Newton's method and a class of meromorphic functions without wandering domains

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Abstract

Let N be the class of meromorphic functions f with the following properties: f has finitely many poles; f' has finitely many multiple zeros; the superattracting fixed points of f are zeros of f' and vice versa, with finitely many exceptions; f has finite order. It is proved that if $f \in N$, then f does not have wandering domains. Moreover, if $f \in N$ and if ∞ is among the limit functions of f^n in a cycle of periodic domains, then this cycle contains a singularity of f^{-1}. (Here f^n denotes the n-th iterate of f.) These results are applied to study Newton's method for entire functions g of the form $g(z) = \int_0^z p(t)e^{q(t)}dt + c$ where p and q are polynomials and where c is a constant. In this case, the Newton iteration function $f(z) = z - g(z)/g'(z)$ is in N. It follows that $f^n(z)$ converges to zeros of g for all z in the Fatou set of f, if this is the case for all zeros z of g''. Some of the results can be extended to the relaxed Newton method.

1 Introduction and main results

Let f be a meromorphic function (but not constant or a linear transformation). The Fatou set $\mathcal{F}(f)$ is the subset of the complex plane where the iterates f^n of f are defined and form a normal family. The complement of $\mathcal{F}(f)$ is called the Julia set. If U is a component of $\mathcal{F}(f)$, then $f^n(U)$ is contained in some component of $\mathcal{F}(f)$ which we denote by U_n. If $U_n \neq U_m$ for all $n \neq m$, then U is called wandering. Otherwise U is called preperiodic. In particular, if $U_n = U$ for some n, then U is called periodic (with period n) and if $n = 1$, that is, if $f(U) \subset U$, then U is called invariant.

Sullivan [35, 36] proved that rational functions do not have wandering domains. Transcendental entire and meromorphic functions, however, may have wandering domains, cf. [1, 2, 3, 7, 15, 16, 24, 36]. On the other hand, certain classes of transcendental functions which do not have wandering domains are known, cf. [2, 9, 14, 15, 18, 21, 33].

In this paper, we consider a class of functions different from the classes considered in the papers cited above and, using some ideas of Stallard [33], we prove that functions of this class do not have wandering domains. The class of functions we shall consider is of some interest for the application of Newton's method. We recall that if g is a non-constant meromorphic function, then Newton's method of finding the zeros of g consists of iterating the function f defined by

$$f(z) = z - \frac{g(z)}{g'(z)}. \tag{1}$$

The zeros of g are then attracting fixed points of f, and the simple zeros of g are even superattracting fixed points of f. (A fixed point z_0 of f is called attracting if $|f'(z_0)| < 1$ and superattracting if $f'(z_0) = 0$.)

Suppose now that g is an entire function of finite order with at most finitely many critical points, that is, g is of the form

$$g(z) = \int_0^z p(t)e^{q(t)}dt + c \tag{2}$$

for certain polynomials p and q and a complex number c. Then the function f defined by (1) is a meromorphic function with the following properties:

(i) f has only finitely many poles

(ii) the fixed points of f are superattracting, with at most finitely many exceptions

(iii) the zeros of f' are simple and fixed points of f, with at most finitely many exceptions

(iv) f has finite order

We denote the class of meromorphic functions f satisfying (i)-(iv) by N. (Because we are dealing with functions meromorphic in the plane there should be no confusion with the Nevanlinna class of functions of bounded characteristic in the unit disk.)

Theorem 1 *Meromorphic functions in class N do not have wandering domains.*

Rational functions are clearly in N, but we shall assume in the proof of Theorem 1 that f is transcendental because otherwise our result follows from Sullivan's theorem.

The limiting behavior of f^n in periodic components is well understood. Let U be a periodic component of $\mathcal{F}(f)$ with period p. Suppose first that f is rational. Fatou [19, §56, p. 249] proved that if all limit functions of $\{f^n|_U\}$ are constant, then there exists z_0 satisfying $f^p(z_0) = z_0$ such that $f^{pn}(z) \to z_0$ for all $z \in U$ as $n \to \infty$, were either $z_0 \in U$ and $|(f^p)'(z_0)| < 1$ or $z_0 \in \partial U$ and $(f^p)'(z_0) = 1$. In the first case, U is called an attracting basin (superattracting if $(f^p)'(z_0) = 0$), and in the second case, U is called a Leau domain. Cremer [13, p. 317] proved that if $\{f^n|_U\}$ has non-constant limit functions, then U is either simply- or doubly-connected and $f^p|_U$ is conjugate to an irrational rotation. In the simply-connected case, U is called a Siegel disc, and in the doubly-connected case, U is called a Herman ring. More recent (and somewhat simpler) proofs of Fatou's and Cremer's classification theorem can be found in [10, Chapter 7], [31], [34], and [37]. For historical reasons, Töpfer's [38, p. 211] remarks about the classification of invariant domains should perhaps also be mentioned. We remark that all possibilities do actually occur for certain rational functions, but this was not known yet when Fatou and Cremer proved their results.

For transcendental meromorphic functions, all these types of periodic components can also occur, and there is one additional possibility (cf. [8, Theorem 2.3]): U is a periodic component of $\mathcal{F}(f)$ with period p, $f^{pn}(z) \to z_0 \in \partial U$ for all $z \in U$ as $n \to \infty$, and f^p is not defined at z_0. If $p = 1$ or if f is entire, this is possible only if $z_0 = \infty$. If $p \geq 2$ and if f has poles, then it is also possible that z_0 is a pole of f^k for some $k < p$. Following Eremenko and Lyubich [17, 18], we call this new type of periodic component *Baker domain.* (Herman [25] uses the notation *infinite Fatou component* instead while Baker, Kotus, and Yinian [8] refer to this type of component as *essentially parabolic domain.*)

Theorem 2 *If $f \in N$, then every cycle of Baker domains of f contains a singularity of f^{-1}.*

We note that if $f \notin N$, then a cycle of Baker domains need not contain a singularity of f^{-1}, cf. [16, Example 3] and [25, p. 609] for examples.

Functions in class N do not occur only in Newton's method for functions of the form (2). For instance, Newton's method for functions of the form $Re^q + c$ where R is rational, q is a polynomial, and c is a complex number also leads to functions of class N. Also, functions of class N need not come from Newton's method. It is not difficult to see that a function $f \in N$ is of the form (1) for some meromorphic function g if and only if all multipliers $f'(z_0)$ of the fixed points z_0 that are not superattracting are of the form $1 - 1/d(z_0)$ for some integer $d(z_0)$ which is different from 0 and 1. If, in addition, $d(z_0)$ is nonnegative for all fixed points z_0, then g is entire.

In our application to Newton's method, however, we shall restrict ourselves for simplicity to functions of the form (2).

Theorem 3 *Let g be of the form (2) but not of the form $g(z) = e^{az+b}$ where $a, b \in \mathbf{C}$ and let f be defined by (1). Denote by $z_1, z_2, \ldots, z_m$ the zeros of g'' that are not zeros of g or g'. If $f^n(z_j)$ converges to a finite limit for all $j = 1, \ldots, m$, then $f^n(z)$ converges to zeros of g on an open dense subset of the complex plane.*

We remark that if g is of the form $g(z) = e^{az+b}$ where $a \neq 0$, then $f(z) = z - 1/a$ so that $f^n(z) \to \infty$ for all $z \in \mathbf{C}$. Some applications of Theorem 3 will be given in §8.

It is perhaps of interest to note that Baker [2, Theorem 6.2] proved that if g has the form (2), then g does not have wandering domains. This result is, however, quite different from ours because we iterate f, not g.

Theorem 3 can be generalized to the relaxed Newton method. This generalization, however, requires some additional arguments while Theorem 3 is a fairly direct application of Theorems 1 and 2. Therefore we shall give a proof of Theorem 3 before we shall state and prove a more general result concerning the relaxed Newton method in §9.

This paper is organized as follows. In §2 we give a generalization of a lemma of Stallard and in §3 we state some properties of functions of class N. Then we use this to prove Theorem 1, distinguishing the cases of multiply-connected (§4) and simply-connected (§5) wandering domains. In §6 and §7 we prove Theorems 2 and 3. As already mentioned, we give some applications of our results and a further discussion of Newton's method in §8 and extend our results to the relaxed Newton method in §9.

Besides the classical papers of Fatou [19, 20] and Julia [28], there are several surveys of and introductions to iteration theory. We mention [10, 12, 30, 31] for rational functions and [4, 17] for rational and transcendental entire functions. Also, [25] contains an appendix concerning iteration of entire functions. For iteration of transcendental meromorphic functions, we refer to [6, 7, 8, 9, 14]. Finally, we mention [22], where Newton's method for rational functions is discussed in detail.

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2 A lemma of Stallard

Stallard [33, Lemma 3.3] considered multiply-connected components of $\mathcal{F}(f)$ for a certain class of meromorphic functions. Examination of the proof yields the following result.

Lemma 1 *Let f be a meromorphic function with at most finitely many poles. If f is not entire, then suppose that $O^-(\infty) = \{z : z \text{ is a pole of } f^n \text{ for some } n \in \mathbf{N}\}$ is infinite. Let U be a component of $\mathcal{F}(f)$ such that U and all the U_n, $n = 1, 2, \ldots$, are multiply-connected and do not contain singularities of f^{-1}. Suppose also that all limit functions of $\{f^n|_U\}$ are constant (possibly ∞). Let γ be a Jordan curve in U which is not null-homotopic in U. Then, for any $M > 0$, there exists $L = L(M) \in \mathbf{N}$ and a continuum $\delta_L \subset f^L(\gamma)$ such that $\delta_L \subset \{z : |z| > M\}$, with one of the components of the complement of δ_L being bounded and containing $\{z : |z| < M\}$.*

3 Some properties of functions of class N

First we note that if $f \in N$ is not entire and not rational, then $O^-(\infty)$ is infinite. In fact, let p be a pole of f and consider the function F defined by

$$F(z) = \frac{f(z) - z}{p - z}.$$

Then the zeros of F have multiplicity two, with at most finitely many exceptions. Using standard terminology and results of Nevanlinna theory (cf. [23, 27, 32]), we deduce that $\Theta(0, F) \geq \frac{1}{2}$. Also, F has only finitely many poles so that $\Theta(\infty, F) = 1$. Nevanlinna's second fundamental theorem now implies that $\Theta(1, F) \leq \frac{1}{2}$. In particular, F takes the value 1 infinitely often. It follows that f takes the value p infinitely often so that $O^-(\infty)$ has infinitely many elements.

Next we note that if $f \in N$, then f^{-1} has only finitely many singularities that are not contained in invariant domains. To see this, recall that the singularities of f^{-1} are the critical values of f, the asymptotic values of f, and limit points of these values. The critical values are, with at most finitely many exceptions, the superattracting fixed points of f, and these lie in invariant domains. And there are only finitely many asymptotic values by the Denjoy-Carleman-Ahlfors theorem [32, p. 307]. (In [32], the theorem is stated for entire functions, but it extends easily to meromorphic functions with finitely many poles.) Also, it follows from the above observations that the set of critical and asymptotic values is discrete, that is, there are no (finite) limit points of critical and asymptotic values. Hence there are only finitely many singularities of f^{-1} not contained in invariant domains.

Finally we note that if $f \in N$, then $f'(z)/(f(z) - z)$ is a function of finite order which has only finitely many poles and zeros. Hence

$$\frac{f'(z)}{f(z) - z} = \frac{P(z)}{Q(z)} e^{R(z)} \tag{3}$$

for polynomials P, Q, and R. This equation may also be used to define the class N. In fact, a meromorphic function f of finite order is in N if and only if it admits a representation of the form (3). We remark that if g has the form (2) and if f is defined by (1), then R is constant.

4 The non-existence of multiply-connected wandering domains

Suppose that a transcendental function $f \in N$ has a wandering domain U. It is well-known that because U is wandering, all limit functions of $\{f^n|_U\}$ are constant (cf. [19, §28] and [8, Lemma 2.1]). As shown in §3, there are only finitely many singularities of f^{-1} that are not contained in invariant domains. It follows that there exists $k \geq 0$ such that if $n \geq k$, then U_n does not contain singularities of f^{-1}. Here, by definition, $U_0 = U$.

In this section, we shall assume that there exists $m \geq k$ such that U_m is multiply-connected. Then U_n is multiply-connected for all $n \geq m$. We shall assume without loss of generality that $m = 0$ so that U satisfies the hypotheses of Lemma 1. We choose γ as in Lemma 1 and following Stallard [33, p. 221] we can show that if $z_1, z_2 \in \gamma$ and if $f^n(z_1)$ or $f^n(z_2)$ is sufficiently large, then

$$|f^n(z_2)| \leq |f^n(z_1)|^A \tag{4}$$

for some positive constant A and all sufficiently large $n \in \mathbf{N}$. We deduce from Lemma 1 that there exist arbitrarily large R such that

$$\delta_L \subset f^L(\gamma) \subset \{z : R \leq |z| \leq R^A\} \tag{5}$$

for some $L \in \mathbf{N}$.

We define $h(z) = f(z) - z$ and denote the right side of (3) by g. Then $(h' + 1)/h = g$, that is, $h' = gh - 1$. We choose $r_0 > 0$ such that the poles of g are contained in $\{z : |z| < r_0\}$. We also choose $\theta_0 \in (-2\pi, 0)$ and define

$$G(z) = \int_{r_0}^{z} g(t)dt$$

for $|z| > r_0$ and $\theta_0 < \arg z \leq \theta_0 + 2\pi$. (Here the path of integration is contained in $\{t : |t| > r_0, \theta_0 < \arg t \leq \theta_0 + 2\pi\}$.) Then

$$h(z) = e^{G(z)} \left(-\int_{r_0}^{z} e^{-G(t)}dt + c \right)$$

for some constant $c \in \mathbf{C}$.

We note that G is unbounded because otherwise we would have $|h(z)| = O(|z|)$ as $|z| \to \infty$, that is, h and hence f would be rational, contradicting our assumption. It follows that G is unbounded.

Suppose first that the polynomial R in (3) is not constant, say $R(z) \sim az^m$ as $z \to \infty$ where $a \neq 0$ and $m \geq 1$. We consider the sector

$$S = \left\{ z : |z| > r_0, \frac{1}{m}\left(-\frac{\pi}{4} - \arg a\right) < \arg z < \frac{1}{m}\left(\frac{\pi}{4} - \arg a\right) \right\}$$

and choose θ_0 such that $\{re^{i\theta_0} : r > r_0\} \cap S = \emptyset$. Integration by parts shows that

$$G(z) \sim \frac{P(z)}{Q(z)R'(z)} e^{R(z)} \tag{6}$$

as $z \to \infty$ in S. We find that if r_0 is sufficiently large, then S contains a curve σ such that $G(z)$ is real and negative and decreases to $-\infty$ as $z \to \infty$ in σ. In fact, we can deduce from (6) that there exists a branch ϕ of G^{-1} defined in $S' = \{z : |z| > R_0, 0 < \arg z < 2\pi\}$ such that $\phi(S') \subset S$, provided R_0 is sufficiently large. We choose σ as the image of the negative real axis under ϕ. It follows from (6) that $\phi(z) = c(\log z)^{1/m}$ as $z \to \infty$ in S', for a suitable branch of the logarithm and the root. Here $c = a^{-1/m}$. We deduce that

$$\phi'(z) \sim \frac{c}{m} \frac{(\log z)^{1/m-1}}{z}.$$

Hence

$$\int_{R_0}^{R} |\phi'(-t)|dt \leq (1+o(1))\frac{|c|}{m} \int_{R_0}^{R} \frac{(\log t)^{1/m-1}}{t} dt = (1+o(1))|c|(\log R)^{1/m} = (1+o(1))|\phi(-R)|.$$

It follows that if $l(r)$ denotes the length of the part of σ that is contained in $|z| \leq r$, then $l(r) = (1 + o(1))r = O(r)$ as $r \to \infty$.

Suppose now that the polynomial R in (3) is constant. Then $G(z) \sim az^m$ as $z \to \infty$ where $a \neq 0$ and where m is a positive integer. Using similar arguments as above we deduce that if θ_0 is suitably chosen and if r_0 is sufficiently large, then there exists an unbounded curve σ contained in $\{z : |z| > r_0, \theta_0 < \arg z \leq \theta_0 + 2\pi\}$ with the properties noted above, that is, $G(z)$ is real and negative and decreases to $-\infty$ as $z \to \infty$ in σ and $l(r) = O(r)$.

In both cases it follows that if $z \in \sigma$, then

$$\int_{r_0}^{z} e^{-G(t)} dt \leq l(|z|)e^{|G(z)|} + O(1) \leq O(|z|)e^{|G(z)|} = O(|z|)e^{-G(z)}$$

as $z \to \infty$. Hence

$$|h(z)| = \left| e^{G(z)} \left(\int_{r_0}^{z} e^{-G(t)} dt + c \right) \right| \leq O(|z|) + |c|e^{G(z)} = O(|z|)$$

as $z \to \infty$ in σ. This implies that $|f(z)| = O(|z|)$ for $z \in \sigma$, say $|f(z)| \leq C|z|$.

We now choose R and L such that (5) holds. If R is sufficiently large, then δ_L intersects σ so that

$$\min_{z \in f^L(\gamma)} |f(z)| \leq \min_{z \in \delta_L} |f(z)| \leq CR^A.$$

Applying (4) for $n = L + 1$, we deduce

$$\max_{z \in \delta_L} |f(z)| \leq \max_{z \in f^L(\gamma)} |f(z)| \leq C^A R^{A^2},$$

provided R is large enough. Since f has only finitely many poles, there exists a rational function T such that $T(z) \to 0$ as $z \to \infty$ and $f - T$ is entire. We deduce that

$$\max_{z \in \delta_L} |f(z)| = \max_{z \in \delta_L} |f(z) - T(z)| + o(1) \geq \max_{|z|=R} |f(z) - T(z)| + o(1)$$

as $R \to \infty$. It follows that

$$\max_{|z|=R} |f(z) - T(z)| \leq C^A R^{A^2} + o(1)$$

for arbitrary large R. This implies that $f - T$ is a polynomial. Hence f is rational, contradicting our assumption.

5 The non-existence of simply-connected wandering domains

Let U and k be as at the beginning of §4, but suppose now that there does not exist $m \geq k$ such that U_m is multiply-connected, that is, U_m is simply-connected for all $m \geq k$. Without loss of generality we shall assume that $k = 0$. We choose $K > 1$ and consider meromorphic functions F that are of the form $F = \Phi \circ f \circ \Phi^{-1}$ where Φ is a K-quasiconformal homeomorphism of $\mathbf{C} \cup \{\infty\}$ that fixes 0, 1, and ∞. We note that F and f have the same number of poles, the same number of fixed points that are not superattracting, and their derivatives have the same number of zeros that are not fixed points. Moreover, the orders $\rho(f)$ and $\rho(F)$ of f and F are related by $\rho(f)/K \leq \rho(F) \leq K\rho(f)$ (cf. [2, p. 570] and [33, p.223]). We deduce that there exists polynomials P_Φ, Q_Φ, and R_Φ such that

$$\frac{F'(z)}{F(z) - z} = \frac{P_\Phi(z)}{Q_\Phi(z)} e^{R_\Phi(z)}. \tag{7}$$

Here the degrees of P_Φ and Q_Φ are equal to the degrees of P and Q, where P and Q are the polynomials in (3), and the degree of R_Φ is at most $[K\rho(f)]$, the largest integer not greater than $K\rho(f)$. We choose K such that $[K\rho(f)] = [\rho(f)]$ and deduce that the family of all meromorphic functions F of this form depends on $\deg(p) + \deg(q) + [\rho(f)] + 2$ parameters. Now we can use the quasiconformal methods of [2, 9, 18, 33] to obtain a contradiction. We omit the details. This, together with the result of the previous section, implies that f has no wandering domains and completes the proof of Theorem 1.

6 Proof of Theorem 2

Let U be Baker domain of f with period p and suppose that $U, U_1, U_2, \ldots, U_{p-1}$ do not contain singularities of f^{-1}. It follows that if U and all the U_n are multiply-connected, then U satisfies the hypothesis of Lemma 1. As in §4 we can show that this is impossible for $f \in N$. Hence U_k is simply-connected for some k. This implies that U_n is simply-connected and that $f|_{U_n}$ is univalent for all n. As observed by Herman [25, p. 609], this implies that the space of quasiconformal deformations of f is infinite dimensional. Arguments similar to those used in §5 show that this is not the case. We indicate how one can use the method of [2, 9] to obtain a contradiction.

In [2, 9], for $\delta > 0$ and a positive integer M, a family $\{\phi_{\underline{t}} : \underline{t} \in \mathbf{R}^M, \|\underline{t}\|_\infty \leq \delta\}$ of quasiconformal maps of the unit disc D is constructed such that there is an open subset of the boundary of D which is fixed by all these maps. For us it is easier to work with the upper half plane H instead of D. Let h be a conformal map of U onto H. In order that the method of [2, 9] can be applied, we have to choose the quasiconformal maps $\phi_{\underline{t}}$ such that the dilatation $\nu_{\underline{t}}$ of $\phi_{\underline{t}}$ satisfies

$$\nu_{\underline{t}}(L(z)) = \nu_{\underline{t}}(z) \frac{\overline{L'(z)}}{L'(z)} \tag{8}$$

almost everywhere, where $L(z) = h^{-1}(f^p(h(z)))$. By a suitable choice of h we can achieve that $L(z) = z + 1$ or $L(z) = az$ for some $a > 1$.

In the first case, we choose δ so small that there exist $\alpha_j \in (\delta, 1-\delta)$, $j = 1, \ldots, M$, satisfying $\alpha_j + \delta < \alpha_{j+1} - \delta$ for $j = 1, \ldots, M-1$. For $x \in [\alpha_j - \delta, \alpha_j + \delta]$ we define

$$\psi_j(x) = \delta \exp\left(-\frac{\delta^2}{(x-\alpha_j)^2 - \delta^2}\right)$$

and for $x \in [0,1] \backslash [\alpha_j - \delta, \alpha_j + \delta]$ we define $\psi_j(x) = 0$. We extend the domain of ψ_j to $\mathbf{R}$ by choosing ψ_j periodic with period 1. Finally, for $\underline{t} = (t_1, \ldots, t_M) \in \mathbf{R}^M$ and $z \in H$, we define

$$\phi_{\underline{t}}(z) = z + \sum_{j=1}^{M} t_j \psi_j(\text{Re } z).$$

It is easy to see that (8) is satisfied. Choosing δ sufficiently small and M large enough, we can now proceed as in [2, 9] and obtain a contradiction. The case that $L(z) = az$ is similar.

Hence a component U with these properties does not exist. This completes the proof of Theorem 2.

7 Proof of Theorem 3

First we remark that f does not have asymptotic values. To see this, we note that if $f(z) \to \alpha \in \mathbf{C}$ as $z \to \infty$ in a curve γ, then $h(z) = g(z)/(zg'(z)) \to 1$ as $z \to \infty$ in γ. On the other hand, using integration by parts one can deduce from (2) that $h(z) \to 0$ in the sectors where Re $q(z) \to \infty$ and that $h(z) \to 0$ or $h(z) \to \infty$ in the sectors where Re $q(z) \to -\infty$, depending on whether $g(z) \to 0$ in these sectors or not. Combining these observations with a theorem of Lindelöf [32, p. 65] which says that h must be unbounded between two paths corresponding to different asymptotic values and with an estimate of Tsuji [39, p. 117] (or the Ahlfors distortion theorem [32, p. 97]), one can conclude that 1 is not an asymptotic value of h, that is, f does not have asymptotic values. This implies that the only singularities of f^{-1} are the critical values of f. Also, it is easy to see that the critical points of f are the zeros of g and g'' that are not zeros of g'.

To prove Theorem 3 we may assume that g is not linear. Together with the hypothesis that g is not of the form $g(z) = e^{az+b}$ this implies that f is not a linear transformation. Let now U be a periodic component of $\mathcal{F}(f)$ and denote by p the period of U.

Suppose that U is a Baker domain. Replacing U by U_k for a suitable k if necessary, we may assume that $f^{pn}(z) \to \infty$ for $z \in U$ as $n \to \infty$. By Theorem 2, there exists l such that U_l contains a singularity of f^{-1}, and this singularity must be a critical value of f. More precisely, $f : U_{l-1} \to U_l$ is not invertible because U_{l-1} contains a critical point of f. This critical point must be a zero of g'', but cannot be a zero of g or g'. Hence $f^{pn-l+1}(z_j) \to \infty$ for some $j \in \{1, \ldots, m\}$. This is a contradiction and we deduce that U is not a Baker domain. It follows that U is one of the domains that occur in the iteration of rational functions.

If U is a Siegel disc or Herman ring, then the forward images of the singularities of f^{-1} are dense in the boundary of U, which is impossible by our hypothesis on the zeros of g''. (We note here that the fact that the forward images of the singularities of f^{-1} are dense in the boundary of a Siegel disc or Herman ring is usually stated only for rational or entire functions, but the proof carries over to the transcendental meromorphic case. An analogous remark applies to the fact that cycles of attracting basins or Leau domains contain a singularity of f^{-1}. This result will be used next.)

Having ruled out the possibility that U is a Baker domain, Siegel disc, or Herman ring, we conclude that U is an attracting (possibly superattracting) basin or a Leau domain. Then the periodic cycle of domains where U belongs to contains a singularity of f^{-1} and hence a critical value of f. Our hypothesis on the zeros of g'' implies that $p = 1$. Since all fixed points of f are attracting and correspond to zeros of g, we deduce U is the basin of an attracting fixed point of f so that $f^n(z)$ converges to a zero of g in U.

We conclude that $f^n(z)$ converges to zeros of g in all periodic (and hence all preperiodic) components of $\mathcal{F}(f)$. Since f has no wandering domains by Theorem 1, we deduce that $f^n(z)$ tends to zeros of g for all $z \in \mathcal{F}(f)$. Because $\mathcal{F}(f)$ is either a dense subset of $\mathbf{C}$ or empty, it remains to be proved that $\mathcal{F}(f)$ is not empty. If g has zeros, then f has (super)attracting fixed points so that $\mathcal{F}(f)$ is not empty in this case. If g has no zeros, then g is of the form $g = e^r$ for some polynomial r of degree greater than 1. It follows that f is rational and that ∞ is a rationally indifferent fixed point of f. Again we conclude that $\mathcal{F}(f)$ is not empty. This completes the proof of Theorem 3.

Remark Using the same methods, one can prove that the conclusion of Theorem 3 remains valid under certain more general hypotheses. For instance, instead of assuming that $f^n(z_j)$ converges to a finite limit we may also suppose that $f^n(z_j)$ is preperiodic but not periodic. Similarly, we may suppose that $f^n(z_j)$ tends to ∞ very fast. In fact, Baker, Kotus, and Yinian [7, Lemma 4.1] proved that if

$$\limsup_{n\to\infty} \frac{\log\log|f^n(z_j)|}{n} = \infty, \tag{9}$$

then z_j cannot be in an invariant (or periodic) domain of $\mathcal{F}(f)$. It follows that we can replace the hypothesis that $f^n(z_j)$ converges to a finite limit by (9).

We mention that (9) can be improved for simply-connected domains. As proved by Baker [5, Lemma 1], we have

$$\limsup_{n\to\infty} \frac{\log|f^n(z_j)|}{n} < \infty,$$

if z_j is in a simply-connected invariant (or periodic) domain of $\mathcal{F}(f)$.

8 Examples and remarks

We state some corollaries to Theorem 2.

Corollary 1 *Suppose that $a \in \mathbf{C}$ and $d \in \mathbf{N}$, $a \neq 0$, $d \geq 2$. Then Newton's method for*

$$g(z) = \int_0^z e^{at^d} dt$$

converges to zeros of g on an open dense subset of the plane.

Corollary 2 *Suppose that $a, b, c \in \mathbf{C}$, $a, c \neq 0$. Then Newton's method for*

$$g(z) = e^{az+b} + c$$

converges to zeros of g on an open dense subset of the plane.

The proofs of Corollaries 1 and 2 are straight forward applications of Theorem 3 and will be omitted.

Corollary 3 *Suppose that* $c \in \mathbf{R}$ *and*

$$g(z) = \int_0^z e^{-t^2} dt + c. \tag{10}$$

Define f *by* (1). *If* $c \in \mathbf{R} \backslash \{-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}\}$, *then* $f^n(z)$ *converges to zeros of* g *on an open dense subset of the plane. If* $c = \frac{\sqrt{\pi}}{2}$ *or* $c = -\frac{\sqrt{\pi}}{2}$, *then* f *has a Baker domain. More precisely,* $f^n(z) \to \infty$ *on some invariant component of* $\mathcal{F}(f)$.

We restrict our discussion to the case $c < 0$ because the case $c > 0$ is similar and the case $c = 0$ is contained in Corollary 1.

Suppose first that $c < -\frac{\sqrt{\pi}}{2}$. Then $g(x) < 0$ and $g'(x) > 0$ for all $x \in \mathbf{R}$. Hence $f(x) > x$ and it follows that $f^n(0)$ is increasing and tends to ∞. Since

$$f(x) > x + e^{x^2}\left(-c - \frac{\sqrt{\pi}}{2}\right) > e^x$$

for sufficiently large x, we see that

$$\lim_{n\to\infty} \frac{\log\log f^n(0)}{n} = \infty.$$

In view of the remarks at the end of §7 we can conclude that $f^n(z)$ converges to zeros of g on an open dense subset of $\mathbf{C}$.

Suppose now that $-\frac{\sqrt{\pi}}{2} < c < 0$. Then g has exactly one real zero which we denote by x_0. Clearly, $x_0 > 0$ and $f(x_0) = x_0$. It is not difficult to show that $x < f(x) < x_0$ for $0 < x < x_0$ and that $0 < f(0) = -c < x_0$. This implies that $f^n(0) \to x_0$. From Theorem 3 we deduce that $f^n(z)$ tends to zeros of g on an open dense subset of $\mathbf{C}$.

Finally, suppose that $c = -\frac{\sqrt{\pi}}{2}$. Using integration by parts we find that

$$f(z) = z + \frac{1}{2z} + O\left(\frac{1}{|z|^3}\right)$$

as $|z| \to \infty$, uniformly for $|\arg z| \le \frac{\pi}{4} - \delta$, provided $\delta > 0$. This implies ([19, §8, §11], see also [11, Theorem 3.3] and [26, Theorem 2]) that f has an invariant Baker domain in which $f^n(z)$ tends to ∞. The proof of Corollary 3 is complete.

If g has the form (10) where c is complex, it is possible that there are periodic cycles of components of $\mathcal{F}(f)$ which have period greater than 1. In fact, if c is a solution of

$$e^{c^2}\left(\int_0^{-c} e^{-t^2} dt + c\right) + c = 0, \tag{11}$$

then $\{0, -c\}$ is a superattracting cycle of f. It is easy to see that (11) has infinitely many solutions.

Finally we note that in general it is possible that Newton's method for entire functions leads to wandering domains, an example where this happens is given in [11].

9 The relaxed Newton method

Let g be an entire (or meromorphic) function and $h \in \mathbf{C}$, $|h-1|<1$. The relaxed Newton method consists of iterating the function f_h defined by

$$f_h(z) = z - h\frac{g(z)}{g'(z)}. \tag{12}$$

The zeros of g are then attractive fixed points of f_h, and if z_0 is a simple zero of g, then $f_h'(z_0) = 1-h$. The unrelaxed Newton method considered in the previous sections corresponds to the choice $h=1$.

Theorem 4 *Let g be of the form (2) but not linear or of the form $g(z) = e^{az+b}$ where $a, b \in \mathbf{C}$ and let f_h be defined by (12). Then f_h has no wandering domains and every cycle of Baker domains of f_h contains a critical value of f_h. If all zeros of f_h' tend to a finite limit under iteration of f_h, then $f_h^n(z)$ converges to zeros of g on an open dense subset of the plane.*

For the proof of Theorem 4, we shall need the following estimate of the logarithmic derivative of entire functions.

Lemma 2 *Let g be an entire function of finite order with zeros z_j. Given $\tau > 0$, there exist $a, R > 0$ such that $|g'(z)/g(z)| \leq |z|^a$ for $z \notin \cup_{j=1}^{\infty} D\left(z_j, |z_j|^{-\tau}\right)$ and $|z| \geq R$.*

Here $D(z,r)$ denotes the disc of radius r around z. To prove Lemma 2, we note that if $|z| = r < \rho$ and $g(0) \neq 0$, then (cf. [27, p. 65], see also [23, p. 37] for a similar estimate)

$$\left|\frac{g'(z)}{g(z)}\right| \leq \frac{4\rho}{(\rho-r)^2} T(\rho, g) + \sum_{|z_j|<\rho} \frac{2}{|z-z_j|},$$

with minor modifications if $g(0) = 0$. We choose $\rho = 2r$ and deduce that if

$$z \notin \bigcup_{j=1}^{\infty} D\left(z_j, |z_j|^{-\tau}\right),$$

then

$$\left|\frac{g'(z)}{g(z)}\right| \leq \frac{8T(2r,g)}{r} + 2(2r)^{\tau} n\left(2r, \frac{1}{g}\right).$$

It follows that if μ is greater than the order of g, then $|g'(z)/g(z)| \leq |z|^{\mu+\tau}$ for sufficiently large $z \notin \cup_{j=1}^{\infty} D\left(z_j, |z_j|^{-\tau}\right)$. Thus we may take $a = \mu + \tau$ if we choose R large enough. This completes the proof of Lemma 2.

Proof of Theorem 4 First we note that f_h does not have asymptotic values. This can be seen as at the beginning of §7 where we proved that f_1 does not have asymptotic values. It follows that the singularities of f_h^{-1} are the critical values of f_h. We shall prove that the critical values of f_h, with at most finitely many exceptions, lie in invariant domains.

To do this, we note that the critical points of f_h are the zeros of

$$\frac{g(z)g''(z)}{g'(z)^2} - \frac{h-1}{h}.$$

We define $k(z) = g(z)g''(z)/g'(z)^2$. Using integration by parts we see that $k(z) \to 1$ in the sectors where Re $q(z) \to \infty$ and $k(z) \to 1$ or $k(z) \to \infty$ in the sectors where Re $q(z) \to -\infty$, depending on whether $g(z) \to 0$ in these sectors or not. As in §7 we conclude that 1 is the only asymptotic value of k.

Let now (c_j) be the sequence of critical points of k and denote by $d_j = k(c_j)$ the corresponding critical values. Then $d_j = 0$ for at most finitely many j. In fact, this is the case if and only if c_j is not a zero of g', but a multiple zero of g'' or a common zero of g and g''. If $d_j \neq 0$, then

$$\frac{k'(c_j)}{k(c_j)} = \frac{g'(c_j)}{g(c_j)} + \frac{g'''(c_j)}{g''(c_j)} - 2\frac{g''(c_j)}{g'(c_j)} = 0$$

so that

$$d_j = k(c_j) = \frac{g(c_j)}{g'(c_j)}\frac{g''(c_j)}{g'(c_j)} = \left(2\frac{g''(c_j)}{g'(c_j)} - \frac{g'''(c_j)}{g''(c_j)}\right)^{-1}\frac{g''(c_j)}{g'(c_j)} = \left(2 - \frac{g'''(c_j)g'(c_j)}{g''(c_j)^2}\right)^{-1}.$$

A straight forward computation shows that $g'''(z)g'(z)/g''(z)^2 \to 1$ as $|z| \to \infty$ so that $d_j \to 1$ as $j \to \infty$. We deduce that the half plane $H = \{z : \text{Re } z < \frac{1}{2}\}$ contains only finitely many critical (and no asymptotic) values of k, that is, only finitely many singularities of k^{-1}. Since $(h-1)/h \in H$ for $|h-1| < 1$ and since the critical points of f_h are the zeros of $k(z) - (h-1)/h$, we conclude that the critical points of f_h can be obtained from the critical points of f_1 via analytic continuation of k^{-1}, and vice versa, with at most finitely many exceptions.

For a zero ζ of g, denote by $A(\zeta, h)$ the basin of attraction of ζ with respect to f_h, that is, $A(\zeta, h)$ is the component of $\mathcal{F}(f_h)$ that contains ζ. It is well-known that $A(\zeta, h)$ contains a critical value of f_h. (Recall that f_h does not have asymptotic values.) Also, it is not difficult to see that $A(\zeta, h)$ depends continuously on h.

Next we note that for each zero ζ of g, $A(\zeta, 1)$ contains at most finitely many critical values of f_1, and there are at most finitely many zeros ζ for which $A(\zeta, 1)$ contains more than one critical value of f_1. Moreover, there are only finitely many critical values of f_1 that are not contained in the union of the $A(\zeta, 1)$.

Combining the above observations, we conclude that f_h has only finitely many critical values that are not contained in the union of the $A(\zeta, h)$. This means that there are only finitely many singularities of f_h^{-1} that are not contained in invariant domains.

Once this is known, the proof that f_h does not have wandering domains U with the property such that U_m is multiply-connected for infinitely many m can be carried out as in §4.

Suppose now that f_h has a simply-connected wandering domain U as in §5. Again, we consider meromorphic functions F of the form $F = \Phi \circ f_h \circ \Phi^{-1}$ where Φ is a K-quasiconformal homeomorphism of the sphere that fixes 0, 1, and ∞. Inspection of the proofs in [2, 9] shows that it suffices to consider maps Φ whose dilatation is different from zero at most in the U_n and preimages of the U_n under iterates of f_h. Hence Φ is conformal in invariant (and periodic) domains. Since all fixed points of f_h are attracting, it follows that the multipliers of the fixed points of F are equal to the multipliers of the corresponding fixed points of f_h. This implies that there exists an entire function G such that

$$F(z) = z - h\frac{G(z)}{G'(z)}. \tag{13}$$

Since F and f_h have the same number of poles, we can conclude that G has the form $G(z) = \int_0^z P(t)e^{Q(t)}dt + C$ where C is a constant, P is a polynomial of the same degree as p, and Q is an entire function.

We wish to show that Q is also a polynomial. In order to do this, we suppose that Q is transcendental. The idea is to consider the sets $E(a, f_h) = \{z : |f_h(z) - z| \leq h|z|^{-a}\}$ and $E(b, F) = \{z : |F(z) - z| \leq h|z|^{-b}\}$ and to prove that $E(a, f_h)$ is small while $E(b, F)$ is large for suitable values of a and b. This will lead to a contradiction.

To this end, we estimate the size of $E(a, f)$ and $E(b, F)$. Let (z_j) be the sequence of zeros of g. Lemma 2 says that if $\tau > 0$ is given, then $E(a, f) \subset \cup_{j=1}^{\infty} D\left(z_j, |z_j|^{-\tau}\right) \cup D(0, R)$ can be achieved by choosing a and R sufficiently large. To estimate the size of $E(b, F)$, we note that G has the form $G(z) = \alpha(z)e^{\beta(z)}$ for certain entire functions α and β of finite order. We define $E_1 = \{z : |\alpha'(z)/\alpha(z)| \geq |z|^b\}$ and $E_2 = \{z : |\beta'(z)| \geq 2|z|^b\}$. Then $E_2 \backslash E_1 \subset E(b, F)$. Let $\theta(r)$ be the measure of $\{\varphi : 0 \leq \varphi < 2\pi, re^{i\varphi} \in E_2\}$ and define $\theta^*(r) = \infty$ if $\{z : |z| = r\} \subset E_2$ and $\theta^*(r) = \theta(r)$ otherwise. From an inequality of Tsuji [39, p. 117] we can deduce that if $\eta > 1$, then

$$\int_1^r \frac{dt}{t\theta^*(t)} \leq \frac{1}{\pi} \log\log M(\eta r, \beta') + O(\log\log r).$$

Since β has finite order and $\theta^*(r) = \theta(r)$ unless $\theta(r) = 2\pi$ and $\theta^*(r) = \infty$, we have

$$\int_1^r \frac{dt}{t\theta(t)} = O(\log r).$$

Let now $A(r)$ be the area of $E(b, F) \cap \{z : |z| \leq r\}$. For sufficiently large b, E_1 has finite measure by Lemma 2 so that $A(r) \geq \int_0^r t\theta(t)dt + O(1)$. By Schwarz's inequality,

$$r - 1 = \int_1^r dt \leq \sqrt{\int_1^r t\theta(t)dt}\sqrt{\int_1^r \frac{dt}{t\theta(t)}} = O(\sqrt{A(r)\log r})$$

which implies that $A(r) \to \infty$ as $r \to \infty$, that is, $E(b, F)$ has infinite area for large b.

Now we suppose that $w \in E(b, F)$ and define $z = \Phi^{-1}(w)$. Then

$$|f_h(z) - z| = |\Phi^{-1}(F(w)) - \Phi^{-1}(w)|. \tag{14}$$

It is well-known [29, p. 73] that K-quasiconformal maps are Hölder continuous (with respect to the spherical metric k) with Hölder exponent $1/K$, that is,

$$k(\Phi^{-1}(u), \Phi^{-1}(v)) \leq Ck(u, v)^{1/K}$$

and

$$k(\Phi(u), \Phi(v)) \leq Ck(u, v)^{1/K} \tag{15}$$

for some positive constant C and all $u, v \in \mathbf{C} \cup \{\infty\}$. In particular,

$$k(\Phi^{-1}(F(w)), \Phi^{-1}(w)) \leq Ck(F(w), w)^{1/K}. \tag{16}$$

From the Hölder continuity of Φ and Φ^{-1} at ∞ we can deduce that

$$|z| = |\Phi^{-1}(w)| \leq L|w|^K \tag{17}$$

and

$$|w| = |\Phi(z)| \leq L|z|^K \tag{18}$$

for some positive constant L and all sufficiently large $|z|$ and $|w|$. (The inequalities (17) and (18) yield a proof for the inequality $\rho(f_h)/K \leq \rho(F) \leq K\rho(f_h)$ which was already used in §5.) Combining (16) and (17) we obtain

$$\begin{aligned}
|\Phi^{-1}(F(w)) - \Phi^{-1}(w)| &\leq C|F(w) - w|^{1/K} \frac{\sqrt{(1 + |\Phi^{-1}(F(w))|^2)(1 + |\Phi^{-1}(w)|^2)}}{\left(\sqrt{(1 + |F(w)|^2)(1 + |w|^2)}\right)^{1/K}} \\
&\leq (1 + o(1))Ch^{1/K}|w|^{-b/K} \frac{L^2|F(w)|^K|w|^K}{|F(w))|^{1/K}|w|^{1/K}} \\
&= (1 + o(1))Ch^{1/K}L^2|w|^{-b/K+2K-2/K} \\
&\leq |w|^{-b/K+2K}
\end{aligned} \tag{19}$$

for sufficiently large $|w|$. Hence

$$|f_h(z) - z| \leq \left(\frac{|z|}{L}\right)^{-b/K^2+2}$$

by (14) and (17), provided $b > 2K^2$. It follows that if $b > K^2(a + 2)$, then $z \in E(a, f)$ for sufficiently large z. Hence

$$\Phi^{-1}(E(b, F)) \subset \bigcup_{j=1}^{\infty} D(z_j, |z_j|^{-\tau}) \cup D(0, R),$$

if R is large enough, that is,

$$E(b, F) \subset \Phi(E(a, f)) \cup \Phi(D(0, R)) \subset \bigcup_{j=1}^{\infty} \Phi(D(z_j, |z_j|^{-\tau})) \cup D(0, R')$$

for some positive R'. Suppose now that $z \in D(z_j, |z_j|^{-\tau})$. Define $w_j = \Phi(z_j)$. Then $k(w, w_j) \leq Ck(z, z_j)^{1/K}$ by (15). Combining this with (18) we deduce as in (19) that

$$|w - w_j| \leq |z - z_j|^{1/K}|z_j|^{2K} \leq |z_j|^{-\tau/K+2K}$$

for large j. For sufficiently large τ, the series $\sum_{j=1}^{\infty} |z_j|^{-2\tau/K+4K}$ converges, since g has finite order so that the zeros of g have a finite exponent of convergence. We deduce that if we choose τ (and corresponding values of a and b) large enough, then $E(b, F)$ has finite area. This is a contradiction and we deduce that Q is a polynomial.

Once this is known, it is not difficult to show that the degree of Q is equal to the degree of q, provided K is small enough. It follows that the family of all such functions F depends on $\deg(p) + \deg(q) + 3$ parameters. Hence the quasiconformal methods used before can be applied again and we see that f_h does not have a wandering domain U with the properties stated in §5. Altogether we obtain that f_h does not have wandering domains at all.

Similarly, combining the above reasoning with the methods used in §6, we can prove that every periodic cycle of Baker domains contains a critical value of f_h.

As before, it follows that all components of $\mathcal{F}(f_h)$ are related to critical values of f_h, and we deduce that if $f_h^n(z)$ converges to a finite limit for all critical points of f_h, then $f_h^n(z)$ tends to zeros of g for all $z \in \mathcal{F}(f_h)$. This completes the proof of Theorem 4.

Remark In order to study the relaxed Newton method it is natural to consider the class N_h of all meromorphic functions f that satisfy conditions (i) and (iv) of the definition of class N, and instead of (ii) and (iii) satisfy the following conditions:

(ii') the fixed points of f have multiplier $1-h$, with at most finitely many exceptions

(iii') the $(1-h)$-points of f' are simple and fixed points of f, with at most finitely many exceptions

Clearly, we have $N = N_1$. I do not know whether functions in N_h can have wandering domains if $0 < |h-1| < 1$. Our proof that functions in N do not have wandering domains does not carry over to this more general case. The difficulty is that if $f \in N_h$ where $h \neq 1$ and if Φ is quasiconformal, then $F = \Phi \circ f \circ \Phi^{-1}$ does have properties (i) and (iv), and if Φ is conformal in periodic components of $\mathcal{F}(f)$, then F has also property (ii'), but it is not clear whether F has property (iii').

We remarked in §3 that if g is of the form (2) and if f is defined by (1), then the function $f'(z)/(f(z)-z)$ is rational. The argument used in the proof of Theorem 4 shows that if $F = \Phi \circ f \circ \Phi^{-1}$, then $F'(z)/(F(z)-z)$ is also rational, while the argument used in the proof of Theorem 1 only gives the weaker statement (7).

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