Transcendency of Local Conjugacies in Complex Dynamics and Transcendency of Their Values*

Paul-Georg Becker†, Köln

and

Walter Bergweiler, Aachen

Let p and q be polynomials of the same degree. A classical result of Böttcher says that there exists a function f conformal in a neighborhood of infinity such that $f(p(z)) = q(f(z))$. We show that f is transcendental and takes transcendental values at algebraic points unless p and q are linearly conjugate to monomials or Chebychev polynomials. As an application, we show that the conformal map from the exterior of the Mandelbrot set onto the exterior of the unit disk takes transcendental values at algebraic points. A second application is the solution of a transcendency problem posed by Golomb.

I. Introduction and main results

Let p and q be polynomials of degree $d \geq 2$. Denote by a and b the leading coefficients of p and q, that is, $p(z) = az^d + \dots$ and $q(z) = bz^d + \dots$, and let λ be a value satisfying $\lambda^{d-1} = a/b$. A classical theorem of Böttcher (see e.g. [Bd, Theorem 6.10.1] or [S1, § 3.3]) says that there exists a unique function f defined and analytic in a neighborhood of ∞ such that $f(z) \sim \lambda z$ as $z \to \infty$ and

$$f(p(z)) = q(f(z)) \tag{1}$$

for all large z, that is, p and q are locally conjugate in a neighborhood of ∞. Such a conjugating function f is called a *Böttcher function* with respect to p and q. It is clear from this definition that there exist precisely $d-1$ different Böttcher functions. We remark that the theorem is usually stated only in the case $p(z) = z^d$ or $q(z) = z^d$, but the version formulated above follows easily from this special case.

Under suitable hypotheses, it was shown in [Be] that if f is a transcendental solution of (1), then f takes on transcendental values at algebraic points. Therefore,

*1991 Mathematics Subject Classification: Primary 11 J 91. Secondary: 30 D 05, 58 F 08.
†Supported by a Habilitationsstipendium of the Deutsche Forschungsgemeinschaft.

it is of interest to know which Böttcher functions are transcendental and which are algebraic. We start with some examples of algebraic Böttcher functions.

1) f is linear, that is, $f(z) = \alpha z + \beta$, where $\alpha, \beta \in \mathbf{C}, \alpha \neq 0$. Then, for any polynomial p, there exists a unique polynomial q satisfying (1). In this case, we say that p and q are *linearly conjugate.*

2) $p(z) = M_d(z)$ and $q(z) = T_d(z)$, where $M_d(z) = z^d$ and where $T_d(z) = 2^{d-1}z^d + \ldots$ denotes the d–th Chebychev polynomial, that is, $\cos dz = T_d(\cos z)$. Here we find the Böttcher functions

$$f(z) = \frac{1}{2}(\rho z + \frac{1}{\rho z}), \tag{2}$$

where ρ runs through E_{d-1}, the set of $(d-1)$–th roots of unity.

3) $p(z) = M_d(z)$ and $q(z) = -T_d(z)$. We have the Böttcher functions (2), but now with those ρ satisfying $\rho^{d-1} = -1$.

4) Since inverse functions of algebraic functions and compositions of algebraic functions are algebraic, it follows from Examples 2 and 3 that the Böttcher functions with respect to $T_d(z)$ and $-T_d(z)$ are also algebraic. In fact, we have

$$f(z) = \frac{1}{2}\left((\rho + \frac{1}{\rho})z + (\rho - \frac{1}{\rho})\sqrt{z^2 - 1}\right), \tag{3}$$

where $\rho^{d-1} = -1$. Of course, if d is even, then $f(z) = -z$ is among these functions.

5) Similarly as in Example 4, the Böttcher functions for $p(z) = q(z) = T_d(z)$ are algebraic. Here the identity is always a Böttcher function, and if d is odd, so is $f(z) = -z$. The other Böttcher functions are nonlinear and given by (3), where $\rho \in E_{d-1} \setminus \{1, -1\}$.
It is easily seen that these are also the Böttcher functions for $p(z) = q(z) = -T_d(z)$.

Our first result is that the above examples provide a list of all algebraic Böttcher functions, except for permutation of p and q and combination with linear conjugations.

Theorem 1. *Let p and q be polynomials of degree $d \geq 2$ and let f be an algebraic Böttcher function with respect to p and q. Then f is linear or both p and q are linearly conjugate to M_d, T_d, or $-T_d$, where $M_d(z) = z^d$ and where T_d is the d–th Chebychev polynomial.*

It is clear from Theorem 1 and the above examples that for fixed p and q there cannot be transcendental as well as nonlinear algebraic Böttcher functions.

Examples 4 and 5 show that one may have linear as well as nonlinear algebraic Böttcher functions. One can check that $p(z) = q(z) = z^d + 1$ allows only one linear Böttcher function, namely $f(z) = z$, while all others are transcendental by our theorem.

Now we are able to prove the following transcendency result for the values of Böttcher functions.

Theorem 2. *Let p and q be polynomials of degree $d \geq 2$ having algebraic coefficients. Suppose that at least one of them is not linearly conjugate to M_d, T_d, or $-T_d$. Let f be a nonlinear Böttcher function with respect to p and q and suppose that f is defined and analytic in a punctured neighborhood G of ∞ such that $p(G) \subset G$ and $p^m|_G \to \infty$ as $m \to \infty$, where p^m denotes the m–th iterate of p. Then $f(\alpha)$ is transcendental for any algebraic $\alpha \in G$.*

We remark that $G = \{z \mid |z| > R\}$ satisfies the hypotheses of Theorem 1 for sufficiently large R.

The proof of Theorem 1 is based on the following general result concerning the algebraic solutions of the functional equation

$$f(p(z)) = Q(z, f(z)), \tag{4}$$

where $Q(z, y)$ is a rational function.

We should remark that $a \in \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is called a singularity of f if and only if a is an algebraic branch point of f. (Thus poles are not considered as singularities.)

Theorem 3. *Let p be a polynomial of degree $d \geq 2$ and let Q be a rational function in the variables z and y. Suppose that f is an algebraic function satisfying the functional equation (4). If f is not a rational function, then one of the following conditions is satisfied.*

(i) f has exactly one finite singularity σ and another singularity at ∞. Furthermore, there is an integer $t \geq 2$ such that $f(\varphi(z^t))$ with $\varphi(z) = z + \sigma$ is a rational function.

(ii) f has exactly two finite singularities σ_1 and σ_2. Furthermore, p is linearly conjugate to T_d or $-T_d$ and the conjugation map is given by $\varphi(z) = \frac{1}{2}((\sigma_1 - \sigma_2)z + (\sigma_1 + \sigma_2))$. There is an integer $t \geq 1$ such that $f(\varphi(\frac{1}{2}(z^t + z^{-t})))$ is a rational function.

Remarks. 1) The following examples show that the situations (i) and (ii) mentioned in Theorem 3 may in fact both occur. Let $d \in \mathbf{Z}$, $d \geq 2$ and $r \in \mathbf{Q} \setminus \mathbf{Z}$. Then $f_1(z) = z^r \notin \mathbf{C}(z)$ is an algebraic function having its only finite singularity at $z = 0$. Clearly, f_1 satisfies the linear functional equation $f_1(z^d) = f_1(z)^d$. Another functional equation satisfied by f_1 is $f_1(z^d) = z^n f_1(z)$, which is of the form (4) if $n = (d-1)r \in \mathbf{Z}$.
To get an example for the situation (ii) one can take the nonlinear Böttcher functions from the Examples 3, 4, and 5. Here one may take $t = 1$. Another type of examples is constructed as follows. Let d and r be as above and take $f_2(z) = (z + \sqrt{z^2 - 1})^r$. Then we have $f_2(T_d(z)) = f_2(z)^d$. Furthermore, $f_2(\frac{1}{2}(z^t + z^{-t}))$ is a rational function for any multiple t of the denominator of r.

2) In [MFP] Mendès–France and van der Poorten discussed some examples of non–rational algebraic functions satisfying functional equations of the form (4) with $Q(z,y) \in \mathbf{C}(z)[y]$ of degree 1 in y. Theorem 3 explains why the functions of their Examples 3.1 – 3.5 have only two singularities and why their transformations $\varphi(x)$, which play the role of our $p(z)$, are always linearly conjugate to a Chebychev polynomial.

3) The question whether the solutions of functional equations of a similar type are algebraic or transcendental has been considered earlier by Mahler [M], Ostrowski [O], Loxton and van der Poorten [LP], Kubota[K], Gramain, Mignotte and Waldschmidt [GMW], and Nishioka [N]. Their analysis, however, was restricted to linear functional equations or to suitable n–dimensional generalizations of the special transformation $p(z) = z^d$.

II. Applications

The Mandelbrot set M is defined to be the set of all $c \in \mathbf{C}$ such that the Julia set of $p_c(z) = z^2 + c$ is connected. Equivalently,

$$M = \{c \mid p_c^n(0) \not\to \infty \text{ as } n \to \infty\}.$$

Douady and Hubbard ([DH1], [DH2, § 8.1], see also [Bd, § 9.10], [S1, § 6.2]) have shown that M is connected. In fact, they constructed a conformal map

$$\Phi(z) = z + c_0 + \frac{c_1}{z} + \dots$$

from the complement of M onto $\{z \mid |z| > 1\}$. This map was given by $\Phi(c) = \varphi_c(c)$, where φ_c is the unique Böttcher function with respect to $z^2 + c$ and z^2. From Theorem 2 we obtain

Corollary 1. *$\Phi(\alpha)$ is transcendental for all algebraic $\alpha \in \mathbf{C} \setminus M$.*

Of course, the analogous result holds for the inverse function Ψ of Φ, that is, $\Psi(\alpha)$ is transcendental for algebraic α, $|\alpha| > 1$. It is a well-known open problem whether the boundary of M is locally connected or, equivalently, whether $\Psi(z)$ has a continuous extension to $|z| = 1$ (see [DH1], [DH2]). Douady and Hubbard have shown that $c_\theta = \lim_{r\to 1} \Psi(re^{2\pi i\theta})$ exists if θ is rational. It is of interest to note that c_θ is algebraic if $\theta = r/s$, where r is an even and s is an odd integer. This does, however, by no means exclude the possibility of a continuous extension of $\Psi(z)$ to $|z| = 1$. An example of a function analytic for $|z| < 1$ and continuous for $|z| \le 1$ which takes on transcendental values at all algebraic points α with $|\alpha| < 1$ but fails to have this property for all roots of unity is given in [Be].

Our second application concerns a question of Golomb [G]. For $r = 0, 1, 2, \dots$ he defines the sequence $\{\beta_n^{(r)}\}_{n\in\mathbf{N}}$ by $\beta_1^{(r)} = 1 + \frac{r}{2}$ and

$$\beta_{n+1}^{(r)} = (\beta_n^{(r)})^2 + c$$

for $n \in \mathbf{N}$, where $c = \frac{r}{2}(1 - \frac{r}{2})$. Since the successive terms of the sequence are approximately obtained by successive squaring, Golomb asks whether there is a positive real number $\Theta(r)$ with the property $\Theta(r)^{2^n} \sim \beta_{n+1}^{(r)}$. He shows that this is true for

$$\Theta(r) = \beta_1^{(r)} \prod_{i=1}^{\infty} (1 + \frac{c}{(\beta_i^{(r)})^2})^{2^{-i}}.$$

Obviously, $\Theta(0) = 1$ and $\Theta(2) = 2$. Golomb poses the question for which values of $r \neq 0, 2$ the number $\Theta(r)$ is transcendental. It was shown in [AS, p. 435] and [FG, p. 456] that $\Theta(4) = (3 + \sqrt{5})/2$. A final answer to Golomb's question is given by

Corollary 2. $\Theta(r)$ *is transcendental for* $r \neq 0, 2, 4$.

We remark that $\Theta(r)$ may also be defined for any complex r with $\beta_n^{(r)} \to \infty$ for $n \to \infty$. Again we find that $\Theta(r)$ is transcendental for such algebraic $r \neq 2, 4$.

III. Proofs

Proof of Theorem 1. Let p, q, and f be as in the hypotheses of the theorem. Because $f(z) \sim \lambda z$, $\lambda \neq 0$, as $z \to \infty$, there exists a branch of f^{-1} analytic in a neighborhood of ∞ such that $f^{-1}(z) \sim z/\lambda$ as $z \to \infty$. From (1) we deduce that

$$f^{-1}(q(z)) = p(f^{-1}(z)). \tag{5}$$

Of course, by Weierstraß's principle on the permanence of functional equations, (1) and (5) remain valid under analytic continuation of f and f^{-1}.

Suppose now that f is nonlinear. Then f or f^{-1} has a singularity. We see from (1) and (5) that replacing f by f^{-1} corresponds to interchanging the roles of p and q. Therefore we may assume without loss of generality that f has a singularity.

Since ∞ is not a singularity of f, it is clear that f satisfies condition (ii) of Theorem 3. Hence p is linearly conjugate to T_d or $-T_d$. If f^{-1} has also singularities, this argument may be repeated with q instead of p thus showing that q is also linearly conjugate to T_d or $-T_d$. Hence we are left with the case that f^{-1} does not have singularities, that is, f^{-1} is rational. Clearly, f^{-1} is nonlinear and this implies that, besides the simple pole at ∞, f^{-1} must have at least one finite pole. From (5) we deduce that its set of poles is completely invariant with respect to q. Thus the poles of f^{-1} are exceptional in the sense of [Bd, Definition 4.1.1]. Now [Bd, Theorem 4.1.2] implies that q is linearly conjugate to M_d. ∎

Proof of Theorem 2. Suppose that p, q, f, G, and α satisfy the assumptions of the theorem. Let $\beta \in \mathbf{C}$ be a fixed point of p. Since f has a simple pole at ∞, we introduce $\tilde{f}(z) = f(z)/(z - \beta)$. Define $U = G \cup \{\infty\}$ and $Tz = p(z)$. We show that $\tilde{f}$, U, and T satisfy the hypotheses of the main theorem in [Be].

Clearly, T is meromorphic in the neighborhood U of $\omega = \infty$, ω is a fixed point of T of order d and $T(U) \subset U$. Since $p^m|_U \to \infty$ as $m \to \infty$, we have $\beta \notin U$. Thus $\tilde{f}$ is holomorphic in U.

The functional equation (1) leads to a recursion formula for the Laurent coefficients of $f(z)$ at $z = \infty$. Hence the algebraicity of the coefficients of p and q

guarantees that f has algebraic Laurent coefficients, too. Thus the power series expansion of $\tilde{f}$ at ω has only algebraic coefficients. Theorem 1 allows to conclude the transcendency of $f(z)$ over $\mathbf{C}(z)$ from the assumption that not both, p and q, are linearly conjugate to M_d, T_d, or $-T_d$.

Let
$$\tilde{P}(z,u,w) = (p(z)-\beta)w - q((z-\beta)u).$$
$\tilde{P}$ is a polynomial with algebraic coefficients and it is easily seen that
$$\tilde{P}(z,\tilde{f}(z),\tilde{f}(Tz)) = 0$$
for $z \in U$.

Now, α is an algebraic number with $T^m\alpha \to \infty$ for $m \to \infty$. Since p is a polynomial, we have $T^m\alpha \neq \infty$ for $m = 0, 1, \ldots$. Furthermore, $T^m\alpha \neq \beta$ implies that $\tilde{P}(T^m\alpha, \tilde{f}(T^m\alpha), w)$ is not identically zero.

Using the terminology of [Be] we have $h(T) = h_w(\tilde{P}) = d$, $d(T) = d_w(\tilde{P}) = 1$, and $\mathrm{ord}_\omega T = d$. Thus condition (2) of the main theorem in [Be] is also satisfied. We conclude that $\tilde{f}(\alpha)$ and hence $f(\alpha)$ is a transcendental number. ∎

Proof of Theorem 3. Let f, p, and Q be as required in the statement of the theorem. Since f is algebraic, but not rational, it has only finitely many, say $n \geq 1$, singularities in $\hat{\mathbf{C}}$. This already implies $n \geq 2$. Otherwise we would have an algebraic function with exactly one singularity, say a, and f would be single–valued in $\hat{\mathbf{C}} \setminus \{a\}$ by the monodromy theorem, a contradiction.

Now, suppose that $p(\sigma) \in \mathbf{C}$ is a singularity of f. Suppose also that $p'(\sigma) \neq 0$. Then σ is a singularity of $f \circ p$ and hence, by the functional equation (1), σ is a singularity of f.

Let $\sigma_1, \ldots, \sigma_n$ be the finite singularities of f. Then there are nd inverse images of these singularities under p, counted according to multiplicity. These inverse images must be singularities of f or zeros of p'. Let $\omega_1, \ldots, \omega_r$ be the zeros of p' and denote by $\nu_1, \ldots, \nu_r$ their respective multiplicities. Then $r \leq d-1$ and $\sum_{j=1}^r \nu_j = d-1$. The above observations yield
$$dn \leq n + \sum_{j=1}^{r} (\nu_j + 1) = n + d - 1 + r.$$
Hence
$$n \leq 1 + \frac{r}{d-1} \leq 2. \tag{6}$$
Thus f has either one or two finite singularities. We treat these two case separately.

(A) f has exactly one finite singularity, say σ. Then $\tilde{f}(z) = f(z+\sigma)$ has its only finite singularity at $z = 0$. Since $\tilde{f}$ must have a second singularity in $\hat{\mathbf{C}}$, it has the singularities 0 and ∞. But, as $\tilde{f}$ is algebraic, there exist integers $t_0, t_\infty \geq 2$ such that $\tilde{f}(z^{t_a})$ is regular at $z = a$ for $a = 0, \infty$. Let $t = t_0 t_\infty$. Clearly, $\tilde{f}(z^t)$ has no singularities in $\hat{\mathbf{C}}$, hence it must be a rational function.

(B) f has exactly two finite singularities, say σ_1 and σ_2. From (6) we conclude $r = d-1$, that is, all zeros of p' are simple. Furthermore, we have

$$p(\omega_1), \ldots, p(\omega_r),\ p(\sigma_1),\ p(\sigma_2) \in \{\sigma_1, \sigma_2\}.$$

Let $\varphi(z) = \frac{1}{2}((\sigma_1 - \sigma_2)z + (\sigma_1 + \sigma_2))$, $\tilde{f}(z) = f(\varphi(z))$, $\tilde{p}(z) = \varphi^{-1}(p(\varphi(z)))$, and $\tilde{Q}(z,y) = Q(\varphi(z), y)$. It is easily seen that

$$\tilde{f}(\tilde{p}(z)) = \tilde{Q}(z, \tilde{f}(z))$$

and that $\tilde{f}$ has the two finite singularities ± 1. Since all zeros of $\tilde{p}'$ are simple, ± 1 are algebraic branch points of $\tilde{f}$ of order 2.

Following an argument of Steinmetz (see [S1, p. 143] or [S2]), we consider the polynomials $d^2(\tilde{p}(z)^2 - 1)$ and $(z^2-1)\tilde{p}'(z)^2$. These polynomials have the same leading coefficients and the same zeros, and hence are equal. Differentiation of the identity $d^2(\tilde{p}(z)^2 - 1) = (z^2-1)\tilde{p}'(z)^2$ yields

$$(z^2-1)\tilde{p}''(z) + z\tilde{p}'(z) - d^2\tilde{p}(z) = 0,$$

i.e. the differential equation for the Chebychev polynomials. The polynomial solutions of this differential equation are given by $\tilde{p}(z) = cT_d(z)$, where $c \in \mathbf{C}$. Since $T_d(1) = 1$ and $\tilde{p}(1) = \pm 1$, we have $\tilde{p} = T_d$ or $\tilde{p} = -T_d$.

From the fact that $(z + z^{-1})/2$ has the fixed points ± 1, each of order 2, and that ± 1 are also algebraic branch points of $\tilde{f}$ of order 2, we conclude that they are regular points of $g(z) = \tilde{f}((z+z^{-1})/2)$. Furthermore, g satisfies the functional equation

$$g(z^d) = Q((z+z^{-1})/2, g(z))$$

and has either no singularity or exactly one finite singularity at $z = 0$. Hence g is either rational or, by the results established in part (A), there exists an integer $t \geq 2$ such that $g(z^t) = f(\varphi((z^t + z^{-t})/2))$ is a rational function. ∎

Proof of Corollary 1. The polynomial $p_{-2} = z^2 - 2$ is linearly conjugate to $T_2(z) = 2z^2 - 1$, and also to $-T_2(z)$. For $c \in \mathbf{C} \setminus \{0, -2\}$, p_c is not linearly conjugate to M_2, T_2, or $-T_2$. Suppose now that $\alpha \in \mathbf{C} \setminus M$ is algebraic. Because $\{0, -2\} \subset M$ we conclude that p_α is not linearly conjugate to M_2, T_2, or $-T_2$. We define

$$A(\infty) = \{z \in \mathbf{C} \mid \lim_{n\to\infty} p_\alpha^n(z) = \infty\}$$

and

$$G = \{z \in A(\infty) \mid g(z) > g(0)\},$$

where g denotes the Green's function of $A(\infty)$. It follows from the analysis in [Bd, § 9.10] that the Böttcher function φ_α with respect to p_α and M_2 is defined and analytic in G. Moreover, $\alpha \in G$, $p_\alpha(G) \subset G$, and $p_\alpha^n|_G \to \infty$ as $n \to \infty$. Thus, by Theorem 2, $\varphi_\alpha(z)$ is transcendental for each algebraic $z \in G$. In particular, $\Phi(\alpha) = \varphi_\alpha(\alpha)$ is transcendental. ∎

Proof of Corollary 2. Let z be a complex number with sufficiently large absolute value. Let $p(z) = z^2 + c$ with $c = \frac{r}{2}(1 - \frac{r}{2})$. We define

$$f(z) = z \prod_{i=0}^{\infty} (1 + \frac{c}{p^i(z)^2})^{2^{-i-1}},$$

where the branch of the 2^{i+1}–th root is chosen in such a way that $(1 + cz^{-1})^{2^{-i-1}}$ takes on the value 1 at $z = \infty$.

It is easily seen that $f(z)$ satisfies the functional equation

$$f(p(z)) = f(z)^2 \tag{7}$$

and has a simple pole at ∞. Thus f has to be the unique Böttcher function with respect to p and M_2.

Now, we are going to apply Theorem 2. Since $p(z) = z^2 + c$ and M_2 are linearly conjugate only for $c = 0$, $f(z)$ is nonlinear in all other cases. Furthermore, let G be a suitable neighborhood of infinity with $p(G) \subset G$ and $p^n|_G \to \infty$ for $n \to \infty$. (Since ∞ is an attractive fixed point of p, such a neighborhood exists.) Clearly, $p^m(1 + \frac{r}{2}) \in G$ if m is sufficiently large. Thus, by (7), the assertion of the corollary is true, if we can show the the transcendency of $f(p^m(1+\frac{r}{2}))$ in the case that $p(z)$ is not linearly conjugate to M_2 or T_2. But that is now immediate from Theorem 2. ∎

Remark. Our corollaries concern only the special case $q(z) = z^d$. It is also possible to give proofs of Corollaries 1 and 2 based on the results established in [GMW, § 4], where the arithmetic properties of the solutions of the functional equation

$$f(z^d) = af(z)^d + bz^h$$

are studied. Here a and b are complex numbers, h is an integer, and f is a power series at $z = 0$. It turns out that such a proof would require the study of suitable modifications of the inverses of the Böttcher functions used in the proofs given above.

Acknowledgement. We are grateful to Professor S. W. Golomb for drawing our attention to the papers [AS] and [FG].

References

[AS] *A. V. Aho, N. J. A. Sloane*, Some doubly exponential sequences, Fibonacci Quart. **11**, (1973), 429–437.

[Bd] *A. Beardon*, Iteration of Rational Functions, Springer, New–York 1991.

[Be] *P.–G. Becker*, Transcendence of the values of functions satisfying generalized Mahler type functional equations (to appear in: J. reine angew. Math.).

[DH1] *A. Douady, J. Hubbard*, Itération des polynômes quadratiques complexes, C. R. Acad. Sci. Paris **294** (1982), 123–126.

[DH2] *A. Douady, J. Hubbard*, Étude dynamique des polynômes complexes I & II, Publ. Math. Orsay 84-02 (1984) & 85-04 (1985).

[FG] *J. N. Franklin, S. W. Golomb*, A function–theoretic approach to the study of nonlinear recurring sequences, Pac. J. Math. **56**, (1975), 455–468.

[G] *S. W. Golomb*, On certain nonlinear recurring sequences, Amer. Math. Monthly **70** (1963), 403–405.

[GMW] *F. Gramain, M. Mignotte, M. Waldschmidt*, Valeurs algébriques de fonctions analytiques, Acta Arith. **47** (1986), 97–121.

[K] *K. K. Kubota*, On the algebraic independence of holomorphic solutions of certain functional equations and their values, Math. Ann. **227** (1977), 9–50.

[LP] *J. H. Loxton, A. J. van der Poorten*, A class of hypertranscendental functions, Aequationes Math. **16** (1977), 93–106.

[M] *K. Mahler*, Über das Verschwinden von Potenzreihen mehrerer Veränderlicher in speziellen Punktfolgen, Math. Ann. **103** (1930), 573–587.

[MFP] *M. Mendès–France, A. J. van der Poorten*, From geometry to Euler identities, Theor. Comput. Sci. **65** (1989), 213–220.

[N] *K. Nishioka*, Algebraic function solutions of a certain class of functional equations, Arch. Math. **44** (1985), 330–335.

[O] *A. Ostrowski*, Über einige Verallgemeinerungen des Eulerschen Produktes $\prod_{v=0}^{\infty}(1+x^{2^v}) = 1/(1-x)$, in: *Ostrowski, Collected Mathematical Papers*, Birkhäuser, Basel 1984.

[S1] *N. Steinmetz*, Rational Iteration, Walter de Gruyter, Berlin 1993.

[S2] *N. Steinmetz*, Dendrites and Jordan arcs (preprint 1992).

Paul–Georg Becker
Mathematisches Institut der Universität zu Köln
D–50923 Köln
Germany

Walter Bergweiler
Lehrstuhl II für Mathematik der RWTH Aachen
D–52056 Aachen
Germany