Erratum to “Maximum Modulus, Characteristic, and Area on the Sphere”

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The proof of Theorem 2 of the paper mentioned in the title [1] is not correct. In fact the argument used there is correct only if $\log T(r)$ is convex in $\log r$, which is not the case in general. I do not know whether Theorem 2 holds in the form it was stated. The purpose of this note is to prove that the conclusion of Theorem 2 holds if the function γ occurring there also satisfies the hypothesis of Theorem 3, that is, if

$$\frac{\log \gamma(r)}{\log r} \to \infty \quad (r \to \infty) \tag{26}$$

and the function $h(x) = \gamma(e^x)$ satisfies

$$\left(\frac{h'(x)}{h(x)}\right)' \leq \left(\frac{h'(x)}{h(x)}\right)^{3/2} \quad (x > x_0) \tag{27}$$

for some sufficiently large x_0. We note that this additional hypothesis is satisfied in the corollaries to Theorem 2.

We refer systematically to the notation, and formulae (1) - (25), of [1]. To prove that the conclusion of Theorem 2 holds with the additional hypotheses (26) and (27), we put $\Phi(x) = \log h(x)$ and start by proving that if a sequence (x_j) satisfying $x_j \to \infty$ is given, then there exist sequences (M_j) and (ε_j) satisfying $M_j \to \infty$ and $\varepsilon_j \to 0$ such that

$$\Phi(x_j + h) \leq \Phi(x_j) + \Phi'(x_j)h + \varepsilon_j \tag{28}$$

for $|h| \leq M_j/\Phi'(x_j)$. To see this, we note that (27) implies that $\Phi''(x) \leq \Phi'(x)^{3/2}$ for large x. It follows that if M is a positive constant, $h = M/\Phi'(v)$, and v is large enough, then

$$1 - \sqrt{\frac{\Phi'(v)}{\Phi'(v+h)}} = \frac{\sqrt{\Phi'(v)}}{2} \int_v^{v+h} \frac{\Phi''(x)}{\Phi'(x)^{3/2}} dx \leq \frac{\sqrt{\Phi'(v)}}{2} \int_v^{v+h} dx = \frac{M}{2\sqrt{\Phi'(v)}}.$$

This implies that (21) holds for large v. Similarly, we see that (22) holds for large v. As in the proof of Lemma 1 we conclude that (16) holds for large v, and this implies the desired conclusion.

Now we define $L(r)$ by $T(r)=\gamma(L(r))$ and put

$$\lambda_L=\liminf_{r\to\infty}\frac{\log L(r)}{\log r}\quad\text{and}\quad \rho_L=\limsup_{r\to\infty}\frac{\log L(r)}{\log r}.$$

Since $T(r)\le\gamma(r)$ for arbitrarily large r, we have $\lambda_L\le 1$.

Next we prove that for given sequences (s_j) and (t_j) which tend to ∞, there exist sequences (r_j) and (K_j) which tend to ∞, a sequence (δ_j) which tends to zero, and a sequence (σ_j) which tends to λ_L such that $r_j\ge t_j$ and

$$L(r)\le\left(1+\delta_j\sigma_j\left|\log\frac{r}{r_j}\right|\right)\left(\frac{r}{r_j}\right)^{\sigma_j}L(r_j)\tag{29}$$

for $\sigma_j|\log r/r_j|\le K_j$ and $r\ge s_j$. Of course, (r_j) is a sequence of Pólya peaks of order λ_L for $L(r)$, and in fact the existence of the above sequences follows easily from Edrei's [2, p. 6] proof of the existence of Pólya peaks if $\lambda_L<\rho_L\le\infty$. In this case, we may even choose $\delta_j=0$. If $\lambda_L=\rho_L>0$, let $\rho(r)$ be a proximate order satisfying $L(r)\le r^{\rho(r)}$ for all large r with equality on a sequence (r_j) which tends to ∞ (cf. [3, p. 35]). Passing to a subsequence if necessary, we may assume that $r_j\ge t_j$. We recall that, by the definition of a proximate order in [3], we have $\rho(r)\to\rho_L$ and $\rho'(r)r\log r\to 0$. We define $\sigma_j=\rho(r_j)$ and deduce that

$$\frac{L(r)}{L(r_j)}\left(\frac{r_j}{r}\right)^{\sigma_j}\le\frac{r^{\rho(r)}}{r^{\rho(r_j)}}=\exp\left(\int_{r_j}^{r}\left(\rho'(t)\log t+\frac{\rho(t)-\rho(r_j)}{t}\right)dt\right)\le\exp\left(\varepsilon_j\left|\log\frac{r}{r_j}\right|\right)$$

if $r\ge s_j$, where $\varepsilon_j\to 0$. Defining $K_j=1/\sqrt{\varepsilon_j}$ and $\delta_j=2\varepsilon_j/\sigma_j$, we see that the desired conclusion holds for large j. Finally, if $\lambda_L=\rho_L=0$, we define $c_j=L(t_j)$ and $\sigma_j=\inf\{\sigma|L(r)\le c_jr^{\sigma}\text{ for all }r\ge 1\}$. Then there exists $r_j\ge 1$ such that $L(r_j)=c_j(r_j)^{\sigma_j}$. We easily see that (r_j) has the desired properties if we choose $\delta_j=0$ and take any sequence (K_j) that tends to ∞.

Now choose (s_j) and (t_j) such that $t_j/s_j\to\infty$, let (r_j) be as above, and define $x_j=\log L(r_j)$. Since $\Phi(x)=\log\gamma(e^x)$ we deduce from (28) that

$$\gamma(L(r))\le(1+o(1))\left(\frac{L(r)}{L(r_j)}\right)^{\mu_j}\gamma(L(r_j))\tag{30}$$

for

$$\left|\log\frac{L(r)}{L(r_j)}\right|\le\frac{M_j}{\mu_j}\tag{31}$$

where $\mu_j=L(r_j)\gamma'(L(r_j))/\gamma(L(r_j))$. Our hypothesis on γ imply that $\mu_j\to\infty$. We may assume that M_j tends to ∞ so slowly that $M_j/2\mu_j\le K_j$ and $\delta_jM_j\to 0$. This, together with (29), implies that if $|\log r/r_j|\le M_j/2\sigma_j\mu_j$ and $r\ge s_j$, then

$$L(r)^{\mu_j}\le\left(1+\frac{\delta_jM_j}{2\mu_j}\right)^{\mu_j}\left(\frac{r}{r_j}\right)^{\sigma_j\mu_j}L(r_j)^{\mu_j}=(1+o(1))\left(\frac{r}{r_j}\right)^{\sigma_j\mu_j}L(r_j)^{\mu_j}.\tag{32}$$

It follows that (31) and hence (30) hold for large j, provided $|\log r/r_j|\le M_j/2\sigma_j\mu_j$ and $r\ge s_j$. Combining (30) and (32) we find that

$$T(r)=\gamma(L(r))\le(1+o(1))\left(\frac{r}{r_j}\right)^{\sigma_j\mu_j}\gamma(L(r_j))=(1+o(1))\left(\frac{r}{r_j}\right)^{\sigma_j\mu_j}T(r_j)\tag{33}$$

for $|\log r/r_j| \leq M_j/2\sigma_j\mu_j$ and $r \geq s_j$. We may assume that $\sigma_j\mu_j \to \infty$ since otherwise $T(r)$ has Pólya peaks of some finite order and the conclusion follows from the remarks made in the introduction of [1]. It follows that if M_j tends to ∞ slowly enough, then $r \geq s_j$ is always satisfied if $|\log r/r_j| \leq M_j/2\sigma_j\mu_j$. As in the proof of Theorem 1 we can now deduce from (33) that

$$\log M(r_j) \leq (1+o(1))\pi\sigma_j\mu_j\gamma(L(r_j)) = (1+o(1))\pi\sigma_j\Psi(T(r_j)).$$

This completes the proof since $\sigma_j \to \lambda_L \leq 1$.

We remark that if $\lambda_L = 0$, then the range of r-values where (29) holds is larger than in the usual definition of Pólya peaks, and for our purposes this larger range is actually needed. To me it seems more natural to work with these larger intervals if $\lambda_L = 0$. We also remark that in order that (29) holds, it is not necessary that $\sigma_j \to \lambda_L$ but we may achieve $\sigma_j \to \sigma$ for any σ satisfying $\lambda_L \leq \sigma \leq \rho_L$. Moreover, we do not need $\lambda_L \leq 1$. In fact, even the case $\lambda_L = \infty$ is not excluded, but can be treated using the methods of the proof of Lemma 1 in [1]. (In this case, the range of r-values where (29) holds is smaller than in the usual definition of Pólya peaks.) Finally, we remark that it was proved in [1, p. 173] that if $L(r)$ is convex in $\log r$ and if $0 < \sigma < \infty$, then we have $r_jL'(r_j)/L(r_j) \sim \sigma$ for any sequence (r_j) of Pólya peaks of order σ for $L(r)$. Since it follows from (29) that $\sigma_j \sim r_jL'(r_j)/L(r_j)$, we see that such peaks also exist if $L(r)$ is only assumed to be increasing and differentiable.

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References

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