Canonical products of infinite order

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1 Introduction and results

Let (z_j) be a sequence of complex numbers satisfying $0 < |z_1| \le |z_2| \le \dots$ and $\lim_{j\to\infty} |z_j| = \infty$. A classical theorem of Weierstraß [47] says that there exists an entire function f whose zeros are precisely the z_j, where a complex number that occurs m times in the sequence (z_j) corresponds to a zero of multiplicity m. Moreover, Weierstraß's theorem asserts that f may be written in the form

$$f(z) = \prod_{j=1}^{\infty} E\left(\frac{z}{z_j}, p_j\right) \tag{1}$$

for a suitable sequence (p_j) of non-negative integers. Here $E(u,p)$ is the Weierstraß primary factor which is defined by $E(u,p) = (1-u)\exp(\sum_{j=1}^{p} u^j/j)$ for $p \ge 1$ and $E(u,0) = 1-u$. It is well-known that the choices $p_j = j$, $p_j = [\log j]$, or $p_j = [\alpha \log j / \log r_j]$ where $\alpha > 1$ always assure the convergence of the infinite product which defines f. Here and in the following $[x]$ denotes the greatest integer not larger than x.

Much more can be said if $\sum_{j=1}^{\infty} |z_j|^{-\alpha} < \infty$ for some positive α. The infimum of all these numbers α is called the exponent of convergence of (z_j) and denoted by σ. An equivalent definition is

$$\sigma = \limsup_{r\to\infty} \frac{\log n(r)}{\log r},$$

where $n(r)$ denotes the number of z_j satisfying $|z_j| \le r$, that is, $n(r) = \max\{j : |z_j| \le r\}$. In this case, we may take $p_j = q$ where q is the smallest integer such that $\sum_{j=1}^{\infty} |z_j|^{-q-1} < \infty$. Clearly, we have $q \le \sigma \le q+1$. In particular, if σ is not an integer, then $q = [\sigma]$. The Weierstraß product formed with $p_j = q$ is called the canonical product and q is called its genus.

A classical result of Pólya [39] and Valiron ([44], §§59-60, [46]) says that if σ is finite but not an integer, then there exists a positive constant K depending only on σ such that the canonical product f satisfies

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \le K,$$

where $M(r, f)$ denotes the maximum modulus of f. Let $K(\sigma)$ be the smallest constant with this property. Pólya and Valiron proved that

$$K(\sigma) \leq \sigma \int_0^\infty \frac{\log M(r, E(\cdot, q))}{r^{\sigma+1}} dr, \tag{2}$$

while functions with zeros regularly distributed on a ray, the so-called Lindelöf functions, show that

$$K(\sigma) \geq \frac{\pi}{|\sin \pi\sigma|}. \tag{3}$$

Pólya ([39], p. 183) noted that if $q = 0$, that is, if $0 < \sigma < 1$, then the right sides of (2) and (3) are equal so that $K(\sigma) = \pi / \sin \pi\sigma$ in this case. He also stated (without proof) that for $\sigma > 1$ the right side of (2) is of the form

$$\frac{\pi}{|\sin \pi\sigma|} + 2\log(\sigma + 1) + h(\sigma)$$

for some bounded function $h(\sigma)$, while Valiron ([44], p. 228) obtained the slightly weaker result that the right side of (2) is equal to

$$\frac{1}{(q+1-\sigma)(\sigma - q)} + 2\log q + h_1(\sigma)\sqrt{\log q}$$

for some bounded function $h_1(\sigma)$. Baernstein [private communication] recently proved that if $\sigma > 1$, then we have strict inequality in (3). Hence we have

$$\frac{\pi}{|\sin \pi\sigma|} < K(\sigma) \leq \frac{\pi}{|\sin \pi\sigma|} + 2\log(\sigma + 1) + A \tag{4}$$

in this case, where A is an absolute constant. The precise value of $K(\sigma)$ is not known for $\sigma > 1$. In fact, it is still open whether $K(\sigma) \to \infty$ as $\sigma \to \infty$, see Remark 3 in § 6 for further discussion. For more information and an improvement of the upper bound in the case that the zeros are distributed on finitely many rays, we refer to Shea and Wainger [43].

In this paper, we consider this problem for sequences (z_j) which do not have a finite exponent of convergence, that is, we assume that

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \infty. \tag{5}$$

Here it turns out to be useful to consider the quantity

$$\mu = \limsup_{r\to\infty} \frac{\log\log n(r)}{\log\log r}. \tag{6}$$

Clearly, (5) implies that $\mu \geq 1$ while $\mu > 1$ implies (5).

Theorem 1 *Let (z_j) be a sequence of complex numbers satisfying $0 < |z_1| \le |z_2| \le \ \ldots$ and $\lim_{j\to\infty} |z_j| = \infty$. Suppose that the number μ defined by (6) satisfies $1 < \mu \le \infty$. Then there exists an entire function f whose zeros are precisely the z_j such that*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \le \begin{cases} \dfrac{3\mu}{4(\mu-1)} & \text{if } 1 < \mu < \infty \\ \dfrac{3}{4} & \text{if } \mu = \infty. \end{cases} \tag{7}$$

Theorem 2 *For all μ satisfying $1 < \mu < \infty$ there exists a sequence (z_j) satisfying the hypothesis of Theorem 1 such that*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \ge (\mu-1)\log\frac{\mu-1}{\mu} + (\mu+1)\log\frac{\mu+1}{\mu} \tag{8}$$

for all entire functions f whose zeros are precisely the z_j.

We remark that the right side of (8) is positive and decreasing for $\mu > 1$. It tends to 0 as $\mu \to \infty$ and to $2\log 2$ as $\mu \to 1$.

For instance, it follows from Theorems 1 and 2 that $\mu \ge 4$ guarantees that we can achieve

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \le 1,$$

while $\mu \ge 1.1$ does not. Similar to $K(\sigma)$, we may define

$$L(\mu) = \sup_{(z_j)} \inf_{f} \liminf_{r\to\infty} \frac{\log M(r,f)}{n(r)},$$

where the infimum is taken over all entire functions f with zeros z_j and the supremum is taken over all sequences (z_j) satisfying (6). Then the last remark takes the form $L(1.1) > 1 \ge L(4)$.

There is a gap between the upper bound for $L(\mu)$ given in Theorem 1 and the lower bound given in Theorem 2. In particular, I have been unable to determine the behavior of $L(\mu)$ as $\mu \to 1$ or $\mu \to \infty$. In view of the fact that even the corresponding problem for sequences with finite exponent of convergence σ, namely the question whether $\lim_{\sigma\to\infty} K(\sigma) = \infty$, is still open, it seems to be difficult to close the gap between the estimates of Theorems 1 and 2.

It is possible, however, to improve the constants a little bit. For instance, it will be apparent from the proof that the number $\frac{3}{4}$ occurring in (7) can be replaced by $a_\infty = 0.7423049\ldots$, the solution of a certain transcendental equation, cf. [13] and §2. The constant on the right side of (8) can also be improved by a refinement of the method, at the expense of more complicated computations, cf. Remark 1 in § 6. It seems unlikely, however, that the methods of this paper will lead to sharp bounds.

The problem to estimate the maximum modulus of a Weierstraß product in terms of the sequence of zeros is quite old. In the case of finite genus the basic result was proved by Borel ([10], p. 361) in 1897 and it says that the order of the canonical product is equal to the exponent of convergence of the sequence of zeros. The results of Pólya and Valiron in the case of finite genus have already been mentioned.

For sequences which do not have a finite exponent of convergence, the problem of estimating the growth of a Weierstraß product in terms of the zeros seems to have been considered first at the beginning of this century by Boutroux ([11], pp. 188-198) and Maillet ([32], pp. 266-280), who studied sequences (z_j) satisfying $n(r) \leq \exp_k r^\sigma$ where $k \geq 1$ and $0 < \sigma < \infty$, and by Kraft ([30], Kapitel C) who avoided such growth restrictions by introducing certain comparison functions ("Vergleichsfunctionen") of regular growth. A few years later, Blumenthal ([8], Chapitre 4) modified and extended Kraft's results and Denjoy ([15], [16]) obtained very precise estimations in the case that $n(r)$ grows regular in some sense (but is allowed to grow arbitrarily fast). More recent contributions to the subject are due to Ahmad [2], Balašov ([3], Theorem 5), Frank, Hennekemper, and Polloczek ([20], Hilfssatz 4), Hellerstein ([24], Lemma 2), Jank and Volkmann ([25], [26], [27], [28]), Jank and Wallner [29], Mues ([37], Hilfssatz 2), Winkler [48], and Yang and Zheng ([49], Lemma 2). Applications to various problems in function theory can also be found in these papers.

In the papers cited above, the emphasis has been on estimating $M(r, f)$ for a large set of r-values, either for all large r or outside small exceptional sets, but at least on a set of infinite measure, while our Theorem 1 gives an upper bound for $M(r, f)$ in terms of $n(r)$ only on a sequence of r-values. It should therefore be remarked that the limes inferior in Theorem 1 cannot be replaced by the limes superior. To see this, we note that if the zeros of an entire function f of infinite order are distributed on a single ray, then f has infinite lower order. This was proved by Gol'dberg ([21], Corollary 1, see also [22], p. 342) and, according to Miles ([35], p. 137), who also gives a proof of this result, independently by Edrei and Fuchs ([18], see also [17], p. 308). It is not difficult to see that for any $\mu > 1$ there exists a sequence (z_j) of positive numbers which satisfies (6), but has the property that $n(r)$ grows slowly on a sequence of r-values, say $n(r) \leq r$ for arbitrarily large r. It follows that if f is an entire function with zeros z_j, then f has infinite order and hence infinite lower order so that

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{n(r)} = \infty.$$

Using these arguments one can in fact show that if $\phi(r)$ is any given increasing function of r, then there exists a sequence (z_j) such that

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{\phi(n(r))} = \infty$$

for all entire functions f whose zeros are the z_j. One may also achieve that $\log M(r, f)/\phi(n(r)) \to \infty$ on a large set of r-values, say on a set of infinite measure or upper density one.

Similarly, one can prove that Hilfssatz 4 in [20], which says that there exists a Weierstraß product f satisfying $\log M(r,f) \le A \exp BN(r)$ outside an exceptional set of finite measure for certain constants A and B, is best possible in the sense that $A \exp BN(r)$ cannot be replaced by $\exp \phi(N(r))$ for any fixed function $\phi(t)$ satisfying $\lim_{t\to\infty} \phi(t)/t = 0$. Here $N(r)$ is the usual counting function of Nevanlinna theory, that is, $N(r) = \int_0^r n(t)/t\,dt$.

In Remark 2 in § 6 we will sketch an alternative method to prove that the limes inferior in Theorem 1 cannot be replaced by the limes superior. This method actually shows that there exist sequences (z_j) with $\mu = \infty$ such that if f is an entire function with zeros z_j, then $\log M(r,f) \ge (1-o(1))\pi n(r)$ for all r outside some exceptional set of finite logarithmic measure.

2 Estimations of primary factors

The following estimation of the Weierstraß primary factors is due to J. H. E. Cohn [13].

Lemma 1 *(i) There exists a decreasing sequence (a_j) of positive constants satisfying $a_1 = 1.2784645\ldots$ and $a_\infty = \lim_{p\to\infty} a_p = 0.7423049\ldots$ such that*

$$\log |E(z,p)| \le a_p |z|^p \quad (|z| \ge 1, z \ne 1) \tag{9}$$

for all $p \ge 1$.

(ii) There exists a decreasing sequence (b_j) of positive constants satisfying $b_1 = 0.5$ *and* $b_\infty = \lim_{p\to\infty} b_p = \mathrm{Ci}(\pi/2) = 0.4720006\ldots$ *such that*

$$\log |E(z,p)| \le b_p |z|^{p+1} \quad (|z| \le 1, z \ne 1) \tag{10}$$

for all $p \ge 1$.

We shall not use the full power of Lemma 1. Instead of (10), the weaker estimate

$$\log |E(z,p)| \le |z|^{p+1} \quad (|z| \le 1, z \ne 1) \tag{11}$$

suffices for our purposes. This estimate is due to Blumenthal ([8], p. 131). It is easier to prove (cf. Jank and Volkmann [28], p. 94) than (10) and also holds for $p = 0$. Also, instead of (9), we shall for simplicity sometimes use the weaker result

$$\log |E(z,p)| \le 2|z|^p \quad (|z| \ge 1, z \ne 1) \tag{12}$$

for $p \ge 1$. We need (9), however, to deduce the following result.

Lemma 2 *There exists a positive integer P such that*

$$\log |E(z,p)| \le \frac{3}{4}|z|^p \quad (|z| \ge 1, z \ne 1)$$

for $p > P$.

We remark that the part of Lemma 1 that is needed for the proof of Lemma 2 was already obtained by Denjoy. More precisely, Denjoy ([16], pp. 24-25) proved that $a_2 = 1$, $a_p < 1$ for $p > 2$, and $a_\infty = 1/\alpha'$ where $\alpha' = 1.34\ldots$.

3 Proof of Theorem 1

First, we put $r_j = |z_j|$. Without loss of generality we may assume that $r_1 > 1$. For $t \geq t_0 = \log(\max\{3, r_3\})$ we define $\mu(t)$ by

$$\mu(\log r) = \frac{\log\log n(r)}{\log\log r}$$

and for $0 \leq t < t_0$ we put $\mu(t) = 0$. Then $n(r) = \exp\left((\log r)^{\mu(\log r)}\right)$ for $r \geq \exp t_0$ and $\limsup_{t\to\infty} \mu(t) = \mu$. If $\mu < \infty$, let $\rho(t)$ be a proximate order for $\mu(t)$, that is, let $\rho(t)$ be a differentiable function satisfying $\lim_{t\to\infty} \rho(t) = \mu$, $\lim_{t\to\infty} \rho'(t) t \log t = 0$, and $\mu(t) \leq \rho(t)$ for all large t, with equality for an unbounded sequence of t-values, cf. [31], p. 35, and [44], §53, for the existence of a function $\rho(t)$ with these properties.

We shall now define sequences (R_m) and (T_m) of positive real numbers and sequences (q_m) and (p_j) of non-negative integers by recursion. The function f will then be defined as the Weierstraß product with primary factors $E(z/z_j, p_j)$.

To this end, choose R_1 and T_1 such that $0 < R_1 < T_1 < r_1$, define $q_1 = 0$, and suppose that $m \geq 2$, that R_k, T_k, and q_k have been defined for $1 \leq k \leq m-1$, and that p_j has been defined for all j satisfying $r_j \leq T_{m-1}$, that is, for $1 \leq j \leq n(T_{m-1})$.

We define $q_m = \max\{p_j : 1 \leq j \leq n(T_{m-1})\}$ if $T_{m-1} \geq r_1$ and $q_m = 0$ otherwise. Next we choose R_m satisfying $R_m > T_{m-1} + 1$ such that the following four conditions are satisfied:

The first requirement is that $\mu(\log R_m) = \rho(\log R_m)$ if $\mu < \infty$ and $\mu(\log R_m) \geq \mu(\log r)$ for all r satisfying $0 < r \leq R_m$ if $\mu = \infty$. Secondly, we suppose that

$$n(T_{m-1})(R_m)^{q_m} \leq \frac{n(R_m)}{m} \tag{13}$$

and the third condition is that

$$\left(1 + \frac{1}{m}\right)(\log t)^{\rho(\log t)-1} \leq \frac{1}{4}(\log R_m)^{\rho(\log R_m)-1} \tag{14}$$

for $t \leq T_{m-1}$ if $\mu < \infty$ and

$$\left(1 + \frac{1}{m}\right)(\log T_{m-1})^{\mu(\log R_m)-1} \leq \frac{1}{4}(\log R_m)^{\mu(\log R_m)-1} \tag{15}$$

if $\mu = \infty$. Finally, we suppose that

$$\frac{1}{m^2}\log n(R_m) \geq \log\log n(R_m). \tag{16}$$

It is clear that the first condition can be satisfied for arbitrarily large values of R_m, and from that the hypothesis that $\mu > 1$ we deduce that if R_m satisfies the first condition and is sufficiently large, then the three other conditions are also satisfied.

For $n(T_{m-1}) < j \le n(R_m)$, that is, for $T_{m-1} < r_j \le R_m$, we define

$$p_j = \left[\left(1+\frac{1}{m}\right)\frac{\log j}{\log r_j}\right]. \tag{17}$$

Next we choose T_m such that $T_m > R_m$ and

$$\sum_{r_j > T_m} \left(\frac{R_m}{r_j}\right)^{\left(1+\frac{1}{m+1}\right)\frac{\log j}{\log r_j}} < \frac{1}{2^m}. \tag{18}$$

That is can be achieved by a suitable choice of T_m follows from the fact that

$$\left(\frac{R_m}{r_j}\right)^{\left(1+\frac{1}{m+1}\right)\frac{\log j}{\log r_j}} = j^{\left(1+\frac{1}{m+1}\right)\left(\frac{\log R_m}{\log r_j}-1\right)} \le j^{-\left(1+\frac{1}{m+2}\right)}$$

for sufficiently large j. In the case that $\mu < \infty$ we shall also assume that T_m is chosen so large that

$$|\rho'(u)u\log u| \le \frac{1}{m+1} \quad \text{and} \quad |\rho(u)-\rho(v)| \le \frac{1}{m+1} \tag{19}$$

for $u, v \ge \log T_m$.

Finally, for $n(R_m) < j \le n(T_m)$, that is, for $R_m < r_j \le T_m$, we choose p_j such that

$$p_j \ge \left(1+\frac{1}{m}\right)\frac{\log j}{\log r_j} \tag{20}$$

and

$$\sum_{R_m < r_j \le T_m} \left(\frac{R_m}{r_j}\right)^{p_j+1} \le 1. \tag{21}$$

It is clear that this can be achieved by choosing p_j sufficiently large.

Let now (R_m), (T_m), (q_m), and (p_j) be defined as above and consider the Weierstraß product f defined by (1). Then

$$\begin{aligned} \log M(R_m, f) &\le \sum_{j=1}^{\infty} \log M\left(\frac{R_m}{r_j}, E(\cdot, p_j)\right) \\ &\le \Bigg(\sum_{\substack{r_j \le R_m \\ p_j = 0}} + \sum_{\substack{r_j \le T_{m-1} \\ p_j \ge 1}} + \sum_{\substack{T_{m-1} < r_j \le R_m \\ 1 \le p_j \le P}} + \sum_{\substack{T_{m-1} < r_j \le R_m \\ p_j > P}} \\ &\quad + \sum_{R_m < r_j \le T_m} + \sum_{r_j > T_m}\Bigg) \log M\left(\frac{R_m}{r_j}, E(\cdot, p_j)\right) \\ &= S_1 + S_2 + S_3 + S_4 + S_5 + S_6 \end{aligned} \tag{22}$$

for $m \geq 2$. First, we have $S_5 \leq 1$ by (11) and (21) and

$$S_2 \leq 2 \sum_{r_j \leq T_{m-1}} \left(\frac{R_m}{r_1}\right)^{q_m} = 2n(T_{m-1})(R_m)^{q_m} \leq \frac{2n(R_m)}{m}$$

by (12) and (13). To estimate S_1, we note that if $p_j = 0$, then p_j is defined by (17) so that $\log j / \log r_j < 1$, that is, $j < r_j$. It follows that

$$\begin{aligned}
S_1 &\leq \sum_{\substack{r_j \leq R_m \\ p_j = 0}} \log\left(1 + \frac{R_m}{r_j}\right) \\
&\leq \sum_{j=1}^{n(R_m)} \log\left(1 + \frac{R_m}{j}\right) \\
&\leq \log(1 + R_m) + \int_1^{n(R_m)} \log\left(1 + \frac{R_m}{t}\right) dt \\
&= n(R_m) \log\left(1 + \frac{R_m}{n(R_m)}\right) + R_m \log\left(\frac{n(R_m) + R_m}{1 + R_m}\right) \\
&= o(n(R_m))
\end{aligned}$$

as $m \to \infty$, since (13) implies that

$$(R_m)^N = o(n(R_m)) \tag{23}$$

for every fixed $N > 0$. From (12), (17), (18), and (23) we deduce that

$$\begin{aligned}
S_3 &\leq 2 \sum_{\substack{T_{m-1} < r_j \leq R_m \\ 1 \leq p_j < P}} \left(\frac{R_m}{r_j}\right)^{p_j} \\
&\leq 2(R_m)^P \sum_{\substack{T_{m-1} < r_j \leq R_m \\ 1 \leq p_j < P}} \left(\frac{1}{r_j}\right)^{\left(1+\frac{1}{m}\right)\frac{\log j}{\log r_j}} \\
&\leq 2^{2-m}(R_m)^P \\
&= o(n(R_m))
\end{aligned}$$

as $m \to \infty$. Moreover, (11), (17), (18), and (20) imply that

$$\begin{aligned}
S_6 &\leq \sum_{l=m}^{\infty} \sum_{T_l < r_j \leq T_{l+1}} \left(\frac{R_m}{r_j}\right)^{p_j} \\
&\leq \sum_{l=m}^{\infty} \sum_{T_l < r_j \leq T_{l+1}} \left(\frac{R_l}{r_j}\right)^{\left(1+\frac{1}{l+1}\right)\frac{\log j}{\log r_j}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{l=m}^{\infty} \frac{1}{2^l} \\
&= \frac{1}{2^{m-1}}.
\end{aligned}
$$

Altogether we see that

$$
S_1 + S_2 + S_3 + S_5 + S_6 = o(n(R_m)) \tag{24}
$$

as $m \to \infty$ and it remains to estimate S_4.

By Lemma 2 and (17) we have

$$
\begin{aligned}
S_4 &\leq \frac{3}{4} \sum_{\substack{T_{m-1} < r_j \leq R_m \\ p_j > P}} \left(\frac{R_m}{r_j} \right)^{p_j} \\
&\leq \frac{3}{4} \sum_{T_{m-1} < r_j \leq R_m} \left(\frac{R_m}{r_j} \right)^{\left(1+\frac{1}{m}\right) \frac{\log j}{\log r_j}}
\end{aligned}
$$

We recall that $\rho(r)$ has already been defined as a proximate order in the case that $\mu < \infty$. For $\mu = \infty$ we define $\rho(r) = \rho(\log R_m)$. Next we put $\alpha(t) = \exp\left((\log t)^{\rho(\log t)} \right)$. Then (14) and (15) may be written in the common form

$$
\left(1 + \frac{1}{m}\right) \frac{\log \alpha(t)}{\log t} \leq \frac{1}{4} \frac{\log \alpha(R_m)}{\log R_m} = \frac{1}{4} \frac{\log n(R_m)}{\log R_m} \tag{25}
$$

for $t \leq T_{m-1}$. In particular, we have

$$
\log \alpha(T_{m-1}) \leq \frac{1}{4} \log n(R_m) \tag{26}
$$

For $n(T_{m-1}) < j \leq n(R_m)$, that is, for $T_{m-1} < r_j \leq R_m$, we define t_j by $\alpha(t_j) = j$. Since $n(t) \leq \alpha(t)$ for $t \leq R_m$ we have $t_j \leq r_j$ for $r_j \leq R_m$. This implies that

$$
S_4 \leq \frac{3}{4} \sum_{n(T_{m-1}) < j \leq n(R_m)} \left(\frac{R_m}{t_j} \right)^{\left(1+\frac{1}{m}\right) \frac{\log j}{\log t_j}}.
$$

Since

$$
\begin{aligned}
\sum_{\substack{t_j \leq T_{m-1} \\ j > n(T_{m-1})}} \left(\frac{R_m}{t_j} \right)^{\left(1+\frac{1}{m}\right) \frac{\log j}{\log t_j}} &\leq \sum_{\substack{t_j \leq T_{m-1} \\ j > n(T_{m-1})}} (R_m)^{\left(1+\frac{1}{m}\right) \frac{\log \alpha(t_j)}{\log t_j}} \\
&\leq \alpha(T_{m-1}) \, (R_m)^{\frac{1}{4} \frac{\log n(R_m))}{\log R_m}} \\
&\leq \sqrt{n(R_m)}
\end{aligned}
$$

by (25) and (26), we find that

$$\begin{aligned} S_4 &\leq \frac{3}{4} \sum_{T_{m-1} < t_j \leq R_m} \left(\frac{R_m}{t_j} \right)^{\left(1+\frac{1}{m}\right) \frac{\log j}{\log t_j}} + \frac{3}{4} \sqrt{n(R_m)} \\ &= \frac{3}{4} \int_{T_{m-1}}^{R_m} \left(\frac{R_m}{t} \right)^{\left(1+\frac{1}{m}\right) \frac{\log \alpha(t)}{\log t}} d[\alpha(t)] + \frac{3}{4} \sqrt{n(R_m)}. \end{aligned} \tag{27}$$

This implies that

$$S_4 \leq \frac{3}{4} \int_{T_{m-1}}^{R_m} \left(\frac{R_m}{t} \right)^{\left(1+\frac{1}{m}\right) \frac{\log \alpha(t)}{\log t}} d\alpha(t) + o(n(R_m)), \tag{28}$$

as can be seen by integrating the difference between the two integrals in (27) and (28) by parts. An elementary computation shows that the integral on the right side of (28) is equal to

$$\int_{T_{m-1}}^{R_m} \beta(t) (\log t)^{\rho(\log t)} \exp \left(\left(\left(1 + \frac{1}{m} \right) \left(\frac{\log R_m}{\log t} - 1 \right) + 1 \right) (\log t)^{\rho(\log t)} \right) dt.$$

where

$$\beta(t) = \frac{\rho'(\log t) \log \log t}{t} + \frac{\rho(\log t)}{t \log t}.$$

Suppose now that $\mu < \infty$. Then we have

$$\beta(t) \leq \frac{1}{m\, t \log t} + \frac{\rho(\log R_m) + \frac{1}{m}}{t \log t} = \left(\rho(\log R_m) + \frac{2}{m} \right) \frac{1}{t \log t}$$

for $T_{m-1} \leq t \leq R_m$ by (19). By the definition of R_m and ρ, this inequality trivially holds for $\mu = \infty$. This bound for $\beta(t)$ leads to

$$S_4 \leq \frac{3}{4} \left(\rho(\log R_m) + \frac{2}{m} \right) I + o(n(R_m)) \tag{29}$$

where

$$I = \int_{T_{m-1}}^{R_m} (\log t)^{\rho(\log t) - 1} \exp \left(\left(\left(1 + \frac{1}{m} \right) \left(\frac{\log R_m}{\log t} - 1 \right) + 1 \right) (\log t)^{\rho(\log t)} \right) \frac{dt}{t}.$$

Substituting $u = \log t$ and defining $L = \log R_m$ and $l = \log T_{m-1}$ we find that

$$I = \int_l^L u^{\rho(u) - 1} \exp \left(\left(\left(1 + \frac{1}{m} \right) \left(\frac{L}{u} - 1 \right) + 1 \right) u^{\rho(u)} \right) du.$$

Finally, the substitution $s = u/L$ leads to

$$I = \int_{l/L}^1 \exp \gamma(s) ds$$

where

$$\begin{aligned}\gamma(s) &= \left(\left(1+\frac{1}{m}\right)\left(\frac{1}{s}-1\right)+1\right)s^{\rho(Ls)}L^{\rho(Ls)}+(\rho(Ls)-1)\log s+\rho(Ls)\log L \\ &= \left(1+\frac{1-s}{m}\right)s^{\rho(Ls)-1}L^{\rho(Ls)}+(\rho(Ls)-1)\log s+\rho(Ls)\log L.\end{aligned}$$

Next we note that (19) implies that

$$|\rho(Ls)-\rho(L)|\leq\frac{1}{m} \tag{30}$$

for $l/L\leq s\leq 1$. It follows that if $l/L\leq s\leq 1/\sqrt{L}$ and if m is large enough, then

$$\begin{aligned}\gamma(s) &\leq \left(1+\frac{1}{m}\right)s^{\rho(Ls)-1}L^{\rho(Ls)}+\rho(Ls)\log L \\ &\leq \left(1+\frac{1}{m}\right)L^{\rho(Ls)/2+1/2}+\rho(Ls)\log L \\ &\leq \left(1+\frac{1}{m}\right)L^{\rho(L)/2+1/2m+1/2}+\left(\rho(L)+\frac{1}{m}\right)\log L \\ &\leq \frac{1}{2}L^{\rho(L)}+2\rho(L)\log L \\ &= \frac{1}{2}\log n(R_m)+2\log\log n(R_m).\end{aligned}$$

This implies that

$$\int_{l/L}^{1/\sqrt{L}}\exp\gamma(s)ds\leq\sqrt{n(R_m)}(\log n(R_m))^2=o(n(R_m)) \tag{31}$$

as $m\to\infty$. We now estimate $\gamma(s)$ for $1/\sqrt{L}\leq s\leq 1$. First we note that if $\mu<\infty$, then (19) implies that

$$\begin{aligned}|\rho(L)-\rho(Ls)| &\leq \int_{Ls}^{L}|\rho'(t)|dt \\ &\leq \frac{1}{m}\int_{Ls}^{L}\frac{dt}{t\log t} \\ &= \frac{1}{m}(\log\log L-\log\log Ls) \\ &= -\frac{1}{m}\log\left(1+\frac{\log s}{\log L}\right) \\ &\leq -\frac{2\log s}{m\log L}\end{aligned}$$

for $1/\sqrt{L} \le s \le 1$. It follows that

$$\rho(Ls)\log L \le \rho(L)\log L - \frac{2}{m}\log s$$

and hence

$$L^{\rho(Ls)} \le L^{\rho(L)} s^{-2/m}$$

for $1/\sqrt{L} \le s \le 1$. Clearly, these inequalities also hold if $\mu = \infty$, since $\rho(Ls) \le \rho(L)$ in this case. Combining the last two inequalities with (30) we find that

$$\begin{aligned} \gamma(s) &\le \left(1 + \frac{1-s}{m}\right) s^{\rho(L)-1-3/m} L^{\rho(L)} \\ &\quad + \left(\rho(L) - 1 - \frac{3}{m}\right)\log s + \rho(L)\log L \\ &\le \left(1 + \frac{1-s}{m}\right) s^{\rho(L)-1-3/m} L^{\rho(L)} + \rho(L)\log L \end{aligned} \tag{32}$$

for large m. Differentiation shows that the right side of (32) is increasing for $0 \le s \le 1$, provided $\rho(L) > 1 + 4/m$, which is the case for sufficiently large m. We define s_m by

$$s_m^{\rho(L)-1-3/m} = 1 - \frac{2}{m}$$

and deduce that if $1/\sqrt{L} \le s \le s_m$ and if m is sufficiently large, then

$$\begin{aligned} \gamma(s) &\le \gamma(s_m) \\ &\le \left(1 + \frac{1}{m}\right)\left(1 - \frac{2}{m}\right) L^{\rho(L)} + \rho(L)\log L \\ &\le \left(1 - \frac{2}{m^2}\right) L^{\rho(L)} + \rho(L)\log L \\ &\le L^{\rho(L)} - \rho(L)\log L \end{aligned}$$

by (16). It follows that

$$\int_{1/\sqrt{L}}^{s_m} \exp\gamma(s)ds \le \exp\left(L^{\rho(L)} - \rho(L)\log L\right) = \frac{n(R_m)}{\log n(R_m)}. \tag{33}$$

Next we note that if $K > 1$ and $0 \le s \le 1$, then

$$\left(1 + \frac{1-s}{m}\right) s^K \le 1 - \left(K - 1 - \frac{1}{m}\right)(1-s)s^K,$$

as can be seen by differentiating the difference between right and left side. Applying this for $K = \rho(L) - 3/m$ and using (32) and the definition of s_m we find that if

$s_m \le s \le 1$, then

$$\begin{aligned}\gamma(s) &\le \left(1 - \left(\rho(L) - 1 - \frac{4}{m}\right)(1-s)s^{\rho(L)-1-3/m}\right) L^{\rho(L)} + \rho(L)\log L \\ &\le \left(1 - \left(\rho(L) - 1 - \frac{4}{m}\right)(1-s)\left(1 - \frac{2}{m}\right)\right) L^{\rho(L)} + \rho(L)\log L\end{aligned}$$

for large m. It follows that

$$\begin{aligned}&\int_{s_m}^{1} \exp\gamma(s)ds \\ \le\ & L^{\rho(L)} \exp L^{\rho(L)} \int_{s_m}^{1} \exp\left(-\left(\rho(L) - 1 - \frac{4}{m}\right)(1-s)\left(1-\frac{2}{m}\right)L^{\rho(L)}\right) ds \\ \le\ & \frac{\exp L^{\rho(L)}}{\left(\rho(L) - 1 - \frac{4}{m}\right)\left(1 - \frac{2}{m}\right)}.\end{aligned}$$

Combining this with (29), (31), and (33) and recalling that $L = \log R_m$ and $n(R_m) = \exp L^{\rho(L)}$, we find that

$$S_4 \le \frac{\frac{3}{4}\left(\rho(\log R_m) + \frac{2}{m}\right)}{\left(\rho(\log R_m) - 1 - \frac{4}{m}\right)\left(1 - \frac{2}{m}\right)} n(R_m) + o(n(R_m)).$$

This, together with (22) and (24), yields the conclusion since $\rho(\log R_m) \to \mu$ as $m \to \infty$.

4 Results about Fourier series

The proof of Theorem 2 will be based on estimations of the Fourier coefficients $c_m = c_m(r, f)$ of $\log|f(re^{i\theta})|$ for an entire function f. By definition,

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{im\theta} \log|f(re^{i\theta})| d\theta$$

for each integer m, so that

$$\log|f(re^{i\theta})| = \sum_{m=-\infty}^{\infty} c_m e^{im\theta}.$$

The following Lemma 3 gives a relation between the zeros of an entire function f and the Fourier coefficients $c_m(r, f)$. Proofs of this lemma, together with applications, can be found in the papers of Edrei and Fuchs ([17], p. 312), Miles and Shea ([36], p. 379), and Rubel ([41], Lemma 1). For further applications to entire functions we refer to the survey of Rubel [42]. We also mention a paper of Miles [35] where the result has been applied to functions of infinite order with positive zeros, as it will be the case in the proof of Theorem 2.

Lemma 3 *Let f be an entire function with zeros $z_1, z_2, \dots$ and suppose that $f(0) \neq 0$. If $\log f(z) = \sum_{m=0}^{\infty} a_m z^m$ in a neighborhood of 0, then*

$$c_m = \frac{1}{2} a_m r^m + \frac{1}{2m} \sum_{|z_j| \le r} \left(\left(\frac{r}{z_j} \right)^m - \left(\frac{\bar{z}_j}{r} \right)^m \right)$$

for each positive integer m.

We also note that $c_{-m} = \overline{c_m}$ since $\log |f(re^{i\theta})|$ is real and that

$$c_0 = \sum_{|z_j| \le r} \log \frac{r}{|z_j|} + \log |f(0)| \tag{34}$$

by Jensen's formula.

In order to estimate $M(r, f)$ in terms of the Fourier coefficients we shall use the following result.

Lemma 4 *Let g be a real, measurable, 2π-periodic function with Fourier coefficients c_m and suppose that $|g(x)| \le M$ for all x. Then*

$$c_0 + 2 \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right) \operatorname{Re} c_m \le M$$

for all positive integers n.

Proof. Let

$$F_{n-1}(x) = 1 + 2 \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right) \cos mx = \sum_{m=-n+1}^{n-1} \left(1 - \frac{m}{n}\right) e^{imx}$$

be Fejér's kernel. It is well-known (cf. e. g. [12], p. 43, or [50], Vol. I, p. 88) that $F_{n-1}(x)$ is non-negative. Hence

$$\begin{aligned} c_0 + 2 \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right) \operatorname{Re} c_m &= \sum_{m=-n+1}^{n-1} \left(1 - \frac{m}{n}\right) c_m \\ &= \frac{1}{2\pi} \int_0^{2\pi} F_{n-1}(x) g(-x) dx \\ &\le M \frac{1}{2\pi} \int_0^{2\pi} F_{n-1}(x) dx \\ &= M. \end{aligned}$$

5 Proof of Theorem 2

Assume that $1 < \mu < \infty$ and define $z_j = \exp\left((\log(j+1))^{1/\mu}\right)$ for $j \geq 1$. Then

$$n(t) = [\exp((\log t)^\mu)] - 1$$

for $t \geq 1$ so that (6) is satisfied. Let f be an entire function whose zeros are precisely the z_j. Since $z_1 > 1$ we deduce that if

$$\log f(z) = \sum_{m=0}^{\infty} a_m z^m$$

near 0, then $a_m \to 0$ as $m \to \infty$. Therefore we may assume without loss of generality that $|a_m| \leq 1$ for all $m \geq 1$, because otherwise we can replace f by fe^p for suitable polynomial p. This does not affect the asymptotic behavior of $\log M(r,f)$, since f has infinite lower order in view of the result of Edrei and Fuchs and Gol'dberg already mentioned in the introduction. Using Lemma 3 and integration by parts we find that if $m \geq 1$, then

$$\begin{aligned}
\operatorname{Re} c_m &\geq -\frac{1}{2}r^m + \frac{1}{2m}\int_1^r \left(\left(\frac{r}{t}\right)^m - \left(\frac{t}{r}\right)^m\right) dn(t) \\
&= -\frac{1}{2}r^m + \frac{1}{2}\int_1^r n(t)\left(\left(\frac{r}{t}\right)^m + \left(\frac{t}{r}\right)^m\right)\frac{dt}{t} \\
&\geq -\frac{1}{2}r^m - \int_1^r \left(\left(\frac{r}{t}\right)^m + \left(\frac{t}{r}\right)^m\right)\frac{dt}{t} \\
&\quad +\frac{1}{2}\int_1^r \exp((\log t)^\mu)\left(\left(\frac{r}{t}\right)^m + \left(\frac{t}{r}\right)^m\right)\frac{dt}{t} \\
&\geq -\frac{3}{2}r^m + \frac{1}{2}\int_1^r \exp((\log t)^\mu)\left(\frac{r}{t}\right)^m\frac{dt}{t} \\
&\quad +\frac{1}{2}\int_1^r \exp((\log t)^\mu)\left(\frac{t}{r}\right)^m\frac{dt}{t} \\
&= -\frac{3}{2}r^m + I_1 + I_2.
\end{aligned}$$

Substituting $u = \log t$ and putting $L = \log r$ we find that

$$\begin{aligned}
I_1 &= \frac{r^m}{2}\int_0^L \exp(u^\mu - mu)\, du \\
&\geq \frac{r^m}{2}\int_0^L \exp\left(L^\mu + \mu L^{\mu-1}(u-L) - mu\right) du \\
&= \frac{1}{2}\frac{\exp(L^\mu)}{\mu L^{\mu-1} - m}\left(1 - \exp(mL - \mu L^\mu)\right) \\
&\geq \frac{1}{2}\frac{n(r)}{\mu L^{\mu-1} - m}\left(1 - \exp((1-\mu)L^\mu)\right)
\end{aligned}$$

for $1 \leq m \leq [L^{\mu-1}]$. Similarly, we have

$$\begin{aligned} I_2 &= \frac{1}{2r^m}\int_0^L \exp\left(u^\mu + mu\right) du \\ &\geq \frac{1}{2}\frac{\exp\left(L^\mu\right)}{\mu L^{\mu-1}+m}\left(1-\exp\left(-mL-\mu L^\mu\right)\right) \\ &\geq \frac{1}{2}\frac{n(r)}{\mu L^{\mu-1}+m}\left(1-\exp\left((1-\mu)L^\mu\right)\right) \end{aligned}$$

for $1 \leq m \leq [L^{\mu-1}]$. It follows that

$$\text{Re } c_m \geq -\frac{3}{2}r^m + n(r)\left(1-\exp((1-\mu)L^\mu)\right)\frac{\mu L^{\mu-1}}{\left(\mu L^{\mu-1}\right)^2 - m^2}.$$

for $1 \leq m \leq [L^{\mu-1}]$. We now apply Lemma 4 for $n = [L^{\mu-1}]+1$. Since $c_0 \geq 0$ for large r by (34), we deduce that

$$\begin{aligned} \log M(r,f) &\geq 2\sum_{m=1}^{[L^{\mu-1}]}\left(1-\frac{m}{[L^{\mu-1}]+1}\right)\text{Re } c_m \\ &\geq -3\sum_{m=1}^{[L^{\mu-1}]}\left(1-\frac{m}{[L^{\mu-1}]+1}\right)r^m \\ &\quad +2n(r)(1-o(1))\mu L^{\mu-1}\sum_{m=1}^{[L^{\mu-1}]}\frac{1-m/L^{\mu-1}}{\left(\mu L^{\mu-1}\right)^2-m^2}. \end{aligned}$$

Now

$$\begin{aligned} \sum_{m=1}^{[L^{\mu-1}]}\left(1-\frac{m}{[L^{\mu-1}]+1}\right)r^m &\leq \sum_{m=1}^{[L^{\mu-1}]-1} r^m + \left(1-\frac{[L^{\mu-1}]}{[L^{\mu-1}]+1}\right)r^{[L^{\mu-1}]} \\ &\leq \left(L^{\mu-1}-1\right)r^{L^{\mu-1}-1} + \frac{1}{L^{\mu-1}}r^{L^{\mu-1}} \\ &= \left(\frac{(\log r)^{\mu-1}-1}{r} + \frac{1}{(\log r)^{\mu-1}}\right)\exp\left((\log r)^{\mu-1}\right) \\ &= o(n(r)). \end{aligned}$$

Moreover,

$$\begin{aligned} L^{\mu-1}\sum_{m=1}^{[L^{\mu-1}]}\frac{1-m/L^{\mu-1}}{\left(\mu L^{\mu-1}\right)^2-m^2} &= \frac{1}{L^{\mu-1}}\sum_{m=1}^{[L^{\mu-1}]}\frac{1-m/L^{\mu-1}}{\mu^2-\left(m/L^{\mu-1}\right)^2} \\ &\to \int_0^1 \frac{1-u}{\mu^2-u^2}du \\ &= \frac{1}{2\mu}\left((\mu-1)\log\frac{\mu-1}{\mu} + (\mu+1)\log\frac{\mu+1}{\mu}\right) \end{aligned}$$

as $r \to \infty$. Altogether we see that

$$\log M(r,f) \geq (1-o(1))n(r)\left((\mu-1)\log\frac{\mu-1}{\mu}+(\mu+1)\log\frac{\mu+1}{\mu}\right)$$

as $r \to \infty$. This completes the proof of Theorem 2.

6 Remarks

Remark 1 In order to estimate the maximum modulus in terms of the Fourier coefficients, we have used Lemma 4 whose proof was based on the positivity of Fejér's kernel. If we use other positive kernels instead of Fejér's kernel, we get different (and possibly better) bounds for the maximum modulus in terms of the Fourier coefficients. In fact, our proof shows that if λ is an integrable function which satisfies $\lambda(0)=1$ and has the property that the trigonometric polynomial p defined by

$$p(t)=1+2\sum_{k=1}^{n}\lambda\left(\frac{k}{n}\right)\cos kt \tag{35}$$

is non-negative for all n, then the right side of (8) may be replaced by

$$2\mu\int_0^1\frac{\lambda(u)}{\mu^2-u^2}du.$$

In particular, if $g(x)$ is an even and non-negative function which is integrable on $(-\infty,\infty)$ and has the property that its Fourier transform $g^\wedge(u)$ vanishes for $|u|\geq 1$, then we may take $\lambda(u)=g^\wedge(u)/g^\wedge(0)$. In this case the kernel defined by (35) is called a kernel of Fejér's type (cf. [1], p. 133, [12], p. 124, and [38], §3), Fejér's kernel itself corresponding to $g(x)=\sin^2\frac{x}{2}/x^2$ and $\lambda(u)=1-u$ (for $0\leq u\leq 1$). We note here that it is not essential to restrict to even kernels (respectively even functions g), but we have done so for the sake of simplicity.

It follows that the right side of (8) may be replaced by

$$M(\mu)=2\mu\sup_{\lambda}\int_0^1\frac{\lambda(u)}{\mu^2-u^2}du,$$

that is, we have $L(\mu)\geq M(\mu)$. Here the supremum is taken over all functions λ which are of the form $\lambda(u)=g^\wedge(u)/g^\wedge(0)$ for some even, non-negative, and integrable function g and vanish for $|u|\geq 1$. An equivalent condition is that λ is an even function satisfying $\lambda(0)=1$ and $\lambda(u)=0$ for $|u|\geq 1$ which has the property that $\lambda^\wedge(u)$ is non-negative. Specializing a result of Boas and Kac ([9], Lemma 5.1) to even functions we see that this means that λ is of the form

$$\lambda(u) = \int_{-1/2}^{1/2} C(x)C(x-u)dx$$

for some real-valued function C which satisfies

$$\int_{-1/2}^{1/2} C(x)^2 dx = 1 \tag{36}$$

and $C(x) = 0$ for $|x| \geq \frac{1}{2}$. (Here Fejér's kernel is obtained by choosing $C(x) = 1$ for $|x| < \frac{1}{2}$.) It follows that

$$\begin{aligned} M(\mu) &= 2\mu \sup_C \int_0^1 \int_{-1/2}^{1/2} \frac{C(x)C(x-u)}{\mu^2 - u^2} dx\, du \\ &= 2\mu \sup_C \int_{-1/2}^{1/2} \int_{-1/2}^{x} \frac{C(x)C(y)}{\mu^2 - (x-y)^2} dy\, dx \end{aligned}$$

where the supremum is taken over all square integrable functions C satisfying (36) and vanishing outside $(-\frac{1}{2}, \frac{1}{2})$.

I have been unable to find an explicit expression for $M(\mu)$, but it can be computed numerically for any fixed $\mu \geq 1$ using Ritz's method (cf. e. g. [34]). For instance, it turns out that $M(2) = 0.52349443\ldots$ while the right side of (8) is equal to $0.52324814\ldots$ if $\mu = 2$, so there is in fact a slight improvement of the lower bound for $L(2)$. Here the convergence is quite fast, already the choice $C(x) = a + bx^2$ for suitable values of a and b yields $M(2) \geq 0.52349433\ldots$. The convergence of Ritz's method is fairly slow in the case $\mu = 1$, but it seems that $M(1) = 1.6\ldots$ so that in particular $M(1) < \infty$. Nevertheless it seems likely to me that $\lim_{\mu\to 1} L(\mu) = \infty$, but our method is apparently not suitable to prove this conjecture. However, we find at least that $\liminf_{\mu\to 1} L(\mu) > 1.6$ which is better than the estimate $\liminf_{\mu\to 1} L(\mu) \geq 2\log 2 = 1.3862943\ldots$ obtained from (8).

Concerning improvements of the lower bound for $L(\mu)$ for large μ, we remark that Fejér ([19], p. 66, see also [9], p. 108, and [40], Vol. II, p. 83) found an extremal property of the kernel that today bears his name, which, if specialized to kernels of the form (35), says that Fejér's kernel maximizes $\int_0^1 \lambda(u)du$ among even positive kernels. This shows that replacing Fejér's kernel by another positive kernel alone can improve the constant on the right side of (8) at most by a factor $\mu^2/(\mu^2-1)$. (We have already seen that the improvement is in fact much smaller if $\mu = 2$.) So using this method we cannot hope to find significantly better lower bounds for $L(\mu)$ if μ is large. On the other hand, we note that the proof of Theorem 1 was based on the inequality

$$\log M(r, f) \leq \sum_{j=1}^{\infty} \log M\left(\frac{r}{r_j}, E(\cdot, p_j)\right).$$

Since $\log M(r, E(\cdot, p)) \geq \log M(1, E(\cdot, p)) \geq \mathrm{Ci}(\pi/2) = 0.4720006\ldots$ for $r \geq 1$ and all p (cf. [13], §3), we see that the method used in the proof of Theorem 1 cannot be used to replace the right side of (7) by an expression which is less than $0.4720006\ldots$ for certain values of μ. Thus the methods of this paper do not seem to be suitable to determine the behavior of $L(\mu)$ as $\mu \to \infty$.

Remark 2 One way to see that the limes inferior in Theorem 1 cannot be replaced by the limes superior is to consider the entire function E_0 constructed by Hayman ([23], p. 81) and Pólya and Szegö ([40], Vol. I, p. 115). This function is large inside a strip $\{z : \mathrm{Re}\ z > 0, |\mathrm{Im}\ z| < \pi\}$ and small outside this strip. It is known (cf. [23], p. 83) that $\log M(r, E_0) \sim e^r$ and $T(r, E_0) \sim e^r/\pi r$ as $r \to \infty$, where $T(r, E_0)$ denotes the Nevanlinna characteristic. Similarly, one can also show that $n(r) \sim e^r/\pi$, if $n(r)$ denotes the number of zeros of E_0 in $|z| \leq r$. In particular, we have $\mu = \infty$ for the sequence of zeros of E_0. By Theorem 1 there exists an entire function f which has the same zeros as E_0 such that

$$\log M(r, f) \leq \left(\frac{3}{4} + o(1)\right) n(r) \tag{37}$$

for arbitrarily large r. On the other hand, f has the form $f(z) = E_0(z)e^{g(z)}$. Applying the Wiman-Valiron theory (cf. [45], Chapter 4) to g one may deduce that

$$\log M(r, f) \geq (1 + o(1)) \log M(r, E_0) = (1 + o(1))\pi n(r)$$

outside an exceptional set of finite logarithmic measure. This shows that (37) holds only on a set of finite logarithmic measure.

Examples similar to E_0 can be constructed for other strips, cf. [7], [14], and [33]. This leads to sequences with the above property for which $n(r)$ grows faster or slower than e^r/π.

Remark 3 The examples of Baernstein mentioned in the introduction have zeros distributed regularly on a logarithmic spiral. He has kindly informed me that entire functions of finite order with zeros on logarithmic spirals have been studied in detail by Balašov and others ([4], [5], [6]) and the inequality $K(\sigma) > \pi/|\sin \pi\sigma|$ for $\sigma > 1$ can be obtained from Balašov's results. One may in fact obtain explicit lower bounds for $K(\sigma)$ by a suitable choice of the spiral. Here we only note that

$$\liminf_{n\to\infty} K(n + \delta) \geq \begin{cases} 2\pi & \text{if } \frac{1}{4} \leq \delta \leq \frac{3}{4} \\ \dfrac{\pi}{|\sin \pi\delta \cos \pi\delta|} & \text{if } 0 < \delta < \frac{1}{4} \text{ or } \frac{3}{4} < \delta < 1 \end{cases}$$

if n runs through the positive integers. If $\frac{1}{4} \leq \delta \leq \frac{3}{4}$, this can be seen by choosing $c = \log \sigma/\sigma$ and $\theta = 2\pi(1 - 1/4\sigma)$ in Theorem 1 of [5]. Here $\sigma = n + \delta$. In the remaining case, we take

$$c = -\frac{\log(\cos 2\pi\delta)}{2\pi\sigma}$$

and choose $\theta = 2\pi(1 - \delta/\sigma)$ if $0 < \delta < \frac{1}{4}$ and $\theta = 2\pi(1 - (\delta - \frac{1}{2})/\sigma)$ if $\frac{3}{4} < \delta < 1$. As far as I can see, we cannot obtain a better asymptotic lower bound for $K(n+\delta)$ from Balašov's functions by choosing different values of c and θ. Hence in particular the question whether $K(\sigma) \to \infty$ as $\sigma \to \infty$ still remains open.

Concerning the upper bound for $K(\sigma)$, Shea [private communication] has shown that the constant A in (4) may be replaced by $2\gamma + 2\log 3 + O(1/\sigma)$, and this result remains valid if $\log M(r,f)$ is replaced by $-\log L(r,f)$ where $L(r,f)$ denotes the minimum modulus.

Occasionally, the Lindelöf functions for which

$$\lim_{r\to\infty} \frac{\log M(r,f)}{n(r)} = \frac{\pi}{|\sin \pi\sigma|}$$

have been conjectured to be extremal for the "$K(\sigma)$–problem". As shown by the above examples, they are not. One may of course ask whether the Balašov functions are extremal for this problem, but I feel that I do not have arguments for a conjecture in either direction.

It seems possible that one can improve the lower bound for $L(\mu)$ by considering sequences (z_j) which are not distributed radially. Such an improvement will, however, require new methods of proof, since the Fourier series method used in the proof of Theorem 2 yields best results in the case that the z_j are distributed on a single ray. As far as I know, entire functions of infinite order with zeros on spirals have not been studied yet.

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