An example concerning iteration and factorization of rational functions

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A rational function R is said to be prime if in any factorization $R = S \circ T$ one of the two factors S and T is a linear transformation. It is not difficult to show that any rational function can be factored into prime functions, that is, given a rational function R there exist an integer n and prime rational functions $R_1, R_2, \ldots, R_n$ such that $R = R_1 \circ R_2 \circ \ldots \circ R_n$. Ritt [6] proved that if a polynomial P has two factorizations $P = P_1 \circ P_2 \circ \ldots \circ P_n$ and $P = Q_1 \circ Q_2 \circ \ldots \circ Q_m$, where the P_i and Q_j are prime, then $m = n$. In fact, he proved that there are even closer relations between the two factorizations (cf. [6], see also [1, Appendix A]). Ritt [6, p. 53] also claimed that no analogue holds for the factorization of rational functions. However, he did not give a proof of this claim but stated that he would return to this matter in a later communication. According to Gross (cf. [2, p. 50], [3, p. 53]), however, there is no further discussion of this assertion in Ritt's subsequent publications. This led Gross (cf. [2, p. 50], [3, Conjecture 1]) to the question, whether Ritt's claim is correct.

In this note, we will show that this is in fact the case. More precisely, we will prove the following result.

Theorem. *There exists a rational function which has two factorizations into prime functions each having a different number of factors.*

To prove the theorem, we consider the $\wp$-function of Weierstraß with primitive periods 1 and $i\sqrt{8}$. For the notation used and the properties of the $\wp$-function needed in the sequel, see e. g. [4]. It is well known that

$\wp(2u) = R(\wp(u))$ where

$$R(z) = \frac{16z^4 + 8g_2z^2 + (g_2)^2 + 32g_3}{16(4z^3 - g_2z - g_3)}.$$

Similarly, we have $\wp(i\sqrt{8}u) = S(\wp(u))$ and $\wp(8u) = T(\wp(u))$ for certain rational functions S and T. Since $\wp$ is an even function, we deduce that $R(R(R(\wp(u)))) = \wp(8u) = \wp(-8u) = S(S(\wp(u))$. It follows that $T = R \circ R \circ R = S \circ S$. Suppose now that R is prime. This implies that T has a factorization with three prime factors. On the other hand, the representation $T = S \circ S$ leads to a factorization with an even number of factors. Hence T has the desired properties.

It remains to prove that R is prime. To do this, we assume that there exist rational functions P and Q of degree 2 such that $R = P \circ Q$. Without loss of generality we may assume that $P(0) = P(\infty) = \infty$, since otherwise we may consider $P \circ L$ and $L^{-1} \circ Q$ instead of P and Q, where L is a suitable linear transformation. It follows that $P(z) = az + b + c/z$ for suitable constants a, b, and c, As usual we denote by e_1, e_2, and e_3 the zeros of $4z^3 - g_2z - g_3$. Since Q maps these zeros and ∞ either to 0 or to ∞, we may assume without loss of generality that $Q(z) = (z - e_1)(z - e_2)/(z - e_3)$. We now consider the equation $R(z) = P(Q(z))$ where we express g_2, g_3, and e_3 in terms of e_1 and e_2. Comparing the coefficients of z^4, z^3, and z^2 in the numerator we find that $a = 1/4$, $b = (e_1 + e_2)/2$, and $c = (e_1 - e_2)^2/4$. Comparing the coefficients of z and the constant terms, we deduce that $e_1e_2(e_1 + e_2) = 0$, that is, $g_3 = 0$.

Now we consider the function f defined by

$$f(h) = 64\pi^6(\frac{1}{216} - \frac{7}{3}\sum_{r=1}^{\infty}\frac{r^5h^{2r}}{1 - h^{2r}}).$$

It is well known [4, p. 210] that $g_3 = f(e^{-\pi\sqrt{8}})$. On the other hand, we have $f(e^{-\pi}) = 0$, since this value corresponds to the g_3-value of the $\wp$-function with periods 1 and i. Since $f(h)$ is decreasing, we see that $g_3 > 0$. This contradiction completes the proof of the theorem.

It seems likely to me that the examples that Ritt had had in mind were of a similar nature. In fact, in [6, p. 53] he explicitly mentioned the rational functions that arise from the multiplication formulas of elliptic functions, and he studied these functions in detail in some of his other papers, cf. e. g. [5], [7], and [8].

References

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