A question of Gol'dberg concerning entire functions with prescribed zeros

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Abstract

Let (z_j) be a sequence of complex numbers satisfying $|z_j| \to \infty$ as $j \to \infty$ and denote by $n(r)$ the number of z_j satisfying $|z_j| \le r$. Suppose that $\liminf_{r\to\infty} \log n(r)/\log r > 0$. Let ϕ be a positive, non-decreasing function satisfying $\int^\infty (\phi(t) t \log t)^{-1} dt < \infty$. It is proved that there exists an entire function f whose zeros are the z_j such that $\log\log M(r, f) = o((\log n(r))^2 \phi(\log n(r)))$ as $r \to \infty$ outside some exceptional set of finite logarithmic measure, and that the integral condition on ϕ is best possible here. These results answer a question by A. A. Gol'dberg.

1 Introduction and results

Let (z_j) be a sequence of complex numbers satisfying

$$\lim_{j\to\infty} |z_j| = \infty. \tag{1}$$

A classical theorem of Weierstraß says that there exists an entire function f whose zeros are the z_j, with multiplicities taken into account. Many authors have studied the problem to choose this function f such that there is a "good" bound for its maximum modulus $M(r, f)$ in terms of the number $n(r)$ of values z_j satisfying $|z_j| \le r$, see [1] for references. A. A. Gol'dberg [6, Theorem 6] proved that if

$$0 < \liminf_{r\to\infty} \frac{\log n(r)}{\log r} \le \infty \tag{2}$$

and if $\varepsilon > 0$, then we may achieve

$$\log\log M(r, f) = o((\log n(r))^{2+\varepsilon}) \quad (r \notin F) \tag{3}$$

as $r \to \infty$, where F has finite logarithmic measure. He noted that the right side of (3) cannot be replaced by $O(\log n(r))$, but he asked whether it is possible to replace the number $2 + \varepsilon$ in (3) by $1 + \varepsilon$. We shall show that this is not the case, although (3) can be improved slightly. More precisely, we have the following results.

Theorem 1 *Let $\phi(t)$ be positive, non-decreasing function satisfying*

$$\int^\infty \frac{dt}{\phi(t) t \log t} < \infty$$

and let (z_j) be a sequence of complex numbers satisfying (1) *and* (2)*. Then there exists an entire function f whose zeros are the z_j and a set $F \subset [1, \infty)$ of finite logarithmic measure such that*

$$\log\log M(r, f) = o((\log n(r))^2 \phi(\log n(r))) \quad (r \notin F). \tag{4}$$

Theorem 2 *Let $\phi(t)$ be positive, non-decreasing function satisfying*

$$\int^{\infty} \frac{dt}{\phi(t) t \log t} = \infty$$

and

$$\phi(t) \leq t^{\beta} \tag{5}$$

for some $\beta > 0$ and all large t. If $0 < \alpha < \infty$, then there exists a sequence (z_j) satisfying (1) *and*

$$\liminf_{r \to \infty} \frac{\log n(r)}{\log r} = \alpha \tag{6}$$

with the following property: if f is an entire function with zeros z_j, then there exists a set $F \subset [1, \infty)$ of infinite logarithmic measure such that

$$(\log n(r))^2 \phi(\log n(r)) = o(\log \log M(r, f)) \quad (r \in F). \tag{7}$$

The condition (5) is probably not necessary in Theorem 2. On the other hand, since the interesting choices for $\phi(t)$ are functions like $\phi(t) = 1$ or $\phi(t) = \log \log t$, the hypothesis (5) seems to be a mild restriction for the applicability of Theorem 2. The choice $\phi(t) = 1$ shows that we cannot replace $2 + \varepsilon$ by 2 in (3).

Gol'dberg [6, Theorem 6] has shown that we can replace $n(r)$ by

$$N(r) = \int_0^r \frac{n(t) - n(0)}{t} dt + n(0) \log r$$

in (3). Similarly, we can replace $n(r)$ by $N(r)$ in Theorems 1 and 2. As far as Theorem 1 is concerned, this follows from the inequality

$$n(r) \leq N(r)^2 \quad (r \notin F)$$

for some set F of finite logarithmic measure. This inequality was also used (and proved) by Gol'dberg [6, p. 14]. Since $N(r) \leq n(r) \log r + O(1) \leq n(r)^2$ for large r by (2), Theorem 2 also holds with $n(r)$ replaced by $N(r)$.

Finally we note that some hypothesis like (2) is necessary for the validity of (3) or (4). In fact, Gol'dberg [6, Theorem 2] proved that if $\psi(t)$ is any given increasing function, then there exists a sequence (z_j) satisfying (1) such that

$$\log M(r, f) \geq \psi(n(r)) \quad (r \in F)$$

for all entire functions f with zeros z_j, where F has upper logarithmic density one.

2 Proof of Theorem 1

First we note that there exists a positive, non-decreasing function $\psi(t)$ satisfying

$$\int^{\infty} \frac{dt}{\psi(t) t \log t} < \infty$$

and

$$\lim_{t\to\infty} \frac{\psi(t)}{\phi(t)} = 0. \tag{8}$$

For example, if we take

$$\psi(t) = \sup_{s\le t} \phi(s)\sqrt{\int_s^\infty \frac{d\tau}{\phi(\tau)\tau\log\tau}},$$

then $\psi(t)$ is non-decreasing, (8) is satisfied, and

$$\int_r^\infty \frac{dt}{\psi(t)t\log t} \le \int_r^\infty \frac{dt}{\phi(t)t\log t\sqrt{\int_t^\infty \frac{d\tau}{\phi(\tau)\tau\log\tau}}} = 2\sqrt{\int_r^\infty \frac{dt}{\phi(t)t\log t}} < \infty.$$

We may assume that $0 < |z_1| \le |z_2| \le |z_3| \le \ldots$. We define $p_j = [2\psi(\log j)\log j]$ and

$$f(z) = \prod_{j=1}^\infty E\left(\frac{z}{z_j}, p_j\right)$$

where $E(z/z_j, p_j)$ is the usual Weierstraß primary factor. It is well-known that f is an entire function with zeros z_j. Blumenthal [2, pp. 131-136] proved that $\log|E(u,p)| \le |u|^{p+1}$ for $p \ge 1$. Clearly, this also holds if $p = 0$. (We remark that Blumenthal's estimate can also be obtained from the results of Denjoy [4, pp. 18-24] and Cohn [3] if $p > 1$, and the case $p = 1$ is not difficult.) It follows that

$$\log M(r,f) \le \sum_{j=1}^\infty \left(\frac{r}{|z_j|}\right)^{p_j+1}.$$

For sufficiently large r, we define

$$r' = r\exp\left(\frac{1}{\psi(\log n(r))}\right).$$

Then

$$\log M(r,f) \le \sum_{|z_j|\le r}\left(\frac{r}{|z_j|}\right)^{p_j+1} + \sum_{r<|z_j|\le r'}\left(\frac{r}{|z_j|}\right)^{p_j+1} + \sum_{|z_j|>r'}\left(\frac{r}{|z_j|}\right)^{p_j+1} = S_1 + S_2 + S_3. \tag{9}$$

Now

$$\begin{aligned}
\log S_1 &\le \log\left(n(r)\left(\frac{r}{|z_1|}\right)^{p_{n(r)}+1}\right) && (10)\\
&\le \log n(r) + \left(p_{n(r)}+1\right)\log\frac{r}{|z_1|} \\
&\le (2+o(1))\psi(\log n(r))\log n(r)\log r && (11)\\
&= o((\log n(r))^2\phi(\log n(r)))
\end{aligned}$$

by our choice of p_j, (2), and (8). To estimate S_2, we use a well-known growth lemma essentially due to Borel and refined by Nevanlinna [9, p. 140]. Applying it to the functions $u(r) = \log n(e^r)$ and $\varphi(t) = 1/\psi(t)$ and choosing $\varepsilon = 1$ we deduce that

$$\log n(r') \le (\log n(r))^2 \quad (r \notin F) \tag{12}$$

for some set F of finite logarithmic measure. We remark that the book by Gol'dberg and Ostrovskii contains the lemma of Borel and Nevanlinna in a slightly different, but equivalent form [7, p. 120]. In fact, Nevanlinna first proved his result in the form stated by Gol'dberg and Ostrovskii and then made a suitable substitution [9, p. 142-143]. To deduce (12) from the version of the Borel-Nevanlinna lemma given in [7], we have to choose $u(r) = \log\log\log n(e^r)/\log 2$. This version of the lemma was also used by Gol'dberg [6], but he applied it to the function $u(r) = \log\log n(e^r)$ and chose $\varphi(u) = e^{-\delta u}$. It follows from (12) that

$$\log S_2 \leq \log n(r') = o((\log n(r))^2 \phi(\log n(r))) \quad (r \notin F). \tag{13}$$

Finally, we have

$$S_3 \leq \sum_{j=n(r')+1}^{\infty} \left(\exp\left(\frac{1}{\psi(\log n(r))} \right) \right)^{-p_j - 1} \leq \sum_{j=n(r')+1}^{\infty} j^{-2} \leq 1 \tag{14}$$

for sufficiently large r. Combining (9), (11), (13), and (14) we obtain (4).

We remark that the above proof essentially follows the argument of Gol'dberg [6], the difference being the way in which the lemma of Borel and Nevanlinna is applied.

3 Proof of Theorem 2

Similarly as in the proof of Theorem 1, there exists a positive, non-decreasing function $\psi(t)$ satisfying

$$\int^{\infty} \frac{dt}{\psi(t) t \log t} = \infty, \tag{15}$$

$$\psi(t) \leq t^{\gamma} \tag{16}$$

for some $\gamma > 0$ and all large t, and

$$\lim_{t\to\infty} \frac{\psi(t)}{\phi(t)} = \infty. \tag{17}$$

With α as in the statement of the theorem, we take $r_1 > 1$ so large that the sequence (r_j) defined recursively by

$$\log r_{j+1} = (\log r_j)^2 \psi(\alpha \log r_j) \tag{18}$$

is increasing and tends to ∞. We define $l_0 = 0$ and $l_j = [(r_{j+1})^{\alpha}]$ for $j \geq 1$. Clearly, $l_{j-1}/l_j \to 0$ as $j \to \infty$. Next we define $z_k = r_j$ for $l_{j-1} + 1 \leq k \leq l_j$ and $j \geq 1$. We deduce that if $r_j \leq r < r_{j+1}$, then

$$n(r) = l_j = (1 + o(1))(r_{j+1})^{\alpha} \tag{19}$$

as $j \to \infty$. Thus (6) holds.

Let now f be an entire function whose zeros are the z_j. By $c_m(r)$ we denote the mth Fourier coefficient of $\log|f(re^{i\theta})|$. First we show that

$$|c_m(r)| \leq \log M(r, f) \tag{20}$$

for $m \geq 1$ and sufficiently large r. Indeed, $c_m(r) = |c_m(r)|e^{i\varphi}$, so

$$\begin{aligned}
|c_m(r)| &= \operatorname{Re}\left(e^{-i\varphi}\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|e^{-im\theta}d\theta\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\cos(m\theta+\varphi)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|(1+\cos(m\theta+\varphi))d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta \\
&\leq \frac{\log M(r,f)}{2\pi}\int_0^{2\pi}(1+\cos(m\theta+\varphi))d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta \\
&= \log M(r,f) - N(r) - \log|f(0)|
\end{aligned}$$

by Jensen's formula, and (20) follows.

It is well-known (see [5, p. 312], [8, p. 379], or [10, Lemma 1]) that if $\log f(z) = \sum_{m=0}^{\infty} a_m z^m$ near $z = 0$, then

$$c_m(r) = \frac{1}{2}a_m r^m + \frac{1}{2m}\sum_{|z_k|\leq r}\left(\left(\frac{r}{z_k}\right)^m - \left(\frac{\bar{z}_k}{r}\right)^m\right)$$

for $m \geq 1$. It follows that if $r_j \leq r < r_{j+1}$, then

$$c_m(r) = \frac{1}{2}a_m r^m + \frac{1}{2m}\sum_{\mu=1}^{j}(l_\mu - l_{\mu-1})\left(\left(\frac{r}{r_\mu}\right)^m - \left(\frac{r_\mu}{r}\right)^m\right).$$

We deduce that if $r_{j-1} < r < r_j < R < r_{j+1}$, then

$$\begin{aligned}
&c_m(R) - \left(\frac{R}{r}\right)^m c_m(r) \\
= \; &\frac{l_j - l_{j-1}}{2m}\left(\left(\frac{R}{r_j}\right)^m - \left(\frac{r_j}{R}\right)^m\right) - \frac{1}{2m}\sum_{\mu=1}^{j-1}(l_\mu - l_{\mu-1})\left(\left(\frac{r_\mu}{R}\right)^m - \left(\frac{r_\mu R}{r^2}\right)^m\right).
\end{aligned}$$

We now define r_j' by

$$\log\frac{r_j}{r_j'} = \frac{1}{2\psi(\log l_j)}. \tag{21}$$

Since $\psi(t) \to \infty$ as $t \to \infty$ it follows that $r_j < r_{j+1}' < r_{j+1}$ for large j. Moreover, $r_{j+1}'/r_j \to \infty$ and r_j/r_j' remains bounded as $j \to \infty$. Hence

$$c_m(R) - \left(\frac{R}{r}\right)^m c_m(r) = \left(\frac{1}{2} + o(1)\right)\frac{l_j}{m}\left(\frac{R}{r_j}\right)^m \tag{22}$$

if $r_j' < r < r_j$ and $r_{j+1}' < R < r_{j+1}$. Now we define

$$m = [\psi(\log l_j)\log l_j]. \tag{23}$$

From (19) we deduce that

$$\begin{aligned}
\log\left(\frac{l_j}{m}\left(\frac{R}{r_j}\right)^m\right) &= \log l_j - \log m + m\log\frac{R}{r_j} \\
&\geq (1-o(1))\psi(\log l_j)\log l_j \log r_{j+1} \qquad (24) \\
&= \left(\frac{1}{\alpha} - o(1)\right)(\log n(R))^2\psi(\log n(R))
\end{aligned}$$

for $r'_{j+1} < R < r_{j+1}$. Suppose now that there exists R satisfying $r'_{j+1} < R < r_{j+1}$ and

$$\log\log M(R,f) \le \frac{1}{2\alpha}(\log n(R))^2\psi(\log n(R)). \tag{25}$$

From (20), (22), (24), and (25) we can deduce that

$$\left(\frac{R}{r}\right)^m |c_m(r)| \ge \left(\frac{1}{2} - o(1)\right)\frac{l_j}{m}\left(\frac{R}{r_j}\right)^m$$

for $r'_j < r < r_j$, that is,

$$|c_m(r)| \ge \left(\frac{1}{2} - o(1)\right)\frac{l_j}{m}\left(\frac{r'_j}{r_j}\right)^m$$

for $r'_j < r < r_j$. Combining this with (16), (18), (19), (20), (21), and (23) we find that

$$\begin{aligned}
\log\log M(r,f) &\ge \log|c_m(r)| \\
&\ge \log l_j - \log m + m\log\frac{r'_j}{r_j} - O(1) \\
&\ge \log l_j - \log\psi(\log l_j) - \log\log l_j - \frac{1}{2}\log l_j - O(1) \\
&\ge \left(\frac{1}{2} - o(1)\right)\log l_j \\
&= \left(\frac{\alpha}{2} - o(1)\right)\log r_{j+1} \\
&= \left(\frac{\alpha}{2} - o(1)\right)(\log r_j)^2\psi(\alpha\log r_j) \\
&\ge \left(\frac{1}{2\alpha} - o(1)\right)(\log n(r))^2\psi(\log n(r))
\end{aligned}$$

for $r'_j < r < r_j$.

We now define F to be the set of all r satisfying

$$\log\log M(r,f) \ge \frac{1}{3\alpha}(\log n(r))^2\psi(\log n(r)).$$

From (17) we deduce that (7) is satisfied, and the argument given above shows that if j is large enough, then one of the intervals (r'_{2j-1}, r_{2j-1}) and (r'_{2j}, r_{2j}) is contained in F. Since

$$\log\frac{r_{2j-1}}{r'_{2j-1}} = \frac{1}{2\psi(\log l_{2j-1})} \ge \frac{1}{2\psi(\log l_{2j})} = \log\frac{r_{2j}}{r'_{2j}},$$

this implies that

$$\int_{F\cap(r_{2j-2},r_{2j})}\frac{dt}{t} \ge \frac{1}{2\psi(\log l_{2j})}.$$

From (16) and (18) we deduce that

$$\log r_{j+1} \le (\log r_j)^2(\alpha\log r_j)^\gamma \le (\log r_j)^{\gamma+3}$$

for sufficiently large j. We define $b = \gamma + 3$. Induction shows that there exists $a > 0$ such that

$$\log r_{j+1} \leq \exp(ab^j).$$

Hence

$$\log l_j \leq \alpha \log r_{j+1} \leq \alpha \exp(ab^j)$$

so that

$$\int_{F \cap (r_{2j-2}, r_{2j})} \frac{dt}{t} \geq \frac{1}{2\psi(\alpha \exp(ab^{2j}))}$$

for sufficiently large j, say $j \geq j_0$. Putting $r_0 = r_{2j_0-2}$ we deduce that

$$\int_{F \cap (r_0, \infty)} \frac{dt}{t} \geq \frac{1}{2} \sum_{j=j_0}^{\infty} \frac{1}{\psi(\alpha \exp(ab^{2j}))} \geq \frac{1}{2} \int_{j_0}^{\infty} \frac{dt}{\psi(\alpha \exp(ab^{2t}))}.$$

Using the substitution $u = \alpha \exp(ab^{2t})$ one easily derives from (15) that the integral on the right side diverges, that is, F has infinite logarithmic measure. This completes the proof of Theorem 2.

Remark The set F in Theorem 2 depends not only on the sequence (z_j), but also on the function f. Using (20) one can also obtain a result where the set F does not depend on f:

Suppose that $\phi(t)$ is a decreasing function which tends to zero as $t \to \infty$. If $0 < \alpha < \infty$, then there exists a sequence (z_j) satisfying (1) and (6) and a set F of upper logarithmic density one such that (7) holds for all entire functions f with zeros (z_j).

This result is best possible in the following sense:

Given a sequence (z_j) satisfying (1) and (2) and an unbounded set $G \subset [1, \infty)$, there exists an unbounded sequence (r_k) satisfying $r_k \in G$ for all k such that $\log \log M(r_k) = o((\log n(r_k))^2)$ as $k \to \infty$.

This can be proved by choosing (r_k) rapidly increasing, $p_j = [2 \log j / \log |z_j|]$ for $s_k < |z_j| \leq r_k$, and p_j very large for $r_k < |z_j| \leq s_{k+1}$, where (s_k) is a suitable sequence satisfying $s_k < r_k < s_{k+1}$.

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