On Factorization of Certain Entire Functions

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Abstract

Let P and Q be polynomials and let α be an entire function. Suppose that Q and α are nonconstant. We show that the function $h(z) = P(z)e^{\alpha(z)} + Q(z)$ has a factorization $h(z) = f(g(z))$ with nonlinear meromorphic left and right factors f and g if and only if P, Q, and α have such a factorization with a common right factor. This confirms conjectures of F. Gross, G. D. Song, and C. C. Yang.

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1 Introduction and Main Result

Let f, g, and h be meromorphic functions and suppose that

$$h(z) = f(g(z)). \tag{1}$$

Following Gross [5] we call the representation (1) a *factorization* of h and the functions f and g are called *left* and *right factors* of h. Note that if g has a pole, then f is necessarily rational. If in any factorization (1) of h at least one of the two factors f and g is bilinear, then h is called *prime*.

In this paper, we determine in which cases an entire function h of the form

$$h(z) = P(z)e^{\alpha(z)} + Q(z), \tag{2}$$

is prime, if P and Q are polynomials and if α is an entire function. Our result is as follows.

Theorem *Let P and Q be polynomials and let α be an entire function. Suppose that Q and α are nonconstant and let h be the entire function defined by* (2). *Then h is prime if and only if P, Q, and α do not have a nonlinear common right factor.*

This result was conjectured by Gross [4, p. 232] and Song and Yang [13, Conjecture A3]. Factorizations of functions of the form (2) have been studied in a number of papers (cf. [1, Thm. 5], [2], [3], [4, Thm. 4], [6, Thm. 4], [9, Satz 5], [10, Cor. 3 and 4], [14, Thm. 8, Cor. 9], [15, Thm. 3]). In particular, we mention the result of Goldberg and Prokopovich [3] who proved that the conclusion of the theorem holds if α is a polynomial. This follows also from a more general result of Gross and Yang [7, Thms. 1 and 2]. Hence we may restrict ourselves to the case that α is transcendental. Also, I proved in [2] that the function h defined by (2) does not have a factorization (1) with entire transcendental factors f and g. Thus it remains to consider only the case that f is transcendental and entire and that g is a polynomial, the case that f is rational, and the case that f is transcendental and meromorphic, but not entire.

2 Lemmas

Lemma 1 *Let f be an entire transcendental function. Then there exists an unbounded sequence (w_j) such that $|f(w_j)| \le 1$ and*

$$|f'(w_j)| \ge \frac{A \log M(|w_j|, f)}{|w_j|}$$

for some positive absolute constant A and all j.

Lemma 2 *If u is analytic in the disc $|z - z_0| < R$ and fails to take the values zero and one there, then*

$$\frac{|u'(z_0)|}{1 + |u(z_0)|^2} \le \frac{B}{R}$$

for some absolute constant B.

Lemma 1 is due to Pommerenke [11, Thm. 4]. Lemma 2 is a version of Landau's theorem which can be found for instance in Hayman's book [8, p. 156]. Pommerenke [11, Cor. 2] proved that we may take $B = 4\sqrt{2}$, but we do not need this result or other sharper versions of Landau's theorem (cf. e. g. [8, p. 169]).

3 Proof of the Theorem

Suppose that P, Q, and α satisfy the hypotheses of the theorem and let h be the function defined by (2). It is clear that if h is prime, then P, Q, and α do not have a nonlinear common right factor. Suppose now that h is not prime, that is, suppose that there exists a factorization (1) where f and g are not bilinear.

First we consider the case that f is entire and transcendental. As remarked in the introduction, we may assume that α is transcendental and that g is a nonlinear polynomial. These assumptions imply that h and f have infinite lower order. Combining this with Lemma 1, we see that if $K > 0$, then there exists an unbounded sequence (w_j) such that

$$|f(w_j)| \leq 1 \quad \text{and} \quad |f'(w_j)| \geq |w_j|^K \tag{3}$$

for all j. Considering a subsequence and $f(-z)$ instead of $f(z)$ if necessary we may assume that $\operatorname{Re} w_j \geq 0$ for all j.

We denote the degrees of g and Q by m and n, respectively. First we prove that $n = rm$ for some positive integer r. Suppose that this is not the case. If we choose R large enough, then there exist two branches $a_1(w)$ and $a_2(w)$ of the inverse function of g which are defined for $|w| > R$ and $|\arg w| < \pi$ and have the property that

$$a_1(w) \sim \sigma w^{1/m} \quad \text{and} \quad a_2(w) \sim \sigma e^{2\pi i/m} w^{1/m}$$

as $|w| \to \infty$. Here σ is a constant and $w^{1/m}$ denotes the principal branch of the m-th root. We define $b_1(w) = Q(a_1(w))$ and $b_2(w) = Q(a_2(w))$. Then

$$b_1(w) \sim \tau w^{n/m} \quad \text{and} \quad b_2(w) \sim \tau e^{2\pi i n/m} w^{n/m} \tag{4}$$

for some constant τ as $|w| \to \infty$. Moreover, we have

$$b_1'(w) \sim \frac{n}{m} \tau w^{n/m-1} \quad \text{and} \quad b_2'(w) \sim \frac{n}{m} \tau e^{2\pi i n/m} w^{n/m-1} \tag{5}$$

as $|w| \to \infty$. We define

$$u(w) = \frac{f(w) - b_1(w)}{b_2(w) - b_1(w)}.$$

If R is large enough, then u is analytic for $|w| > R$ and $|\arg w| < \pi$ and u does not take the values zero and one there. Lemma 2 implies that

$$\frac{|u'(w_j)|}{1 + |u(w_j)|^2} \le \frac{B}{|w_j| - R}, \tag{6}$$

provided $|w_j| > R$. On the other hand, we have

$$|u(w_j)| \sim \frac{|b_1(w_j)|}{|b_2(w_j) - b_1(w_j)|} \sim \frac{1}{|e^{2\pi in/m} - 1|}$$

by (3) and (4). Moreover, we can deduce from (3), (4), and (5) that

$$|u'(w_j)| \ge \Big|\frac{f'(w_j) - b_1(w_j)}{b_2(w_j) - b_1(w_j)}\Big| - \Big|\frac{(f(w_j) - b_1(w_j))(b_2'(wj) - b_1'(w_j))}{(b_2(w_j) - b_1(w_j))^2}\Big| \ge 1$$

for a suitable choice of K and sufficiently large j. It follows that

$$\frac{|u'(w_j)|}{1 + |u(w_j)|^2} \ge C$$

for some positive constant C and sufficiently large j. This contradicts (6). Hence we have $n = rm$ for some integer r.

It follows that there exists a polynomial R of degree r such that if $Q_0(z) = Q(z) - R(g(z))$, then the degree n_0 of Q_0 is less than n. We define $f_0(z) = f(z) - R(z)$. Then we have

$$f_0(g(z)) = Q_0(z) + P(z)e^{\alpha(z)}.$$

Again, we have $n_0 = r_0 m$ for some integer r_0 and induction shows that $Q(z) = q(g(z))$ for some polynomial q, that is, g is a right factor of Q. Once this is known, it is not hard to show that g is also a right factor of P and α.

Next we consider the case that f is rational but not bilinear. If a is a pole of f, then g does not take the value a since h is entire. Similarly, if f is a polynomial, then g has no poles. In any case, we can deduce from a result of

Prokopovich [12, p. 200, Cor.] that the equation $h(z) = Q(z)$ has infinitely many solutions, contradicting (2).

It remains to consider the case that f is transcendental and meromorphic, but not entire. Let a be a pole of f. Then g is an entire function which does not take the value a, that is, g is of the form $g(z) = a + e^{G(z)}$ for some entire function G. We define $F(z) = f(a + e^z)$. Then F and G are entire functions and we have $F(G(z)) = f(g(z)) = h(z)$. As shown in [2], G is a polynomial. The argument used above shows that G is a common right factor of P, Q, and α. It remains to be proved that G is nonlinear. Suppose that this is not the case, that is, suppose that G is of the form $G(z) = cz + d$. Then g is periodic with period $2\pi i/c$. It follows that h is periodic with period $2\pi i/c$. This, together with (2), implies that the equation $h(z) = Q(z+2\pi ik/c)$ has only finitely many solutions, if k is an integer. Since Q is a nonconstant polynomial, the polynomials $Q(z + 2\pi ik/c)$ are pairwise distinct. This contradicts the second fundamental theorem of Nevanlinna theory in its version for three 'small' functions [8, p. 47]. This contradiction completes the proof of the theorem.

References

[1] I. N. Baker and F. Gross, Further results on factorization of entire functions, in *Entire functions and related parts of analysis*, Proc. Symp. Pure Math. 11, Amer. Math. Soc., Providence, R. I., 1968, 30-35.

[2] W. Bergweiler, Proof of a conjecture of Gross concerning fix-points, *Math. Z.* 204 (1990), 381-390.

[3] A. A. Goldberg and G. S. Prokopovich, On the simplicity of certain entire functions, *Ukrain. Math. J.* 22 (1970), 701-704 (translation from *Ukrain. Mat. Zh.* 22, 813-817).

[4] F. Gross, Prime entire functions, *Trans. Amer. Math. Soc.* 161 (1971), 219-233.

[5] F. Gross, *Factorization of meromorphic functions*, U. S. Government Printing Office, Washington, D. C., 1972.

[6] F. Gross und C. C. Yang, The fix-points and factorization of meromorphic functions, *Trans. Amer. Math. Soc.* 168 (1972), 211-219.

[7] F. Gross und C. C. Yang, Further results on prime entire functions, *Trans. Amer. Math. Soc.* 192 (1974), 347-355.

[8] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford 1964.

[9] E. Mues, Über faktorisierbare Lösungen Riccatischer Differentialgleichungen, *Math. Z.* 121 (1971), 145-156.

[10] M. Ozawa, On certain criteria for the left-primeness of entire functions, *Kodai Math. Sem. Rep.* 26 (1975), 304-317.

[11] Ch. Pommerenke, Normal functions, in *Proceedings of the NRL conference on classical function theory*, U. S. Government Printing Office, Washington, D. C., 1970.

[12] G. S. Prokopovich, Fix-points of meromorphic functions, *Ukrain. Math. J.* 25 (1973), 198-208 (translation from *Ukrain. Mat. Zh.* 25, 249-260).

[13] G. D. Song and C. C. Yang, Research problems in factorization theory of entire and meromorphic functions, *Southeast Asian Bull. Math.* 13 (1989), 63-67.

[14] H. Urabe, Uniqueness of the factorization under composition of certain entire functions, *J. Math. Kyoto Univ.* 18 (1978), 95-120.

[15] C. C. Yang, On the factorization of entire functions, *Ill. J. Math.* 21 (1977), 898-905.