ON THE ZEROS OF THE SECOND DERIVATIVE OF REAL ENTIRE FUNCTIONS

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1. Introduction and Preliminaries.

An entire (or meromorphic) function $f(z)$ is called **real**, if for all $z \in \mathbb{C}$, $f(\overline{z}) = \overline{f(z)}$.

The study of real entire functions all of whose zeros are real has a long history. [See [**SS**] for a brief account.] In particular A. Wiman raised the question whether for such functions of order > 2, $f''(z)$ always has non–real zeros. He formulated a precise conjecture about the number $c(\rho)$ of non–real zeros for an $f(z)$ of finite order ρ, giving an explicit formula for $c(\rho)$ which showed that $c(\rho) \to \infty$ as $\rho \to \infty$.

Wiman's conjecture was made in 1914, it remained open until the work of Sheil–Small in 1989 [**SS**].

For the case of infinite order Sheil–Small made the

Conjecture. *Let $f(z)$ be a real entire function of infinite order all of whose zeros are real. Then $f''(z)$ has infinitely many non–real zeros.*

The conjecture is known to be true, if one of the following conditions holds:

(1) All zeros of $f'(z)$ are real. [**HW**]

(2) $\limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \infty$. [**LO**]

In this paper we shall add to these results the

Theorem. *The conjecture is true, if $f(z)$ has only a finite number of zeros.*

This was already proved by Sheil–Small for functions of the form $Q(z)\exp(ez + P(z))$, [**SS**], where P and Q are polynomials [**SS**].

We write C for a positive number which depends only on the choice of the function $f(z)$. The value of C can vary from one occurrence to the next.

From now on $f(z)$ denotes a real entire function of finite order all of whose zeros are real, which has only a finite number of zeros and whose second derivative has only a finite number of non–real zeros. Also $f'(z)$ has a non–real zero.

We shall show that such an $f(z)$ can not exist. We shall need the 'Levin–Ostrovski representation' [**LO**]

$$\frac{f'(z)}{f(z)} = \psi(z)\varphi(z), \tag{3}$$

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where $\psi(z)$ is a real meromorphic function with poles at the zeros of $f(z)$ which satisfies

$$\Im\, \psi(z) > 0$$

in the half–plane

$$H = \{z = x + iy \mid y > 0\}.$$

The function $\varphi(z)$ is a real entire function. By (2) we may assume that

$$\limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} < \infty.$$

It is known that under this condition

(4) $$\log M(r,\varphi) = 0(r\log r) \qquad (r \to \infty)$$

and that there are positive constants c_1, c_2 such that

(5) $$c_1\frac{|\sin\theta|}{r} < |\psi(rei\theta)| < c_2\frac{r}{|\sin\theta|} \quad (r > 0, -\pi < \theta < \pi).$$

[For details see [**HW**; p. 499].]

Also, since $f(z)$ has only a finite number of zeros,

(6) $$\psi(z) = 0(|z|) \quad (|z| > R).$$

[See **HW**; (1.17); p. 500]

2. Lemmas.

Notation. For $0 < \delta < \pi/2$ we put

$$S_{\delta,R} = \{z \mid \delta \le \arg z \le \pi - \delta,\ |z| \ge R\}.$$

For a function $g(z)$ defined in a domain D

$$M(r,g,D) = \sup_{z\in D, |z|=r} |g(z)|.$$

We define

$$F(z) = z - \frac{f(z)}{f'(z)} = U(z) + iV(z).$$

Then

$$F'(z) = f(z)f''(z)/f'2(z),$$

so that, for suitably large R,

$$F'(z) \ne 0 \quad (|z| \ge R).$$

Lemma 1. [Slight extension of **SS**, Theorem 4, with the same proof.] *$F(z)$ has an asymptotic path L in the upper half plane*

$$H = \{z = x + iy \mid y > 0\}$$

on which $F(z)$ tends to $\alpha \in H$.

Lemma 2. [**S**, Theorem 5.1] *Let $D \subset \mathbb{C}$ be a domain. Let $\mathcal{G}$ be the family of functions $g(z)$ meromorphic in D and satisfying $gg'' \neq 0$ in D.*

Then

$$\mathcal{H} = \{g'(z)/g(z) \mid g \in \mathcal{G}\}$$

is a normal family in D.

Lemma 3. *If*

$$|rei\theta f'(rei\theta)/f(rei\theta)| < A \quad (> 1)$$

for a θ satisfying $\delta \leq \theta \leq \pi - \delta$ and an $r > 2R$, then for ζ in

$$K = \left\{\zeta \mid \frac{1}{2} \leq |\zeta| \leq 2,\ \delta \leq \arg\zeta \leq \pi - \delta\right\}$$

$$|r\zeta f'(\zeta)/f(r\zeta)| < C_1 A,$$

where C_1 is a constant depending only on the choice of f and of δ.

Proof: Suppose the lemma were false. Then we can find an increasing sequence of positive numbers a_n and a sequence of complex numbers $\zeta_n \in K$ such that

$$|a_n\zeta_n f'(a_n\zeta_n)/f(a_n\zeta_n)| > n, \tag{7}$$

while

$$|a_n f'(a_n ei\theta_n)/f(a_n ei\theta_n)| < A. \tag{8}$$

By going over to a subsequence, if necessary, we may suppose that $\zeta_n \to \zeta_0$ $(n \to \infty)$. And by applying Lemma 2 with $D = S_{\delta/2,R}$ and $\mathcal{G} = \{f(a\zeta) \mid a > 1\}$ we may also suppose that *either* $h_n(\zeta) = a_n f'(a_n\zeta)/f(a_n\zeta)$ tends uniformly to a holomorphic limit theorem $h(\zeta)$ for $\zeta \in K$ *or* that $h_n(\zeta)$ tends uniformly to ∞ in K. The second possibility is excluded by (8). But then it follows from the uniform convergence of the $h_n(\zeta)$ that

$$h_n(\zeta_n) \to h(\zeta_0) \quad (n \to \infty),$$

contradicting (7). □

Lemma 4. *If the asymptotic path L of Lemma 1 lies in $S_{\delta/2,K}$ for some $\delta \in (0,\pi)$, then the conclusion of the Theorem holds.*

Proof: Since $F(z) = z - \frac{f(z)}{f'(z)} = \alpha + o(1)$ as $z \to \infty$ on L, $z\frac{f'(z)}{f(z)} = \frac{z}{z-F(z)} = 1 + 0(1/|z|)$ on L. Therefore the hypothesis of Lemma 3 is satisfied for all $r > 2R$ and it follows from Lemma 3 that, in $S_{\delta,R}$,

$$|zf'(z)/f(z)| < C. \tag{9}$$

By (3), (5) and (9)

$$|\varphi(z)| = |f'(z)/f(z)\psi(z)| < C \quad (z \in S_{\delta,R}). \tag{10}$$

Since φ is real entire (10) remains true, if z is replaced by $\overline{z}$. In particular (10) holds for $z = rei\beta$ $(\beta = e\pm i\frac{\pi}{4}, e\pm i\frac{3\pi}{4})$.

Therefore, by (4), (10) and a well–known Phragmén–Lindelöf Theorem

$$|\varphi(z)| < C.$$

And so, by Liouville's Theorem,

$$\varphi(z) = \text{constant}.$$

But this contradicts the fact that $f'(z)$ has a non–real zero.

Lemma 5. [Special case of **T**, Theorem III.68] *Let Γ_1 and Γ_2 be two, non–intersecting Jordan curves tending to ∞. Let $E \subset \mathbb{C}$ be a domain such that for sufficiently large R*

$$\partial E \cap \{z \mid |z| \geq R\} = (\Gamma_1 \cap \Gamma_2) \cap \{z \mid |z| \geq R\}.$$

Let $t\Theta(t)$ be the linear measure of the intersection of E with $|z| = t$.

Then, if $h(z)$ is holomorphic in $\overline{E}$, and $A > e$,

$$|h(z)| < A \quad (z \in \partial E)$$

implies that either

$$|h(z)| < A \quad (z \in E)$$

or

$$Q(r,h,E) = \pi \int_R r/2 \frac{dt}{t\Theta(t)} - \log\log M(r,h,E)$$

satisfies

$$\limsup_{r\to\infty} Q(r,h,E) < \infty. \tag{11}$$

Lemma 6. [**T**, Theorem VIII.14] *Let Γ_1, Γ_2 and E have the same meaning as in Lemma 5.*

Let $g(z)$ be holomorphic in E and continuous and bounded in $\overline{E} \setminus \{\infty\}$.

If $g(z) \to c_j$ $(j = 1,2)$ as $z \to \infty$ along Γ_j, then $c_1 = c_2$.

3. Completion of the Proof of the Theorem.

The rays

$$\Lambda_1 = \left\{ z \mid \arg z = \frac{\pi}{5},\ |z| \geq R \right\},\ \Lambda_2 = \left\{ z \mid \arg z = \frac{4\pi}{5},\ |z| = R \right\}$$

divide

$$H_R = \{ z \in H \bigm| |z| \geq R \}$$

into three parts $S = S_{\frac{\pi}{5},R}$ and Δ_1, Δ_2, sectors adjacent to the positive and negative real axis, respectively.

We still need to prove the Theorem in the case that there are arbitrarily large r such that there exists $z \in L$ with $|z| = r$ and $z \in \Delta_1$ or $z \in \Delta_2$ [Lemma 4].

Let $A > e$ be an upper bound for $|zf'(z)/f(z)|$ on L.

Our first step is the construction of a path $L' \subset H$ on which

$$|zf'(z)/f(z)| < CA = C \tag{12}$$

and which divides H_R into two domains A_1 and A_2 in one of which, at least, (11) does not hold when $h = zf'(z)/f(z)$, $E = A_1$ or A_2. We shall prove that (12) holds in A_1 or in A_2.

We then show that the same is true of one of the domains into which L and $\overline{L}$ divide H_R. The Theorem follows by an application of Lemma 6 [$L = \Gamma_1$, $\overline{L} = \Gamma_2$, $c_1 = \alpha$, $c_2 = \overline{\alpha} \neq c_1$.]

Construction of L'. The rays Λ_1 and Λ_2 divide L into 3 parts:

$$L_0 = \{ z \in L \cap S \}, \quad L_1 = \{ z \in L \cap \Delta_1 \}, \quad L_2 = \{ z \in L \cap \Delta_2 \}.$$

If L_0 is bounded, we choose $L' = L$.

If L_0 is unbounded, let

$$\widetilde{L} = \left\{ z \in \Lambda_1 \mid \exists\ z' \in L_0,\ \frac{1}{2}|z| < |z'| < 2|z| \right\}.$$

By Lemma 3, (9) holds on $\widetilde{L}$ and for all z in S with $|z| e\pi i/5 \in \widetilde{L}$.

If the complement of $\widetilde{L}$ on Λ_1 is bounded, choose $L' = \widetilde{L}$.

If the complement of $\widetilde{L}$ is unbounded, $\widetilde{L}$ is the union of components each one of which is a straight line segment on Λ_1 of length $\geq R$. If $ri\pi/5 \notin \widetilde{L}$, then all points $z \in L$ with $|z| = r$ must lie on "intervals" of L belonging to $L_1 \cup L_2$. Choose one of these intervals, I, with endpoints z_1, z_2, $|z_1| < |z_2|$. Discard all the others. Both endpoints of I are either on Λ_1 or on Λ_2.

If they are on Λ_1, they both belong to $\widetilde{L}$, by the definition of $\widetilde{L}$. If they are both on Λ_2, we join the segments $|z| = |z_1|$, $z \in S$, and $|z| = |z_2|$, $z \in S$, to I, forming I'. The curve I' joins two components of $\widetilde{L}$. We can now describe L': Choose an I or I' with an endpoint as close to the origin as possible. Move along this I (or I') to its endpoint (on Λ_1) in $\widetilde{L}$. Move along $\widetilde{L}$ in the direction of increasing $|z|$ to the first endpoint of an I or I'. Then move along this I (I') to another component of $\widetilde{L}$ and so on $\ldots$. On L' $\arg z$ lies either in $(0, \frac{\pi}{5}]$ or in $[\frac{4\pi}{5}, \pi)$, except for a denumerable set of arcs $|z| =$ constant belonging to the I' which occur in the construction.

Let A_1 be that domain in $\{z \mid |z| \geq R\}$ bounded by L' and its conjugate complex curve $\overline{L'}$ which contains $[R, \infty)$, A_2 the domain bounded by L' and $\overline{L'}$ which contains $[-R, -\infty)$. Let $t\Theta_j(t)$ be the linear measure of the intersection of $|z| = t$ with A_j $(j = 1, 2)$. By construction $\min(\Theta_1(t), \Theta_2(t)) \leq \frac{2\pi}{5}$ and so

$$\pi \int r/2_{R_1} \frac{dt}{t\Theta_1(t)} + \pi \int r/2_{R_1} \frac{dt}{t\Theta_2(t)} \geq \frac{5}{2} \int r/2_{R_1} > \frac{5}{2} \log r - C. \tag{13}$$

We apply Lemma 5 to

$$h(z) = z\,\frac{f'(z)}{f(z)}, \qquad E = A_j.$$

By (3), (4) and (6),

$$\log |h(z)| \leq Cr \log r \qquad (r > R). \tag{14}$$

On $L' \cup \overline{L'}$ (12) holds.

By (13) [notation of Lemma 5] and (14)

$$Q(r, h, A_1) + Q(r, h, A_2) \geq \left(\frac{1}{2} - \varepsilon\right) \log r \qquad (r > R),$$

so that (11) is false for at least one A_j. By Lemma 5, (12) holds in one A_j. Let B be the domain bounded by $L \cup \overline{L}$ which contains such an A_j. In addition to A_j, B contains all or part of sectors

$$\widetilde{S} = \{z \in S \cap \{r_1 \leq |z| \leq r_2\}\}$$

where the segment of Λ_1 with endpoints $r_j e i\pi/5$ belongs to $\widetilde{L}$. As remarked above, (12) holds in $\widetilde{S}$, so that (12) holds in B.

On L,

$$F(z) = \alpha + o(1)$$

as $z \to \alpha$ on L. Therefore

$$g(z) = z2\frac{f'(z)}{f(z)} - z = \frac{zF(z)}{z - F(z)} = \alpha + o(1)$$

as $z \to \infty$ on L. Since g is an even holomorphic function in A,

$$g(z) \to \overline{\alpha} + o(1)$$

as $z \to \infty$ on $\overline{L}$.

In B we apply Lemma 5 to

$$h(z) = g(z); \quad E = B.$$

Using $\Theta(t) \leq 2\pi$ and (12),

$$Q(r, g, A) > \frac{1}{2} \log r - 0(\log\log r),$$

so that, by Lemma 5,

$$|g(z)| < C \qquad (z \in A).$$

Now Lemma 6 leads to the contradiction $\alpha = \overline{\alpha}$.

This completes the proof.

References

[**HW**] S. Hellerstein and J. Williamson, *Derivatives of entire functions and a question of Pólya II*, TAMS **234** (1977), 497–503.

[**LO**] B. Ya. Levin and I. V. Ostrovski, *On the dependence of the growth of entire functions on the location of the zeros of its derivative (On a problem of Pólya and Wiman)*, (In Russian), Sibir. M. J. **1** (1960), 427–455.

[**S**] W. Schwick, *Normality criteria for families of meromorphic functions*, J. Analyse Math. **52** (1989), 241–289.

[**SS**] T. Sheil–Small, *On the zeros of the derivatives of real entire functoins and Wiman's Conjecture*, Ann. of Math. **129** (1989), 179–193.

[**T**] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co. Ltd., Tokyo, 1959.