ON THE ZEROS OF THE SECOND
DERIVATIVE OF REAL ENTIRE FUNCTIONS

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1. Introduction and Preliminaries.

An entire (or meromorphic) function $f(z)$ is called real, if for all $z \in \mathbb{C}$, $f(\mathbb{C}) = \overline{f(z)}$.

The study of real entire functions all of whose zeros are real has a long history. [See [SS] for a brief account.] In particular A. Wiman raised the question whether for such functions of order $> 2$, $f''(z)$ always has non-real zeros. He formulated a precise conjecture about the number $c(\rho)$ of non-real zeros for an $f(z)$ of finite order $\rho$, giving an explicit formula for $c(\rho)$ which showed that $c(\rho) \to \infty$ as $\rho \to \infty$.

Wiman’s conjecture was made in 1914, it remained open until the work of Sheil–Small in 1989 [SS].

For the case of infinite order Sheil–Small made the

**Conjecture.** Let $f(z)$ be a real entire function of infinite order all of whose zeros are real. Then $f''(z)$ has infinitely many non-real zeros.

The conjecture is known to be true, if one of the following conditions holds:

1. All zeros of $f'(z)$ are real. [HW]
2. $\limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r} = \infty$. [LO]

In this paper we shall add to these results the

**Theorem.** The conjecture is true, if $f(z)$ has only a finite number of zeros.

This was already proved by Sheil–Small for functions of the form $Q(z) \exp(\varepsilon z + P(z))$, [SS], where $P$ and $Q$ are polynomials [SS].

We write $C$ for a positive number which depends only on the choice of the function $f(z)$. The value of $C$ can vary from one occurrence to the next.

From now on $f(z)$ denotes a real entire function of finite order all of whose zeros are real, which has only a finite number of zeros and whose second derivative has only a finite number of non-real zeros. Also $f'(z)$ has a non-real zero.

We shall show that such an $f(z)$ can not exist. We shall need the ‘Levin–Ostrovskii representation’ [LO]

$$\frac{f''(z)}{f(z)} = \psi(z)\varphi(z),$$

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*The authors gratefully acknowledge support by the Alexander von Humboldt Foundation and the National Science Foundation.
where $\psi(z)$ is a real meromorphic function with poles at the zeros of $f(z)$ which satisfies

$$\Im \psi(z) > 0$$

in the half–plane

$$H = \{ z = x + iy \mid y > 0 \}.$$  

The function $\varphi(z)$ is a real entire function. By (2) we may assume that

$$\limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r} < \infty.$$  

It is known that under this condition

$$(4) \quad \log M(r, \varphi) = 0(r \log r) \quad (r \to \infty)$$

and that there are positive constants $c_1$, $c_2$ such that

$$(5) \quad c_1 \frac{|\sin \theta|}{r} < |\psi(re^{i\theta})| < c_2 \frac{r}{|\sin \theta|} \quad (r > 0, -\pi < \theta < \pi).$$

[For details see [HW; p. 499].]  

Also, since $f(z)$ has only a finite number of zeros,  

$$(6) \quad \psi(z) = 0(|z|) \quad (|z| > R).$$

[See HW; (1.17); p. 500]

2. **Lemmas.**

*Notation.* For $0 < \delta < \pi/2$ we put

$$S_{\delta,R} = \{ z \mid \delta \leq \arg z \leq \pi - \delta, \ |z| \geq R \}.$$ 

For a function $g(z)$ defined in a domain $D$

$$M(r, g, D) = \sup_{z \in D, |z|=r} |g(z)|.$$ 

We define

$$F(z) = z - \frac{f(z)}{f'(z)} = U(z) + iV(z).$$

Then

$$F'(z) = f(z)f''(z)/f'2(z),$$

so that, for suitably large $R$,

$$F'(z) \neq 0 \quad (|z| \geq R).$$
Lemma 1. [Slight extension of SS, Theorem 4, with the same proof.] $F(z)$ has an asymptotic path $L$ in the upper half plane

$$H = \{z = x + iy \mid y > 0\}$$

on which $F(z)$ tends to $\alpha \in H$.

Lemma 2. [S, Theorem 5.1] Let $D \subset \mathbb{C}$ be a domain. Let $\mathcal{G}$ be the family of functions $g(z)$ meromorphic in $D$ and satisfying $gg'' \neq 0$ in $D$.

Then

$$\mathcal{H} = \{g'(z)/g(z) \mid g \in \mathcal{G}\}$$

is a normal family in $D$.

Lemma 3. If

$$|\text{re}i\theta f'(\text{re}i\theta)/f(\text{re}i\theta)| < A \quad (>1)$$

for a $\theta$ satisfying $\delta \leq \theta \leq \pi - \delta$ and an $r > 2R$, then for $\zeta$ in

$$K = \{\zeta \mid \frac{1}{2} \leq |\zeta| \leq 2, \; \delta \leq \text{arg} \zeta \leq \pi - \delta\}$$

$$|r\zeta f'(\zeta)/f(r\zeta)| < C_1 A,$$

where $C_1$ is a constant depending only on the choice of $f$ and of $\delta$.

Proof: Suppose the lemma were false. Then we can find an increasing sequence of positive numbers $a_n$ and a sequence of complex numbers $\zeta_n \in K$ such that

(7) $$|a_n\zeta_n f'(a_n\zeta_n)/f(a_n\zeta_n)| > n,$$

while

(8) $$|a_n f'(a_n e^{i\theta_n})/f(a_n e^{i\theta_n})| < A.$$

By going over to a subsequence, if necessary, we may suppose that $\zeta_n \to \zeta_0$ ($n \to \infty$). And by applying Lemma 2 with $D = S_{\delta/2,R}$ and $\mathcal{G} = \{f(a\zeta) \mid a > 1\}$ we may also suppose that either $h_n(\zeta) = a_n f'(a_n\zeta)/f(a_n\zeta)$ tends uniformly to a holomorphic limit theorem $h(\zeta)$ for $\zeta \in K$ or that $h_n(\zeta)$ tends uniformly to $\infty$ in $K$. The second possibility is excluded by (8). But then it follows from the uniform convergence of the $h_n(\zeta)$ that

$$h_n(\zeta_n) \to h(\zeta_0) \quad (n \to \infty),$$

contradicting (7). \qed
Lemma 4. If the asymptotic path $L$ of Lemma 1 lies in $S_{\delta/2,K}$ for some $\delta \in (0, \pi)$, then the conclusion of the Theorem holds.

Proof: Since $F(z) = z - f(z) = \alpha + o(1)$ as $z \to \infty$ on $L$, $z f'(z) / f(z) = 1 + o(1)$ on $L$. Therefore the hypothesis of Lemma 3 is satisfied for all $r > 2R$ and it follows from Lemma 3 that, in $S_{\delta,R}$, 

$$|zf'(z)/f(z)| < C.$$  

By (3), (5) and (9) 

$$|\varphi(z)| = |f'(z)/f(z)\psi(z)| < C \quad (z \in S_{\delta,R}).$$  

Since $\varphi$ is real entire (10) remains true, if $z$ is replaced by $\overline{z}$. In particular (10) holds for $z = re^{i\beta}$ ($\beta = e^i \frac{\pi}{4}, e^{3i} \frac{\pi}{4}$).

Therefore, by (4), (10) and a well–known Phragmén–Lindelöf Theorem 

$$|\varphi(z)| < C.$$  

And so, by Liouville’s Theorem, 

$$\varphi(z) = \text{constant}.$$  

But this contradicts the fact that $f'(z)$ has a non–real zero.

Lemma 5. [Special case of T, Theorem III.68] Let $\Gamma_1$ and $\Gamma_2$ be two, non–intersecting Jordan curves tending to $\infty$. Let $E \subset \mathbb{C}$ be a domain such that for sufficiently large $R$ 

$$\partial E \cap \{z \mid |z| \geq R\} = (\Gamma_1 \cap \Gamma_2) \cap \{z \mid |z| \geq R\}.$$  

Let $t\Theta(t)$ be the linear measure of the intersection of $E$ with $|z| = t$.

Then, if $h(z)$ is holomorphic in $\overline{E}$, and $A > e$, 

$$|h(z)| < A \quad (z \in \partial E)$$  

implies that either 

$$|h(z)| < A \quad (z \in E)$$  

or 

$$Q(r, h, E) = \pi \int_0^r \frac{dt}{t\Theta(t)} = \log \log M(r, h, E)$$  

satisfies 

$$\limsup_{r \to \infty} Q(r, h, E) < \infty.$$  

Lemma 6. [T, Theorem VIII.14] Let $\Gamma_1$, $\Gamma_2$ and $E$ have the same meaning as in Lemma 5.

Let $g(z)$ be holomorphic in $E$ and continuous and bounded in $\overline{E} \setminus \{\infty\}$.

If $g(z) \to c_j$ ($j = 1, 2$) as $z \to \infty$ along $\Gamma_j$, then $c_1 = c_2$. 
3. Completion of the Proof of the Theorem.

The rays

\[ \Lambda_1 = \left\{ z \mid \arg z = \frac{\pi}{3}, \ |z| \geq R \right\}, \ \Lambda_2 = \left\{ z \mid \arg z = \frac{4\pi}{3}, \ |z| = R \right\} \]

divide

\[ H_R = \{ z \in H \mid |z| \geq R \} \]

into three parts \( S = S_{\frac{2}{3}R} \) and \( \Delta_1, \Delta_2 \), sectors adjacent to the positive and negative real axis, respectively.

We still need to prove the Theorem in the case that there are arbitrarily large \( r \) such that there exists \( z \in L \) with \( |z| = r \) and \( z \in \Delta_1 \) or \( z \in \Delta_2 \) [Lemma 4].

Let \( A > e \) be an upper bound for \( |zf'(z)/f(z)| \) on \( L \).

Our first step is the construction of a path \( L' \subset H \) on which

\[ (12) \quad \left| \frac{zf'(z)}{f(z)} \right| < CA = C \]

and which divides \( H_R \) into two domains \( A_1 \) and \( A_2 \) in one of which, at least, (11) does not hold when \( h = zf'(z)/f(z) \), \( E = A_1 \) or \( A_2 \). We shall prove that (12) holds in \( A_1 \) or in \( A_2 \).

We then show that the same is true of one of the domains into which \( L \) and \( \bar{L} \) divide \( H_R \). The Theorem follows by an application of Lemma 6 [\( L = \Gamma_1, \ \bar{L} = \Gamma_2 \), \( c_1 = \alpha, \ c_2 = \bar{\alpha} \neq c_1 \).]

**Construction of \( L' \).** The rays \( \Lambda_1 \) and \( \Lambda_2 \) divide \( L \) into 3 parts:

\[ L_0 = \{ z \in L \cap S \}, \quad L_1 = \{ z \in L \cap \Delta_1 \}, \quad L_2 = \{ z \in L \cap \Delta_2 \}. \]

If \( L_0 \) is bounded, we choose \( L' = L \).

If \( L_0 \) is unbounded, let

\[ \tilde{L} = \left\{ z \in \Lambda_1 \mid \exists z' \in L_0, \ \frac{1}{2}|z| < |z'| < 2|z| \right\}. \]

By Lemma 3, (9) holds on \( \tilde{L} \) and for all \( z \) in \( S \) with \( |z|e^{\pi i/5} \in \tilde{L} \).

If the complement of \( \tilde{L} \) on \( \Lambda_1 \) is bounded, choose \( L' = \tilde{L} \).

If the complement of \( \tilde{L} \) is unbounded, \( \tilde{L} \) is the union of components each one of which is a straight line segment on \( \Lambda_1 \) of length \( \geq R \). If \( r\pi/5 \notin \tilde{L} \), then all points \( z \in L \) with \( |z| = r \) must lie on “intervals” of \( L \) belonging to \( L_1 \cup L_2 \). Choose one of these intervals, \( I \), with endpoints \( z_1, z_2, |z_1| < |z_2| \). Discard all the others. Both endpoints of \( I \) may be on \( \Lambda_1 \) or on \( \Lambda_2 \).

If they are on \( \Lambda_1 \), they both belong to \( \tilde{L} \), by the definition of \( \tilde{L} \). If they are both on \( \Lambda_2 \), we join the segments \( |z| = |z_1|, \ z \in S \), and \( |z| = |z_2|, \ z \in S \), to \( I \), forming \( I' \). The curve \( I' \) joins two components of \( \tilde{L} \). We can now describe \( L' \): Choose an \( I \) or \( I' \) with an endpoint as close to the origin as possible. Move along this \( I \) (or \( I' \)) to its endpoint (on \( \Lambda_1 \)) in \( \tilde{L} \). Move along \( \tilde{L} \) in the direction of increasing \( |z| \) to the first endpoint of an \( I \) or \( I' \). Then move along this \( I \) (\( I' \)) to another component of \( \tilde{L} \) and so on . . . . On \( L' \) \( \arg z \) lies either in \( (0, \frac{\pi}{3}) \) or in \( (\frac{4\pi}{3}, \pi) \), except for a denumerable set of arcs \( |z| = \) constant belonging to the \( I' \) which occur in the construction.
Let $A_1$ be that domain in $\{ z \mid |z| \geq R \}$ bounded by $L'$ and its conjugate complex curve $\overline{L'}$ which contains $|R, \infty)$, $A_2$ the domain bounded by $L'$ and $\overline{L'}$ which contains $[-R, -\infty)$. Let $t\Theta_j(t)$ be the linear measure of the intersection of $|z| = t$ with $A_j$ $(j = 1, 2)$. By construction $\min(\Theta_1(t), \Theta_2(t)) \leq \frac{2\pi}{5}$ and so

$$\pi \int \frac{dt}{t\Theta_1(t)} + \pi \int \frac{dt}{t\Theta_2(t)} \geq \frac{5}{2} \int r/2R, > \frac{5}{2} \log r - C. \tag{13}$$

We apply Lemma 5 to

$$h(z) = \frac{f'(z)}{f(z)}, \quad E = A_j.$$

By (3), (4) and (6),

$$\log |h(z)| \leq C \log r \quad (r > R). \tag{14}$$

On $L' \cup \overline{L'}$ (12) holds.

By (13) [notation of Lemma 5] and (14)

$$Q(r, h, A_1) + Q(r, h, A_2) \geq \left( \frac{1}{2} - \varepsilon \right) \log r \quad (r > R),$$

so that (11) is false for at least one $A_j$. By Lemma 5, (12) holds in one $A_j$. Let $B$ be the domain bounded by $L \cup \overline{L}$ which contains such an $A_j$. In addition to $A_j$, $B$ contains all or part of sectors

$$\tilde{S} = \{ z \in S \cap \{ r_1 \leq |z| \leq r_2 \} \}$$

where the segment of $\Lambda_1$ with endpoints $r_j \epsilon i \pi / 5$ belongs to $\overline{L}$. As remarked above, (12) holds in $\tilde{S}$, so that (12) holds in $B$.

On $L$,

$$F(z) = \alpha + o(1)$$

as $z \to \alpha$ on $L$. Therefore

$$g(z) = z^2 \frac{f'(z)}{f(z)} - z = \frac{zF(z)}{z - F(z)} = \alpha + o(1)$$

as $z \to \infty$ on $L$. Since $g$ is an even holomorphic function in $A$,

$$g(z) \to \overline{\alpha} + o(1)$$

as $z \to \infty$ on $\overline{L}$.

In $B$ we apply Lemma 5 to

$$h(z) = g(z); \quad E = B.$$

Using $\Theta(t) \leq 2\pi$ and (12),

$$Q(r, g, A) > \frac{1}{2} \log r - O(\log \log r),$$

so that, by Lemma 5,

$$|g(z)| < C \quad (z \in A).$$

Now Lemma 6 leads to the contradiction $\alpha = \overline{\alpha}$.

This completes the proof.
References


