all $z \in \hat{\mathbb{C}}$ whose spherical distance to S is less than ε.

Lemmas 2, 3, and 4 can be deduced from Lemma 1. This will be done in the next section.

3. Proofs of the lemmas

3.1. **Proof of Lemma 2.** By hypothesis, there exists a simple closed curve σ satisfying $\infty \in \operatorname{ext}(\sigma)$, $\operatorname{int}(\gamma) \subset \operatorname{ext}(\sigma)$, and $f(\gamma) \subset \operatorname{int}(\sigma)$ such that σ does not contain critical values of f. Define $V_1 = \operatorname{int}(\sigma)$ and let $G \subset \operatorname{int}(\gamma)$ be a component of $f^{-1}(\hat{\mathbb{C}}\backslash\overline{V_1})$ and V_0 be the component of $\hat{\mathbb{C}}\backslash\overline{G}$ that contains ∞. One can check that the hypotheses of Lemma 1 are satisfied (case (b) with $k = 1$). Hence f has a weakly repelling fixpoint $z_0 \in G \subset \operatorname{int}(\gamma)$.

3.2. **Proof of Lemma 3.** First we note that $\sigma \subset f(\gamma) \subset f(\operatorname{int}(\sigma))$. Because $f(\sigma) \subset \operatorname{ext}(\sigma)$ by hypothesis and because $\partial f(\operatorname{int}(\sigma)) \subset f(\sigma)$, we deduce $\operatorname{int}(\sigma) \subset f(\operatorname{int}(\sigma))$. Hence we can find a simple closed curve $\tau \subset f(\operatorname{int}(\sigma))$ which does not contain critical values of f such that

$$\overline{\operatorname{int}(\sigma)} \subset \operatorname{int}(\tau) \subset f(\operatorname{int}(\sigma)).$$

Hence there exists a component G of $f^{-1}(\operatorname{int}(\tau))$ satisfying $G \subset \operatorname{int}(\sigma)$. It is easy to check that the hypotheses of Lemma 1 are satisfied (case (b) with $k = 1$) if we choose V_0 as the component of $\hat{\mathbb{C}}\backslash\overline{G}$ that contains ∞ and $V_1 = \operatorname{ext}(\tau) \cup \{\infty\}$. As in the proof of Lemma 2 this gives the existence of a weakly repelling fixpoint in $\operatorname{int}(\sigma)$.

3.3. **Proof of Lemma 4.** If all $f^j(V_1)$ are defined and contained in $\hat{\mathbb{C}}\backslash\overline{V_0}$, then Lemma 1, case (a), shows that there is a weakly repelling fixpoint in $\hat{\mathbb{C}}\backslash\overline{V_0}$, a contradiction.

Hence there exists $k \geq 2$ such that $f^{k-1}(V_1) \not\subset \hat{\mathbb{C}}\backslash\overline{V_0}$. (If $f^n(V_1)$ is not defined, then $\infty \in f^{n-1}(V_1)$ so that $f^{n-1}(V_1) \not\subset \hat{\mathbb{C}}\backslash\overline{V_0}$.)

If $\partial f^{k-1}(V_1) \subset V_0$, then either

$$\overline{f^{k-1}(V_1)} \subset V_0 \tag{α}$$

or

$$\overline{f^{k-2}(V_1)} \subset \hat{\mathbb{C}}\backslash\overline{V_0} \subset f^{k-1}(V_1). \tag{β}$$

In case (α), Lemma 1, case (b), yields the existence of a weakly repelling fixpoint in $\hat{\mathbb{C}}\backslash\overline{V_0}$, a contradiction.

In case (β) let H be a component of $f^{-1}(\hat{\mathbb{C}}\backslash\overline{V_0})$ satisfying $H \subset f^{k-2}(V_1)$. (Note that $f(f^{k-2}(V_1)) = f^{k-1}(V_1) \supset \hat{\mathbb{C}}\backslash\overline{V_0}$.) Let W_0 be the component of $\hat{\mathbb{C}}\backslash\overline{H}$ that contains ∞ and let $W_1 = V_0$. Then W_0 and W_1 satisfy the hypotheses of Lemma 1 case (b), with $k = 1$; again we obtain the existence of a weakly repelling fixpoint in $\hat{\mathbb{C}}\backslash\overline{W_0} \subset \hat{\mathbb{C}}\backslash\overline{V_0}$, a contradiction.

Hence we may assume that

$$\partial f^{k-1}(V_1) \subset \hat{\mathbb{C}} \setminus \overline{V_0}.$$

Together with

$$f^{k-1}(V_1) \not\subset \hat{\mathbb{C}} \setminus \overline{V_0}$$

this implies that $\infty \in f^{k-1}(V_1)$ so that $f^{k-2}(V_1)$ contains a pole.

If $\partial f^{k-2}(V_1)$ and ∞ are in the same component of $\hat{\mathbb{C}} \setminus \partial f^{k-1}(V_1)$ while at the same time $\partial f^{k-1}(V_1)$ and ∞ are in the same component of $\hat{\mathbb{C}} \setminus \partial f^{k-2}(V_1)$, one can find a curve γ in $\partial f^{k-2}(V_1)$ which satisfies the hypotheses of Lemma 2. This implies the existence of a weakly repelling fixpoint in $\operatorname{int}(\gamma) \subset \hat{\mathbb{C}} \setminus \overline{V_0}$.

There remain two possibilities:

(I) $\partial f^{k-2}(V_1)$ and ∞ are in the same component of $\hat{\mathbb{C}} \setminus \partial f^{k-1}(V_1)$, but $\partial f^{k-1}(V_1)$ and ∞ are in different components of $\hat{\mathbb{C}} \setminus \partial f^{k-2}(V_1)$;

(II) $\partial f^{k-2}(V_1)$ and ∞ are in different components of $\hat{\mathbb{C}} \setminus \partial f^{k-1}(V_1)$, but $\partial f^{k-1}(V_1)$ and ∞ are in the same component of $\hat{\mathbb{C}} \setminus \partial f^{k-2}(V_1)$.

In case (I), we choose a simple closed curve σ contained in a bounded component of $\hat{\mathbb{C}} \setminus \partial f^{k-2}(V_1)$ with the property that $\partial f^{k-1}(V_1) \subset \operatorname{int}(\sigma)$ and such that σ does not contain critical values of f. We note that if $\delta > 0$ is given, then we may achieve $\sigma \subset N_\delta(\partial f^{k-1}(V_1))$. Here we choose δ so small that $N_\delta(\partial f^{k-1}(V_1)) \subset f^k(N')$. We define $V_1' = \operatorname{int}(\sigma)$ and denote by H a component of $f^{-1}(\hat{\mathbb{C}} \setminus \overline{V_1'})$ contained in $f^{k-2}(V_1)$. Next we define V_0' to be the component of $\hat{\mathbb{C}} \setminus \overline{H}$ that contains ∞.

It can be checked that V_0' and V_1' have the properties stated in the lemma. In fact, the only property that is not obvious is that $\overline{V_1'} \subset \hat{\mathbb{C}} \setminus \overline{V_0'}$. But otherwise we would have $\overline{V_1'} \subset V_0'$, and hence a weakly repelling fixpoint in $\hat{\mathbb{C}} \setminus \overline{V_0'} \subset \hat{\mathbb{C}} \setminus \overline{V_0}$ by Lemma 1, case (b), $k = 1$.

In case (II), we choose a simple closed curve σ such that $\sigma \in \hat{\mathbb{C}} \setminus \overline{V_0}$, $\partial f^{k-1}(V_1) \subset \operatorname{int}(\sigma)$, and $\sigma \subset N_\delta(\partial f^{k-1}(V_1)) \subset f^k(N')$. We consider two subcases:

(IIa) $f(\sigma) \subset \operatorname{int}(\sigma)$,

(IIb) $f(\sigma) \not\subset \operatorname{int}(\sigma)$.

In case (IIa), we choose a simple closed curve σ' containing no critical values of f such that $\sigma' \subset \operatorname{int}(\sigma)$, $f(\sigma) \subset \operatorname{int}(\sigma')$, and $\sigma' \subset N_\epsilon(f(\sigma)) \subset f^{k+1}(N')$. We define $V_1' = \operatorname{int}(\sigma')$, and denote by H a component of $f^{-1}(\hat{\mathbb{C}} \setminus \overline{V_1'})$ contained in $\operatorname{int}(\sigma)$, and define V_0' to be the component of $\hat{\mathbb{C}} \setminus \overline{H}$ that contains ∞. As in case (I), we can check that V_0' and V_1' have the desired properties.

In case (IIb), we conclude as in the proofs of Lemmas 2 and 3 that there is a weakly repelling fixpoint in $\operatorname{int}(\sigma) \subset \hat{\mathbb{C}} \setminus \overline{V_0}$, contradicting the hypothesis.

4. Proofs of the theorems

4.1. **Proof of Theorem 1.** Suppose that f has a multiply-connected wandering domain U. In view of a theorem of Baker [6, Theorem 1], f is either entire or satisfies Assumption A of [7], that is, $O^-(\infty) = \bigcup_{n=1}^{\infty} f^{-n}(\infty)$ is infinite. Because we shall obtain a stronger conclusion for entire f in Theorem 2, we consider only the latter case.

As proved by Baker, Kotus, and Lü [7, Lemma 1], we have $J = \overline{O^-(\infty)}$. We choose a simple closed curve γ in U such that $J \cap \operatorname{int}(\gamma) \neq \emptyset$. We may assume that $\operatorname{int}(\gamma)$ contains a pole of f. Because otherwise, since $J = \overline{O^-(\infty)}$, there exists a minimal $n \geq 1$ such that $f^n(\text{int}(\gamma))$ contains a pole of f. Therefore $f^n(\gamma)$ contains a simple closed curve σ such that f has a pole in $\text{int}(\sigma)$. Clearly, σ is also contained in a wandering domain and we may replace γ by σ.

If γ satisfies the hypotheses of Lemma 2 or 3, then we are done. Otherwise, we have $f(\gamma) \subset \text{int}(\gamma)$ or there exists a simple closed curve $\sigma \subset f(\gamma)$ such that $f(\sigma) \subset \text{int}(\sigma)$. Without loss of generality we may assume that $f(\gamma) \subset \text{int}(\gamma)$.

We now suppose that $\text{int}(\gamma)$ does not contain a weakly repelling fixpoint and seek a contradiction.

We choose a simple closed curve $\sigma \subset \text{int}(\gamma)$ such that $f(\gamma) \subset \text{int}(\sigma)$. Moreover, we assume that $\sigma \subset N_\delta(f(\gamma))$ where $\delta > 0$ is chosen so small that σ is contained in the component of F that contains $f(\gamma)$.

Let $H = \text{ext}(\sigma) \cup \{\infty\}$ and let G be a component of $f^{-1}(H)$ which is contained in $\text{int}(\gamma)$. We define $V_1 = \text{int}(\sigma)$ and denote by V_0 the component of $\hat{\mathbb{C}} \backslash \overline{G}$ that contains ∞.

If $\overline{V_1} \subset V_0$, then Lemma 1 (case (b), $k = 1$) yields the existence of a weakly repelling fixpoint in $\hat{\mathbb{C}} \backslash \overline{V_0}$, a contradiction. Hence we may assume that $\overline{V_1} \subset \hat{\mathbb{C}} \backslash \overline{V_0}$. Then the hypotheses of Lemma 4 are satisfied. We choose $V_0^{(1)}$, $V_1^{(1)}$ as in the conclusion of Lemma 4, that is:

$V_0^{(1)}, V_1^{(1)}$ are simply-connected domains with $\overline{V_0} \subset V_0^{(1)}$ and

(i) $f(\partial V_0^{(1)}) = \partial V_1^{(1)}$,
(ii) $f(N \cap V_0^{(1)}) \subset V_1^{(1)}$ for some neighborhood N of $\hat{\mathbb{C}} \backslash V_0^{(1)}$,
(iii) $\infty \in V_0^{(1)}$,
(iv) $\overline{V_1^{(1)}} \subset \hat{\mathbb{C}} \backslash \overline{V_0^{(1)}}$,
(v) f has a pole in $\hat{\mathbb{C}} \backslash \overline{V_0^{(1)}}$,
(vi) $f^m(N') \cap f^n(N') = \emptyset$ for $m \neq n$ and some neighborhood N' of $\partial V_0^{(1)}$.

Moreover, $\hat{\mathbb{C}} \backslash \overline{V_0^{(1)}}$ does not contain a weakly repelling fixpoint and

$$f(\partial V_0^{(1)}) = \partial V_1^{(1)} \subset \overline{V_1^{(1)}} \subset \hat{\mathbb{C}} \backslash \overline{V_0^{(1)}}.$$

Iterating this procedure, we obtain a sequence $\left(V_0^{(k)}\right)$ satisfying $\overline{V_0^{(k)}} \subset V_0^{(k-1)}$ with the above properties.

Given $\varepsilon > 0$, we may achieve $\partial V_0^{(k)} \subset N_\varepsilon(f^{m_k}(\gamma))$ for some sequence (m_k). Denote by $\text{diam}(S)$ the spherical diameter of a subset S of $\hat{\mathbb{C}}$. Since limit functions of iterates in wandering domains are constant ([9, Lemma 2.1], [21, II, p. 55]), $\text{diam}(f^{m_k}(\gamma)) \to 0$ as $k \to \infty$. Hence $\text{diam}(\partial V_0^{(k)}) < 3\varepsilon$ for sufficiently large k. Since $\hat{\mathbb{C}} \backslash \overline{V_0^{(k)}}$ contains one of the poles of f in $\hat{\mathbb{C}} \backslash \overline{V_0}$, we can achieve $f(\partial V_0^{(k)}) \subset N_\delta(\infty)$ for any given $\delta > 0$ by choosing k large and ε small. On the other hand,

$$f(\partial V_0^{(k)}) \subset \hat{\mathbb{C}} \backslash \overline{V_0^{(k)}} \subset \hat{\mathbb{C}} \backslash \overline{V_0}.$$

Clearly, this is a contradiction if δ is sufficiently small.

4.2. **Proof of Theorem 2.** Suppose that f has a multiply-connected wandering domain U, let γ be a simple closed curve in U which is not nullhomotopic in U and define $\Gamma_n = f^n(\gamma)$. Since U is multiply-connected, $f^n|_U \to \infty$ as $n \to \infty$. By a theorem of Baker [3, Theorem 3.1], $0 \in \text{int}(\Gamma_n)$ for sufficiently large n. Denote by

$l_H(\Gamma_n, G)$ the length of Γ_n with respect to the hyperbolic metric of a hyperbolic domain G containing Γ_n.

Define $\Omega = \mathbb{C}\backslash\{0,1\}$. Clearly,

$$l_H(\Gamma_n, \Omega) \leq l_H(\Gamma_n, U_n) \leq l_H(\gamma, U) \tag{5}$$

if n is chosen so large that $\{0,1\} \cap U_n = \emptyset$. It is well-known that there exists a positive constant c such that the hyperbolic metric function $\rho_\Omega(z)$ satisfies

$$\rho_\Omega(z) \geq \frac{c}{|z| \log |z|} \tag{6}$$

for sufficiently large z, compare [1, §1.8]. From (5) and (6) we can deduce that there exists $K > 1$ such that

$$\Gamma_n \subset \operatorname{ann}(r_n, r_n^K) \tag{7}$$

for some sequence (r_n) tending to ∞. (Here $\operatorname{ann}(r, R)$ denotes the annulus around 0 with radii r and R.) For sufficiently large n, there exist simple closed curves $\gamma_n \subset \Gamma_n$ such that $0 \in \operatorname{int}(\gamma_n)$. Combining this with (5), (6), and (7) we also find that the euclidian length $l_E(\gamma_n)$ of γ_n satisfies

$$l_E(\gamma_n) \leq l_E(\Gamma_n) = O(r_n^K \log r_n)$$

as $n \to \infty$. We also note that

$$\min_{z \in \gamma_n} |f(z)| \geq r_{n+1} \geq \left(\max_{z \in \gamma_n} |f(z)|\right)^{1/K} \geq M(r_n, f)^{1/K}$$

and hence

$$\min_{z \in \gamma_n} |f(z) - z| \geq M(r_n, f)^{1/K} - r_n^K \geq r_n^{K+1} \tag{8}$$

for sufficiently large n. (Here $M(r, f)$ denotes the maximum modulus of the function f on the circle of radius r around 0.) It is a simple consequence of (8) and Nevanlinna's first fundamental theorem [25, 28, 31] that f has infinitely many fixpoints. We denote them by $z_1, z_2, \ldots$. Then

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{dz}{f(z) - z} = \sum_{z_j \in \operatorname{int}(\gamma_n)} \operatorname{Res}\left(\frac{1}{f(z) - z}, z_j\right) \tag{9}$$

by the residue theorem. (Here $\operatorname{Res}(g(z), a)$ denotes the residue of the function g at the point a.) If z_j is not weakly repelling, then

$$\operatorname{Re}\left(\operatorname{Res}\left(\frac{1}{f(z) - z}, z_j\right)\right) = \operatorname{Re}\left(\frac{1}{f'(z_j) - 1}\right) \leq -\frac{1}{2}. \tag{10}$$

On the other hand,

$$\left|\frac{1}{2\pi i} \int_{\gamma_n} \frac{dz}{f(z) - z}\right| \leq \frac{l_E(\gamma_n)}{2\pi \min_{z \in \gamma_n} |f(z) - z|} = O\left(\frac{\log r_n}{r_n}\right) = o(1) \tag{11}$$

as $n \to \infty$.

Combining (9), (10), and (11) we deduce that f has not only infinitely many fixpoints, but in fact infinitely many weakly repelling fixpoints. Similar arguments were used in [21, I, p. 168], [29, p. 85, p. 243], and [40, p. 532].

4.3. **Proof of Theorem 3 for functions satisfying (1).** We restrict ourselves to the case that f is transcendental. It follows from (1) that the zeros of f' have multiplicity 2 and are fixpoints of f, with at most finitely many exceptions. In particular, all but finitely many critical points of f are superattracting fixpoints. Similarly, all but finitely many fixpoints of f are double zeros of f' and hence superattracting fixpoints. In particular, f has at most finitely many weakly repelling fixpoints.

Next we note that f has finite order [28, Satz 22.4, (i)]. To determine the order of f, which we denote by $\rho(f)$, we define $h(z) = f(z) - z$. Clearly, $\rho(h) = \rho(f)$ and $h' = qh^2 - 1$. Suppose that $q(z) \sim az^d$ as $z \to \infty$ where $d \in \mathbb{Z}$ and $a \in \mathbb{C}\backslash\{0\}$. From [28, Satz 22.4, (ii)] we deduce that if q is a polynomial, then $\rho(h) = 1 + \frac{d}{2}$. Hence $\rho(f) = 1 + \frac{d}{2}$ where $d \geq 0$ in this case. The argument used in [28] shows that if q is rational, then we have at least $\rho(f) = \rho(h) \leq 1 + \frac{d}{2}$ and $d \geq -1$.

We also note that [28, Satz 24.1] implies that h has infinitely many zeros, that is, f has infinitely many fixpoints.

Now we show that f does not have asymptotic values. Suppose to the contrary that $f(z) \to \alpha \in \mathbb{C}$ as $z \to \infty$ along some curve γ. Then $f'(z) \sim az^{d+2}$ as $z \to \infty$ on γ by (1). On the other hand, by a result of Gundersen [24, Corollary 2], there exist for any $\varepsilon > 0$ arbitrarily large r such that $|f'(z)/f(z)| \leq r^{\rho(f)-1+\varepsilon}$ for $|z| = r$. Hence $|f'(z)| \leq (|\alpha| + o(1))|z|^{\rho(f)-1+\varepsilon}$ for arbitrarily large z in γ. We deduce that $d+2 \leq \rho(f) - 1 + \varepsilon$, that is, $\rho(f) \geq d + 3 - \varepsilon$. Clearly, this contradicts the previous findings that $\rho(f) \leq 1 + \frac{d}{2}$ and $d \geq -1$ if ε is sufficiently small.

Suppose now that U is a wandering domain of f. Since f does not have asymptotic values and since all but finitely many critical values are superattracting fixpoints, there exists m such that U_k does not contain critical or asymptotic values of f for $k \geq m$.

First we suppose that there exists $n \geq m$ such that U_n is multiply-connected. Then f cannot be entire by Theorem 2.

Let γ be a simple closed curve in U_n which is not nullhomotopic in U_n. Then $f^k(\gamma)$ is not nullhomotopic in U_{n+k} for $k \geq 1$ because the singularities of f^{-1} are the critical and asymptotic values of f so that U_j does not contain such singularities for $j \geq n$.

As in the proof of Theorem 1 we deduce that there exists a sequence (n_k) tending to ∞ with the property that $f^{n_k}(\gamma)$ contains a simple closed curve γ_k whose interior contains a pole p_k and a weakly repelling fixpoint z_k. If $0 \notin \operatorname{int}(\gamma_k)$ for all large k, then the spherical distance from p_k to z_k tends to zero because $\operatorname{diam}(\gamma_k) \to 0$ as $k \to \infty$. Hence infinitely many of the z_k are distinct so that f has infinitely many weakly repelling fixpoints, a contradiction. But if there are arbitrarily large k with $0 \in \operatorname{int}(\gamma_k)$, then $\mathbb{C} = \bigcup_k \operatorname{int}(\gamma_k)$ because $\operatorname{diam}(\gamma_k) \to 0$ as $k \to \infty$. Hence all components of F are bounded. In particular, the immediate basins of attraction of the infinitely many superattracting fixpoints are bounded. By a classical result of Fatou [21, II, p. 81], each of these immediate basins of attraction contains a weakly repelling fixpoint in its boundary. (Fatou proved this for rational functions and Bhattacharyya [16] extended this to transcendental entire functions. Their argument remains valid for transcendental meromorphic functions.) Again, infinitely many of these weakly repelling fixpoints are distinct, a contradiction.

Hence we may now assume that U_k is simply-connected for all $k \geq m$. This allows us to use the quasiconformal methods introduced by Sullivan [38, 39]. We sketch the argument only briefly.

We consider K-quasiconformal self-maps Φ of the sphere that fix 0, 1, and ∞ such that $f_\Phi = \Phi \circ f \circ \Phi^{-1}$ is meromorphic. Since the fixpoints of f correspond to the double zeros of f', with at most finitely many exceptions, the same is true for f_Φ. From (1) we can deduce that all but finitely many poles of f are simple. Again, the same is true for f_Φ. We conclude that $f'_\Phi(z)/(f_\Phi(z)-z)^2$ has only finitely many zeros and poles, in fact, as many as q has. Also, by the Hölder continuity of Φ at ∞, we have $|\Phi(z)| = O(|z|^K)$ and $|\Phi^{-1}(z)| = O(|z|^K)$ as $z \to \infty$. It is not difficult to see that this implies that $\rho(f_\Phi) \leq K\rho(f)$. It follows that the order of $f'_\Phi(z)/(f_\Phi(z) - z)^2$ is at most $K\rho(f)$. Altogether we see that

$$\frac{f'_\Phi(z)}{(f_\Phi(z) - z)^2} = q_\Phi(z)e^{p_\Phi(z)}$$

for some rational function q_Φ of the same degree as q and some polynomial p_Φ of degree at most $K\rho(f)$. Thus the family of all such functions f_Φ depends on only finitely many parameters. This contradicts the existence of a wandering domain with the properties described above, see [3, 10, 12] for details.

4.4. **Proof of Theorem 3 for functions satisfying (2), (3) or (4).** The idea of the proof is the same as for functions satisfying (1) and most arguments go through with only minor modifications. For example, we now have that the critical points of f correspond to the fixpoints, σ-points, and τ-points of f, with finitely many exceptions. Thus we find that the critical points of f_Φ correspond to the fixpoints, $\Phi(\sigma)$-points, and $\Phi(\tau)$-points of f_Φ, with finitely many exceptions. Again, the family of all such functions f_Φ depends on only finitely many parameters.

There are only two arguments that do not carry over, namely the ones used to show that f has no asymptotic values and infinitely many fixpoints.

We note, however, that it suffices to show that f has only finitely many asymptotic values.

If f satisfies (2), then this follows from the Denjoy-Carleman-Ahlfors theorem [31, p. 307] because nonconstant solutions of (2) have finite order and do not take the value σ.

We now show that f has only finitely many asymptotic values if f satisfies (3) or (4). In fact, we shall show that if $\alpha \in \mathbb{C}\backslash\{\sigma, \tau\}$, then α is not an asymptotic value of f.

Suppose again that $q(z) \sim az^d$ as $z \to \infty$. It follows from (3) or (4), and by taking logarithmic derivatives there, that there exist positive constants R, η, c_1, c_2, and c_3 with the following properties: if $|z| > R$ and $|f(z) - \alpha| < \eta$, then

$$c_1|z|^{d+2} \leq |f'(z)| \leq c_2|z|^{d+2} \tag{12}$$

and

$$\left|\frac{f''(z)}{f'(z)}\right| \leq c_3|z|^{d+2}. \tag{13}$$

It follows from (12) that there exists a positive constant $\delta < \eta/2c_2$ such that if z_0 is sufficiently large and if $|f(z_0)-\alpha| < \eta/2$, then $|f(z)-\alpha| < \eta$ for $|z-z_0| < \delta|z_0|^{-d-2}$. Thus (12) and (13) hold for $|z - z_0| < \delta|z_0|^{-d-2}$ if $|f(z_0) - \alpha| < \eta/2$. From (13) we deduce that if $0<\gamma\leq\delta$, then

$$\left|\log\frac{f'(z)}{f'(z_0)}\right| = \left|\int_{z_0}^{z}\frac{f''(t)}{f'(t)}dt\right| \leq (1+o(1))c_3|z_0|^{d+2}|z-z_0| \leq 2\gamma c_3$$

for $|z-z_0|<\gamma|z_0|^{-d-2}$ and sufficiently large z_0 satisfying $|f(z_0)-\alpha|<\eta/2$. We choose $\gamma<\pi/4c_3$ and define $r=\gamma|z_0|^{-d-2}$. Then

$$\left|\arg\frac{f'(z)}{f'(z_0)}\right| < \frac{\pi}{2}$$

for $|z-z_0|<r$. Thus the values of $f'(z)$ in $|z-z_0|<r$ are contained in a half-plane which implies that $f(z)$ is univalent in $|z-z_0|<r$. Therefore the function h defined by

$$h(w) = f(z_0+rw) - f(z_0)$$

is univalent in the unit disc. Since $|h'(0)| = r|f'(z_0)| \geq rc_1|z_0|^{-d-2} = \gamma c_1$ we obtain from Koebe's distortion theorem $|h(w)| \geq 2\gamma c_1/9$ for $|w| = 1/2$. Thus $|f(z)-f(z_0)| \geq 2\gamma c_1/9$ for $|z-z_0| = r/2$. We deduce that α cannot be an asymptotic value.

We note that the above arguments can also be used to show that if f satisfies (2) and $\alpha\neq\sigma$, then α is not an asymptotic value of f. It also yields that if f satisfies (1), then f does not have asymptotic values at all. The arguments used above for the cases (1) and (2) are, however, much shorter (although less elementary).

It remains to prove that nonconstant solutions of (2), (3), and (4) have infinitely many fixpoints. In case (2) this follows from [28, Satz 24.1] as in case (1). To prove that f has infinitely many fixpoints if f satisfies (3) or (4) we define $g(z) = 1/(f(z)-z)$. In case (3) we have

$$1-2\frac{g'(z)}{g(z)^2}+\frac{g'(z)^2}{g(z)^4} = f'(z)^2 = q(z)\frac{1}{g(z)^2}\left(\frac{1}{g(z)}+z-\sigma\right)$$

As in [28, p. 230] we write this differential equation in the form

$$g(z) = 2\frac{g'(z)}{g(z)}-\frac{g'(z)^2}{g(z)^3}+q(z)\frac{1}{g(z)}\left(\frac{1}{g(z)}+z-\sigma\right) \tag{14}$$

and deduce that if $|g(z)|\geq 1$, then

$$|g(z)| \leq 2\left|\frac{g'(z)}{g(z)}\right| + \left|\frac{g'(z)}{g(z)}\right|^2 + |q(z)|\left(|z|+|\sigma|+1\right).$$

Hence

$$\log^+|g(z)| \leq 2\log^+\left|\frac{g'(z)}{g(z)}\right| + O(\log|z|)$$

as $z\to\infty$. Using the standard terminology of Nevanlinna theory [25, 28, 31] we obtain

$$\mathrm{m}(r,g) \leq 2\mathrm{m}\left(r,\frac{g'}{g}\right)+O(\log r).$$

Thus $\mathrm{m}(r,g) = S(r,g)$ by the lemma on the logarithmic derivative. Nevanlinna's first fundamental theorem now implies that $\mathrm{N}(r,g)\sim\mathrm{T}(r,g)$. In particular, g has infinitely many poles, that is, f has infinitely many fixpoints. We remark that the fact that transcendental meromorphic solutions of (14) have infinitely many poles can also be proved using Wiman-Valiron theory (see [28, §21]).

The case that f satisfies (4) is analogous.

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Lehrstuhl II für Mathematik, RWTH Aachen, D-52056 Aachen, Germany

Current address: Fachbereich Mathematik, Sekr. MA 8–2, TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: bergweil@math.tu-berlin.de

Lehrstuhl II für Mathematik, RWTH Aachen, D-52056 Aachen, Germany

E-mail address: terglan@math2.rwth-aachen.de