

# The Hausdorff dimension of Julia sets of meromorphic functions in the Speiser class

Walter Bergweiler and Weiwei Cui\*

## Abstract

We show that for each  $d \in (0, 2]$  there exists a meromorphic function  $f$  such that the inverse function of  $f$  has three singularities and the Julia set of  $f$  has Hausdorff dimension  $d$ .

## 1 Introduction and main results

We will be concerned with the iteration of transcendental meromorphic functions  $f: \mathbb{C} \rightarrow \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . The main objects studied here are the Fatou set  $F(f)$ , consisting of all  $z \in \mathbb{C}$  for which the iterates  $f^k$  of  $f$  are defined and form a normal family in some neighborhood of  $z$ , and the Julia set  $J(f) := \widehat{\mathbb{C}} \setminus F(f)$ . For an introduction to the dynamics of transcendental meromorphic functions we refer to [10].

The Speiser class  $\mathcal{S}$  consists of all transcendental meromorphic functions  $f$  for which the set  $\text{sing}(f^{-1})$  of singularities of the inverse, i.e., the set of critical and asymptotic values of  $f$ , is finite. More precisely, if  $q$  denotes the cardinality of  $\text{sing}(f^{-1})$ , then we write  $f \in \mathcal{S}_q$ . Equivalently,  $f \in \mathcal{S}$  if there exist finitely many points  $a_1, \dots, a_q \in \widehat{\mathbb{C}}$  such that

$$f: \mathbb{C} \setminus f^{-1}(\{a_1, \dots, a_q\}) \rightarrow \widehat{\mathbb{C}} \setminus \{a_1, \dots, a_q\}$$

is a covering map. And if  $q$  is the minimal number with this property, then  $f \in \mathcal{S}_q$ . The monodromy theorem implies that we always have  $q \geq 2$ .

---

\*W. Cui expresses his gratitude to the Centre for Mathematical Sciences of Lund University for providing a nice working environment.

The Eremenko-Lyubich class  $\mathcal{B}$  consists of all transcendental meromorphic functions  $f$  for which  $\text{sing}(f^{-1}) \setminus \{\infty\}$  is a bounded subset of  $\mathbb{C}$ . Thus

$$\mathcal{S} = \bigcup_{q=2}^{\infty} \mathcal{S}_q \subset \mathcal{B}.$$

Both the Speiser and the Eremenko-Lyubich class play an important role in transcendental dynamics. The similarities and differences between these classes are addressed in a number of recent papers [2, 12, 13, 14, 21]. A survey of some of these as well as many other results concerning the dynamics of functions in  $\mathcal{S}$  and  $\mathcal{B}$  is given in [41].

Considerable attention has been paid to the Hausdorff dimension of Julia sets; see [30] and [45] for surveys. We will denote the Hausdorff dimension of a subset  $A$  of  $\mathbb{C}$  by  $\dim A$ .

Baker [4] showed that if  $f$  is a transcendental entire function, then  $J(f)$  contains continua so that  $\dim J(f) \geq 1$ . Stallard [44, Theorem 1.1] showed that for all  $d \in (1, 2)$  there exists a transcendental entire function  $f$  with  $\dim J(f) = d$  while Bishop [15] constructed an example with  $\dim J(f) = 1$ . Stallard's examples are actually in the Eremenko-Lyubich class  $\mathcal{B}$ . Previously she had shown that  $\dim J(f) > 1$  for entire  $f \in \mathcal{B}$ . In particular, this is the case for entire functions in  $\mathcal{S}$ .

Albrecht and Bishop [1] showed that given  $\delta > 0$  there exists an entire function  $f \in \mathcal{S}$  such that  $\dim J(f) < 1 + \delta$ . In fact, these were the first examples of entire functions in  $\mathcal{S}$  for which the Julia set has dimension strictly less than 2. In their examples the inverse has three finite singularities. Since every non-constant entire function has the asymptotic value  $\infty$  by Iversen's theorem, and since we include  $\infty$  in  $\text{sing}(f^{-1})$ , their examples are in  $\mathcal{S}_4$ .

For transcendental meromorphic functions  $f$  we have  $\dim J(f) > 0$  by a result of Stallard [42]. On the other hand, she showed [43, Theorem 5] that for all  $d \in (0, 1)$  there exists a transcendental meromorphic function  $f$  such that  $\dim J(f) = d$ . Again her examples are in  $\mathcal{B}$ . Together with her result covering the interval  $(1, 2)$  mentioned above, and since  $J(\exp z) = \mathbb{C}$  by a result of Misiurewicz [38] and  $J(\tan z) = \mathbb{R}$ , it follows that for all  $d \in (0, 2]$  there exists  $f \in \mathcal{B}$  such that  $\dim J(f) = d$ .

We shall show that such examples also exist in the Speiser class  $\mathcal{S}$  and in fact in  $\mathcal{S}_3$ .

**Theorem 1.1.** *Let  $d \in (0, 2]$ . Then there exists a function  $f \in \mathcal{S}_3$  such that  $\dim J(f) = d$ .*

We note that if  $f \in \mathcal{S}_2$ , then  $\dim J(f) > 1/2$ ; see [8, Theorem 3.11] and [34, Remark 3.2 and Section 4.3]. On the other hand, Barański [8, Section 4] showed that for  $f_\lambda(z) = \lambda \tan z \in \mathcal{S}_2$  the function  $\lambda \mapsto \dim J(f_\lambda)$  maps  $(0, 1]$  monotonically and continuously onto  $(1/2, 1]$ . It seems likely that for  $d \in (1, 2]$  there also exists  $f \in \mathcal{S}_2$  such that  $\dim J(f) = d$ . The main interest of Theorem 1.1 thus lies in the case where  $0 < d \leq 1/2$ .

## 2 Preliminary results

The escaping set  $I(f)$  of a meromorphic function  $f$  is defined as the set of all  $z \in \mathbb{C}$  for which  $f^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$ . We always have  $I(f) \neq \emptyset$  and  $J(f) = \partial I(f)$ . This was shown by Eremenko [22] for transcendental entire  $f$  and by Domínguez [17] for transcendental meromorphic  $f$ .

For  $f \in \mathcal{B}$  we have  $I(f) \subset J(f)$ . This is due to Eremenko and Lyubich [24, Theorem 1] for entire  $f$  and to Rippon and Stallard [40, Theorem A] for meromorphic  $f$ .

Theorem 1.1 complements the result of [2] where it was shown that for all  $d \in [0, 2]$  there exists a meromorphic function  $f \in \mathcal{S}$  such that  $\dim I(f) = d$ .

An important concept in the theory of meromorphic functions is the order; see, e.g., [28]. The following result was proved in [11, Theorem 1.1].

**Lemma 2.1.** *Let  $f \in \mathcal{B}$  be of finite order  $\rho$ . Suppose that  $\infty$  is not an asymptotic value of  $f$  and that there exists  $M \in \mathbb{N}$  such that all but finitely many poles of  $f$  have multiplicity at most  $M$ . Then*

$$\dim I(f) \leq \frac{2M\rho}{2 + M\rho}. \quad (2.1)$$

We will not actually use this lemma. Instead we will use the following closely related result, which can be proved by the same method.

**Lemma 2.2.** *Let  $f$  be as in the Lemma 2.1. Suppose, furthermore, that  $f(0) \neq 0$ . For  $\lambda \in \mathbb{C} \setminus \{0\}$  define  $f_\lambda(z) = f(\lambda z)$ . Then*

$$\limsup_{\lambda \rightarrow 0} \dim J(f_\lambda) \leq \frac{2M\rho}{2 + M\rho}.$$

*Proof.* We proceed as in [11]. Let  $(a_j)$  be the sequence of poles of  $f$  and let  $m_j$  be the multiplicity of  $a_j$ . Let  $b_j \in \mathbb{C} \setminus \{0\}$  be such that

$$f(z) \sim \left( \frac{b_j}{z - a_j} \right)^{m_j} \quad \text{as } z \rightarrow a_j.$$

Let

$$t > \frac{2M\rho}{2 + M\rho}.$$

By [11, Lemma 3.1] we have

$$\sum_{j=1}^{\infty} \left( \frac{|b_j|}{|a_j|^{1+1/M}} \right)^t < \infty. \quad (2.2)$$

Let  $a_j^\lambda$  and  $b_j^\lambda$  the corresponding values for  $f_\lambda$ . Then  $a_j^\lambda = a_j/\lambda$  and  $b_j^\lambda = b_j/\lambda$  so that

$$\sum_{j=1}^{\infty} \left( \frac{|b_j^\lambda|}{|a_j^\lambda|^{1+1/M}} \right)^t = |\lambda|^{t/M} \sum_{j=1}^{\infty} \left( \frac{|b_j|}{|a_j|^{1+1/M}} \right)^t. \quad (2.3)$$

As in [11] we choose  $R_0 > |f(0)|$  such that  $\text{sing}(f_\lambda^{-1}) = \text{sing}(f^{-1}) \subset D(0, R_0)$  and  $R \geq 2^M R_0$ . Here and in the following  $D(a, r)$  denotes the open disk of radius  $r$  around a point  $a \in \mathbb{C}$ . Choosing  $\lambda$  small we can achieve that  $f(D(0, 3R)) \subset D(0, R)$  so that  $F(f_\lambda) \subset D(0, 3R)$ .

We put  $B(R) := \{z: |z| > R\} \cup \{\infty\}$  and, as in [11, pp. 5376f.], consider the collection  $E_l$  of all components  $V$  of  $f^{-l}(B(R))$  for which  $f^k(V) \subset B(3R)$  for  $0 \leq k \leq l-1$ . Then  $E_l$  is a cover of  $J(f_\lambda)$  and we have

$$\sum_{V \in E_l} (\text{diam}_\chi(V))^t = \frac{1}{M} \left( \frac{32}{(2R)^{1/M} 24} \right)^t \left( M(2^{1/M} 24)^t \sum_{j=1}^{\infty} \left( \frac{|b_j^\lambda|}{|a_j^\lambda|^{1+1/M}} \right)^t \right)^l.$$

Here  $\text{diam}_\chi(V)$  denotes the spherical diameter of  $V$ . Together with (2.2) and (2.3) the last estimate yields that if  $\lambda$  is sufficiently small, then

$$\sum_{V \in E_l} (\text{diam}_\chi(V))^t \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

Hence  $\dim J(f_\lambda) \leq t$  for small  $\lambda$ . □

It follows from a recent result of Mayer and Urbański [35] that Lemma 2.2 and (2.1) can be sharpened to

$$\lim_{\lambda \rightarrow 0} \dim J(f_\lambda) = \dim I(f).$$

In fact, their result says that  $\dim I(f)$  is the infimum of the set of all  $t > 0$  for which (2.2) holds.

For entire functions, the following result can be found in [24, Section 3] and [21, Proposition 2.3].

**Lemma 2.3.** *Let  $f, g \in \mathcal{S}_3$  and suppose that there exist homeomorphisms  $\psi: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  and  $\phi: \mathbb{C} \rightarrow \mathbb{C}$  such that  $\psi \circ f = g \circ \phi$ . Then there exist a fractional linear transformation  $\alpha: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  and an affine map  $\beta: \mathbb{C} \rightarrow \mathbb{C}$  such that  $\alpha \circ f = g \circ \beta$ .*

*Proof.* Since  $f \in \mathcal{S}_3$  it follows from [21, Observation 1.10] that there exists a fractional linear transformation  $\alpha: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  which is isotopic to  $\psi$  relative to  $\text{sing}(f^{-1})$ . We follow the argument in the proof of [21, Proposition 2.3(a)]. Let  $(\psi_t)_{t \in [0,1]}$  be the isotopy between  $\psi_0 = \psi$  and  $\psi_1 = \alpha$ . By the isotopy lifting property, there exists a unique isotopy  $(\phi_t)_{t \in [0,1]}$  in

$$U := f^{-1}\left(\widehat{\mathbb{C}} \setminus \text{sing}(f^{-1})\right)$$

such that  $\phi_0 = \phi$  and  $\psi_t \circ f = g \circ \phi_t$  in  $U$  for all  $t \in [0, 1]$ . It remains to show that  $\phi_t$  extends continuously to the preimages of the singular values of  $f$  and coincides with  $\phi$  there for all  $t$ .

Let  $z_0 \in \mathbb{C}$  be a preimage of  $v_0 \in \text{sing}(f^{-1})$ . We have to show that  $\phi_t(z) \rightarrow \phi(z_0)$  as  $z \rightarrow z_0$ . We take a small neighborhood  $D$  of  $z_0$  such that  $f: D \rightarrow f(D)$  is a proper map, with no critical points except possibly at  $z_0$ . We may also assume that  $f(z) \neq v_0$  for  $z \in \overline{D} \setminus \{z_0\}$ . It then suffices to show that  $\phi_t(z) \in \phi(D)$  if  $z$  is sufficiently close to  $z_0$ . By the continuity of  $f$  and the properties of an isotopy, we have

$$\psi_t(f(z)) \rightarrow \psi_t(f(z_0)) = \psi(v_0) \quad \text{as } z \rightarrow z_0,$$

uniformly in  $t \in [0, 1]$ . It follows that

$$g(\phi_t(z)) = \psi_t(f(z)) \in \psi(f(D)) = g(\phi(D))$$

if  $z$  is sufficiently close to  $z_0$ . Thus  $\phi_t(z)$  can be obtained by analytic continuation of  $g^{-1}$  along the curve  $t \mapsto \psi_t(f(z))$ . It follows that  $\phi_t(z) \in \phi(D)$  if  $z$  is sufficiently close to  $z_0$ .

With  $\beta := \phi_1$  we have  $\alpha \circ f = g \circ \beta$ . If  $z$  is not a critical point of  $f$ , then  $\beta$  is of the form  $g^{-1} \circ \psi \circ f$  near  $z$  for some branch of the inverse of  $g$ . Hence  $\beta$  is holomorphic. Since  $\beta: \mathbb{C} \rightarrow \mathbb{C}$  is a homeomorphism, this implies that  $\beta$  is affine.  $\square$

The following result is due to Kotus and Urbański [29].

**Lemma 2.4.** *Let  $f$  be an elliptic function. Let  $q$  be the maximal multiplicity of the poles of  $f$ . Then*

$$\dim J(f) > \frac{2q}{q+1}.$$

We shall need a variation of this result.

**Lemma 2.5.** *Let  $f$  be as in Lemma 2.4. If  $(f_n)$  is a sequence of meromorphic functions which converges locally uniformly to  $f$ , then*

$$\dim J(f_n) > \frac{2q}{q+1}$$

for all large  $n$ .

Lemma 2.5 can be deduced from the work of Kotus and Urbański; see Remark 2.1 below. But for the convenience of the reader we will include a proof of Lemma 2.5, thereby reproving Lemma 2.4. Here we will use the following result [25, Proposition 9.7].

**Lemma 2.6.** *Let  $S_1, \dots, S_m$  be contractions on a closed subset  $K$  of  $\mathbb{R}^d$  such that there exists  $b_1, \dots, b_m \in (0, 1)$  with*

$$b_k |u - v| \leq |S_k(u) - S_k(v)| \quad \text{for } u, v \in K \quad (2.4)$$

and  $1 \leq k \leq m$ . Suppose that  $K_0$  is a non-empty compact subset of  $K$  with

$$K_0 = \bigcup_{k=1}^m S_k(K_0)$$

and  $S_j(K_0) \cap S_k(K_0) = \emptyset$  for  $j \neq k$ . Let  $t > 0$  with

$$\sum_{k=1}^m b_k^t = 1.$$

Then  $\dim K_0 \geq t$ .

*Proof of Lemmas 2.4 and 2.5.* Let  $(a_j)$  be the sequence of poles of multiplicity  $q$ . For sufficiently large  $R$  there is a neighborhood  $U_j$  of  $a_j$  such that

$$f: U_j \setminus \{a_j\} \rightarrow \{z \in \mathbb{C} : |z| > R\}$$

is a covering map of degree  $q$ . There exists  $r_1 > 0$  such that  $D(a_j, r_1) \subset U_j$  for all  $j$ . Let  $0 < r_0 < r_1$ . Then there exists  $M \in \mathbb{N}$  such that

$$\overline{D(a_k, r_0)} \subset \overline{U_k} \subset f(D(a_j, r_0))$$

for all  $j, k \geq M$ . Thus for  $j, k \geq M$  there exists  $V_{j,k} \subset D(a_j, r_0)$  such that  $f: V_{j,k} \rightarrow D(a_k, r_0)$  is biholomorphic.

Let  $W_k := f^{-1}(V_{k,M}) \cap D(a_M, r_0)$ . Then

$$f^2: W_k \rightarrow D(a_M, r_0)$$

is biholomorphic. Moreover,  $f^2$  extends to a bijective map from  $\overline{W_k}$  to  $K := \overline{D(a_M, r_0)}$ . Let  $S_k: K \rightarrow \overline{W_k}$  be the inverse function of  $f^2: \overline{W_k} \rightarrow K$ . Then  $S_k$  extends to an injective map  $S_k: D(a_M, r_1) \rightarrow \mathbb{C}$ . Choosing  $r_0 \leq (2 - \sqrt{3})r_1$  we conclude that  $W_k = S_k(D(a_M, r_0))$  is convex; see [19, Theorem 2.13].

In order to apply Lemma 2.6 we note that since  $f$  has a pole of multiplicity  $q$  at  $a_j$ , there exists  $c_1 > 0$  such that

$$|f'(z)| \leq c_1 |f(z)|^{(q+1)/q} \quad \text{for } z \in D(a_j, r_0).$$

This implies that there exists  $c_2 > 0$  such that

$$|f'(z)| \leq c_2 |a_k|^{(q+1)/q} \quad \text{if } z \in D(a_j, r_0) \text{ and } f(z) \in D(a_k, r_0).$$

With  $c_3 := c_2^2 |a_M|^{(q+1)/q}$  this yields that

$$|(f^2)'(z)| = |f'(f(z))f'(z)| \leq c_3 |a_k|^{(q+1)/q} \quad \text{for } z \in W_k.$$

Since  $W_k$  is convex this yields that

$$|u - v| = |f^2(S_k(u)) - f^2(S_k(v))| \leq c_3 |a_k|^{(q+1)/q} |S_k(u) - S_k(v)| \quad \text{for } u, v \in K.$$

It follows that (2.4) holds with

$$b_k := \frac{1}{c_3 |a_k|^{(q+1)/q}}.$$

Let now  $N \geq M$  and define  $t > 0$  by

$$\sum_{k=M}^N b_k^t = 1.$$

It follows from Lemma 2.6 that the limit set of the iterated function system  $\{S_k : M \leq k \leq N\}$  has Hausdorff dimension at least  $t$ . It is easily seen that this limit set is contained in the Julia set. Thus  $\dim J(f) \geq t$ . Since

$$\sum_{k=M}^{\infty} \frac{1}{|a_k|^2} = \infty$$

we have  $t > 2q/(q+1)$  if  $N$  is large enough. This yields Lemma 2.4.

To prove Lemma 2.5 we note that for large  $n$  there are domains  $W_k^n$  and  $U_M^n$  close to  $W_k$  and  $U_M$  such that

$$f_n^2 : W_k^n \rightarrow U_M^n$$

is biholomorphic, and the inverse function  $S_k^n$  satisfies (2.4) with a constant  $b_k^n$  instead of  $b_k$ , with  $b_k^n \rightarrow b_k$  as  $n \rightarrow \infty$ . The conclusion then follows again from Lemma 2.6.  $\square$

*Remark 2.1.* The argument of Kotus and Urbański [29] is similar to the one used above, but they use an infinite iterated function system and apply results of Mauldin and Urbański [33] concerning such systems. However, it would suffice to consider a sufficiently large finite subsystem. This would yield Lemma 2.5 as above.

The proof actually yields that the hyperbolic dimension of  $f$  and  $f_n$ , and not only the Hausdorff dimension of their Julia sets, have the given lower bound; see [9, 39] for a discussion of the hyperbolic dimension of meromorphic functions.

**Lemma 2.7.** *Let  $f \in \mathcal{S}$ . Suppose that  $f$  has an attracting fixed point whose immediate attracting basin contains all finite singularities of  $f^{-1}$ . Suppose also that  $\infty$  is not an asymptotic value of  $f$  and that there exists a uniform bound on the multiplicities of the poles of  $f$ . Then  $J(f)$  is totally disconnected and  $F(f)$  is connected.*

There are several closely related results in the literature, see [5, Theorem G], [18, Theorem A and Corollary 3.2] and [47, Theorem 2.7]. However, none of these results seems to apply exactly to the situation we have.

Zheng [47, Theorem 2.7] showed that the conclusion of Lemma 2.7 holds if  $\infty \notin \text{sing}(f^{-1})$ . Thus his result would apply if the poles are assumed to be simple, while we allow multiple poles. Hawkins and Koss ([27, Theorem 3.12]; see also [26, Theorem 3.2]) do not require that the poles are simple, but they restrict to elliptic functions.

Our proof of Lemma 2.7 will use some ideas from the papers mentioned. A difference to the methods employed there, however, is that we will use the following consequence of the Grötzsch inequality; see [16, Section 5.2] and [36, Corollary A.7]. Here and in the following  $\text{mod}(A)$  denotes the modulus of an annulus  $A$ .

**Lemma 2.8.** *Let  $(G_k)$  be a sequence of simply connected domains in  $\mathbb{C}$  such that  $A_k := G_k \setminus \overline{G_{k+1}}$  is an annulus for all  $k \in \mathbb{N}$ . Suppose that*

$$\sum_{k=1}^{\infty} \text{mod}(A_k) = \infty.$$

*Then  $\bigcap_{k=1}^{\infty} G_k$  consists of a single point.*

*Proof of Lemma 2.7.* Let  $\xi$  be the attracting fixed point whose attracting basin  $W$  contains all finite singularities of  $f^{-1}$ . Then the postsingular set

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}) \setminus \{\infty\})}$$

is a compact subset of  $W$ . It is not difficult to see that there exist Jordan domains  $U$  and  $V$  such that

$$\{\xi\} \cup P(f) \subset U \subset \overline{U} \subset V \subset \overline{V} \subset W.$$

Then  $A := V \setminus \overline{U}$  is an annulus. Clearly,  $A \subset W$ .

Let  $(a_j)$  be the sequence of poles of  $f$  and let  $m_j$  denote the multiplicity of  $a_j$ . By hypothesis, there exists  $M \in \mathbb{N}$  such that  $m_j \leq M$  for all  $j$ . Let  $Y_j$  be the component of the preimage of  $\widehat{\mathbb{C}} \setminus \overline{U}$  that contains  $a_j$ . Then  $f: Y_j \setminus \{a_j\} \rightarrow \mathbb{C} \setminus \overline{U}$  is a covering of degree  $m_j$ . Putting  $B_j := f^{-1}(A) \cap Y_j$  we find that  $B_j$  is an annulus with

$$\text{mod}(B_j) = \frac{1}{m_j} \text{mod}(A) \geq \frac{1}{M} \text{mod}(A).$$

To prove that  $J(f)$  is totally disconnected, let  $z \in J(f)$ . We want to show that the component of  $J(f)$  containing  $z$  consists of the point  $z$  only. First we note that for all  $n \in \mathbb{N}$  there exists  $j(n) \in \mathbb{N}$  such that  $f^n(z) \in Y_{j(n)}$ . Let  $X_n$  be the component of  $f^{-n}(Y_{j(n)})$  containing  $z$ . Note that  $Y_{j(n)}$  and  $X_n$  are Jordan domains. Since  $\partial U$  and hence  $\partial X_n$  are contained in  $F(f)$ , the component of  $J(f)$  containing  $z$  is contained in the intersection of the  $X_n$ . It thus suffices to prove that this intersection consists of only one point.

In order to do so we note that since  $P(f) \subset U$ , the map  $f^n: X_n \rightarrow Y_{j(n)}$  is biholomorphic. This implies that

$$C_n := f^{-n}(B_{j(n)}) \cap X_n = f^{-n-1}(A) \cap X_n \quad (2.5)$$

is an annulus satisfying

$$\text{mod}(C_n) = \text{mod}(B_{j(n)}) \geq \frac{1}{M} \text{mod}(A).$$

Moreover,  $X_n$  is equal to the union of  $C_n$  and the component of  $\mathbb{C} \setminus C_n$  that contains  $z$ .

Since the closure of  $A$  is a compact subset of the attracting basin  $W$  of  $\xi$ , there exists  $p \in \mathbb{N}$  such that  $f^p(A) \subset U$ . In particular,

$$f^p(A) \cap A = \emptyset. \quad (2.6)$$

This implies that  $C_{n+p} \cap C_n = \emptyset$  for all  $n \in \mathbb{N}$ . In fact, if  $w \in C_{n+p} \cap C_n$ , then  $f^{n+p+1}(w) \in f^p(A) \cap A$  by (2.5), contradicting (2.6). It follows that

$$X_{n+p} \subset X_n \setminus C_n,$$

and hence that

$$\text{mod}(X_n \setminus \overline{X_{n+p}}) \geq \text{mod}(C_n) \geq \frac{1}{M} \text{mod}(A).$$

As already mentioned above it now follows from Lemma 2.8, applied with  $G_k = X_{1+pk}$ , that  $J(f)$  is totally disconnected. Of course, this yields that  $F(f)$  is connected.  $\square$

### 3 Proof of Theorem 1.1

Let  $G$  be the conformal map from the triangle with vertices  $0$ ,  $\pi/2$  and  $i\pi/2$  onto the lower half-plane such that

$$G(\pi/2) = 0, \quad G(0) = 1 \quad \text{and} \quad G(i\pi/2) = \infty.$$

Extending this to the whole plane by reflections we obtain an elliptic function  $G$ . The critical values of  $G$  are  $0$ ,  $1$  and  $\infty$  so that  $G \in \mathcal{S}_3$ . The zeros and poles of  $G$  have multiplicity 4 while the 1-points have multiplicity 2. We also note that  $G(\mathbb{R}) = [0, 1]$ .

We may express  $G$  in terms of the Weierstrass  $\wp$ -function with periods  $\pi$  and  $\pi i$ . The critical values of  $\wp$  are  $e_1$ ,  $e_2$ ,  $e_3$  and  $\infty$ , with

$$e_2 = \wp((\pi + i\pi)/2) = 0 \quad \text{and} \quad e_1 = \wp(\pi/2) = -e_3 = -\wp(i\pi/2).$$

It follows from this that

$$G(z) = \left( \frac{\wp(z + i\pi/2)}{e_1} \right)^2.$$

First we construct an example of a function  $f \in \mathcal{S}_3$  where the Julia set is the whole sphere and thus has Hausdorff dimension 2. To this end we consider the function

$$f(z) := iG(\pi z/2).$$

Then  $f \in \mathcal{S}_3$ . The critical values of  $f$  are  $0$ ,  $i$  and  $\infty$  and we have  $f(0) = i$  and  $f(i) = \infty$ . To prove that  $J(f) = \widehat{\mathbb{C}}$  we note that  $f$  has no wandering domains [6] and no Baker domains [40, Corollary to Theorem A]. All other types of components of  $F(f)$  are related to the singularities of  $f^{-1}$ ; see [10, Theorem 7] for the exact statement. Since the points in  $\text{sing}(f^{-1}) \cap \mathbb{C} = \{0, i\}$  are mapped to  $\infty$  by  $f$  or  $f^2$  this yields that  $F(f) = \emptyset$  and hence  $J(f) = \widehat{\mathbb{C}}$  as claimed.

To construct functions in  $\mathcal{S}_3$  for which the Julia set has Hausdorff dimension  $d \in (0, 2)$ , we consider, for  $p \in \mathbb{N}$  and small  $\eta \in (0, \pi/2)$ , the function  $H$

defined by  $H(z) := \eta G(z)^p$ . The critical values of  $H$  are  $0$ ,  $\eta$  and  $\infty$ . Thus  $H \in \mathcal{S}_3$ . We have  $H(0) = \eta$  and  $H(\pi/2) = 0$ . Also,  $H$  decreases in the interval  $(0, \pi/2)$ . Choosing  $\eta$  sufficiently small we can achieve that  $H$  has an attracting fixed point  $\xi \in (0, \pi/2)$  whose attracting basin contains  $[0, \eta]$  and thus, since  $H(\mathbb{R}) = [0, \eta]$ , also contains  $\mathbb{R}$ . Lemma 2.7 implies that this attracting basin coincides with  $F(H)$  and that  $J(H)$  is totally disconnected.

Since  $H((0, \pi/2)) = (0, \eta)$  we actually have  $\xi \in (0, \eta)$ . Choosing  $\eta$  small we can also achieve that  $H''(z) \neq 0$  for  $0 < |z| \leq \eta$ .

Let now  $m$  be a (large) odd integer. Then

$$h_m(z) = H(m \arcsin(z/m)).$$

defines a meromorphic function  $h_m \in \mathcal{S}_3$ . Similar examples were already considered by Teichmüller [46, p. 734], and later by Bank and Kaufman [7, Section 5], Langley [31, Section 2] and Eremenko [23].

The elliptic function  $H(z) = h_m(m \sin(z/m))$  has order 2. A result of Edrei and Fuchs [20, Corollary 1.2] thus yields that  $h_m$  has order 0. In fact, as in the papers cited above we find that there exists a constant  $c$  such that the Nevanlinna characteristic satisfies  $T(r, h_m) \sim c(\log r)^2$  as  $r \rightarrow \infty$ .

For large  $m$  the function  $h_m$  has an attracting fixed point  $\xi_m$ , with  $\xi_m \rightarrow \xi$  as  $m \rightarrow \infty$ , such that the attracting basin of  $\xi_m$  contains the interval  $[0, \eta]$  and hence  $\mathbb{R}$ . Lemma 2.7 implies that this attracting basin is connected and coincides with  $F(h_m)$ . Choosing  $m$  large we can also achieve that  $h_m$  decreases in the interval  $[0, \eta]$  and that  $h_m''(z) \neq 0$  for  $0 < |z| \leq \eta$ . This implies that  $h_m'$  decreases in the interval  $[0, \eta]$ .

The poles of  $H$  and  $h_m$  have multiplicity  $4p$ . Except for the zeros at  $\pm m$ , which have multiplicity  $2p$ , the zeros of  $h_m$  also have multiplicity  $4p$ . The  $\eta$ -points on the real axis have multiplicity 2, but if  $p \geq 2$ , then  $H$  and  $h_m$  also have simple  $\eta$ -points (corresponding to the points where  $G$  takes the  $p$ -th roots of unity).

As the poles of  $H$  have multiplicity  $4p$ , Lemma 2.4 yields that

$$\dim J(H) > \frac{8p}{4p+1}.$$

Moreover, this lemma says that if  $m$  is sufficiently large, then

$$\dim J(h_m) > \frac{8p}{4p+1}. \tag{3.1}$$

We fix such a value of  $m$  and, for  $\lambda \in (0, 1]$ , we put  $f_\lambda(z) := h_m(\lambda z)$  so that  $f_1 = h_m$ .

Since  $h_m$  has order 0, Lemma 2.2 yields that

$$\lim_{\lambda \rightarrow 0} \dim J(f_\lambda) = 0. \quad (3.2)$$

We will show that the function  $\lambda \mapsto \dim J(f_\lambda)$  is continuous in the interval  $(0, 1]$ . Since  $f_1 = h_m$  it then follows from (3.1) and (3.2) that for all  $d \in (0, 8p/(4p+1)]$  there exists  $\lambda \in (0, 1]$  such that  $\dim J(f_\lambda) = d$ . Since  $p$  can be chosen arbitrarily large, this yields the conclusion.

It remains to prove that  $\lambda \mapsto \dim J(f_\lambda)$  is continuous. Recalling that  $h_m$  and  $h'_m$  are decreasing in the interval  $[0, \eta]$  we can deduce that  $f_\lambda$  has an attracting fixed point  $\zeta_\lambda \in (0, \eta)$  and that the multiplier

$$m_\lambda := f'_\lambda(\zeta_\lambda) = \lambda h'_m(\lambda \zeta_\lambda)$$

is a decreasing function of  $\lambda$  in the interval  $(0, 1]$ . Note here that  $\zeta_1 = \xi_m$  and  $m_1 = h'_m(\xi_m) < 0$ . As before it follows from Lemma 2.7 that the attracting basin of  $\zeta_\lambda$  is connected and coincides with the Fatou set of  $f_\lambda$ .

Let  $\lambda \in (0, 1]$ . Koenigs' theorem [37, Theorem 8.2] yields that there exists a function  $g$  holomorphic and injective in some neighborhood  $U$  of  $\zeta_\lambda$  such that  $g(\zeta_\lambda) = 0$ ,  $g'(\zeta_\lambda) = 1$  and

$$g(f_\lambda(g^{-1}(z))) = m_\lambda z$$

for all  $z \in g(U)$ . For  $\kappa \in (0, 1]$  we put

$$\gamma = \frac{\log m_\kappa}{\log m_\lambda} - 1$$

and define  $h: \mathbb{C} \rightarrow \mathbb{C}$ ,  $h(z) = z|z|^\gamma$ . Then

$$h(m_\lambda h^{-1}(z)) = m_\kappa z.$$

With  $\phi = h \circ g: U \rightarrow \mathbb{C}$  we then have

$$\phi(f_\lambda(\phi^{-1}(z))) = m_\kappa z \quad (3.3)$$

for  $z \in \phi(U)$ . The maps  $h$  and  $\phi$  are  $K$ -quasiconformal with

$$K = \max \left\{ \frac{\log m_\kappa}{\log m_\lambda}, \frac{\log m_\lambda}{\log m_\kappa} \right\}. \quad (3.4)$$

For a detailed account of quasiconformal mappings, we refer to [32]. The complex dilatation  $\mu(z) := \phi_{\bar{z}}(z)/\phi_z(z)$  satisfies

$$\mu(f_\lambda(z)) = \mu(z) \frac{f'_\lambda(z)}{\overline{f'_\lambda(z)}} \quad (3.5)$$

if  $z, f_\lambda(z) \in U$ . We may use (3.5) to extend  $\mu$  to  $\mathbb{C}$ . More precisely, we put  $\mu(z) = 0$  for  $z \in J(f_\lambda)$  while for  $z \in F(f_\lambda)$  we define

$$\mu(z) = \mu(f_\lambda^n(z)) \frac{\overline{(f_\lambda^n)'(z)}}{(f_\lambda^n)'(z)}, \quad (3.6)$$

where  $n$  is chosen so large that  $f_\lambda^n(z) \in U$ . Using (3.5) it is easily seen that  $\mu$  is well-defined, i.e., the definition does not depend on the value of  $n$  chosen in (3.6). We find that (3.5) holds for all  $z$ .

Let  $\psi: \mathbb{C} \rightarrow \mathbb{C}$  be the solution of the Beltrami equation

$$\mu(z) = \frac{\psi_{\bar{z}}(z)}{\psi_z(z)},$$

normalized by  $\psi(0) = 0$  and  $\psi(\eta) = \eta$ . It follows from (3.5) that

$$k := \psi \circ f_\lambda \circ \psi^{-1} \quad (3.7)$$

is meromorphic. Since  $f_\lambda$  is symmetric with respect to the real axis, the same applies to  $g$ ,  $\phi$ ,  $\mu$ ,  $\psi$  and  $k$ . By definition,  $f_\lambda$  is even. In order to show that  $k$  is also even, we first note that since  $f_\lambda$  is even, it follows from (3.5) that  $\mu$  is even. This implies that  $\psi(z)$  and  $\psi(-z)$  have the same complex dilatation. Hence there exists an affine map  $L$  such that  $\psi(z) = L(\psi(-z))$ . Since  $\psi(0) = 0$  we have  $L(0) = 0$  so that  $L$  has the form  $L(z) = az$  for some  $a \in \mathbb{C} \setminus \{0\}$ . We also see that  $L$  is real on the real axis so that  $a \in \mathbb{R} \setminus \{0\}$ . Since  $a\psi(i) = L(\psi(i)) = \psi(-i) = \overline{\psi(i)}$  we find that  $|a| = 1$ . As  $\psi$  is injective this implies that  $a = -1$  so that  $\psi$  is odd. Hence  $k$  is even.

Since the complex dilatations of  $\phi$  and  $\psi$  agree in  $U$ , we have  $\phi = \tau \circ \psi$  for some function  $\tau$  holomorphic and injective on  $\psi(U)$ . Together with (3.3) this implies that

$$k(z) = \psi(f_\lambda(\psi^{-1}(z))) = \tau^{-1}(m_\kappa \tau(z)) \sim m_\kappa z$$

as  $z \rightarrow \tau^{-1}(0) = \psi(\zeta_\lambda)$ . Thus  $k'(\psi(\zeta_\lambda)) = m_\kappa$ . Moreover,  $k(\psi(\zeta_\lambda)) = \psi(\zeta_\lambda)$ . In other words,  $\psi(\zeta_\lambda)$  is a fixed point of  $k$  of multiplier  $m_\kappa$ .

Another function with a fixed point of multiplier  $m_\kappa$  is  $f_\kappa$ . We will show that  $k = f_\kappa$ . In order to do so we note that both  $k$  and  $f_\kappa$  are in  $\mathcal{S}_3$ , with critical values  $0$ ,  $\eta$  and  $\infty$ . It follows from Lemma 2.3 and (3.7) that there exist a fractional linear transformation  $\alpha$  and an affine map  $\beta$  such that  $\alpha \circ k = f_\lambda \circ \beta$ . Since all poles of  $k$  and  $f_\kappa$  have multiplicity  $4p$ , all but two zeros of both functions have multiplicity  $4p$ , and all  $\eta$ -points of both functions have multiplicity 2 or 1, we find that  $\alpha(0) = 0$ ,  $\alpha(\eta) = \eta$  and  $\alpha(\infty) = \infty$ . Hence  $\alpha(z) \equiv z$  so that  $k = f_\lambda \circ \beta$ .

As  $\beta$  is affine we have  $-\beta(-z) = \beta(z) - 2\beta(0)$ . Noting that  $k$  and  $f_\lambda$  are even we deduce that

$$f_\lambda(\beta(z) - 2\beta(0)) = f_\lambda(-\beta(-z)) = f_\lambda(\beta(-z)) = k(-z) = k(z) = f_\lambda(\beta(z)).$$

Since periodic functions have order at least 1 while  $f_\lambda$  has order 0, this implies that  $\beta(0) = 0$  so that  $\beta$  has the form  $\beta(z) = cz$  for some constant  $c$ . Thus  $k(z) = f_\lambda(cz) = h_m(\lambda cz)$ . As  $k$  has an attracting fixed point of multiplier  $m_\kappa$  this yields that  $c = \kappa/\lambda$  so that  $k(z) = h_m(\kappa z) = f_\kappa(z)$ .

Inserting  $k = f_\kappa$  in (3.7), we obtain

$$f_\kappa = \psi \circ f_\lambda \circ \psi^{-1}.$$

This implies that

$$J(f_\kappa) = \psi(J(f_\lambda)).$$

As  $\psi$  is  $K$ -quasiconformal, and thus Hölder continuous with exponent  $1/K$ , we deduce that

$$\frac{1}{K} \dim J(f_\lambda) \leq \dim J(f_\kappa) \leq K \dim J(f_\lambda). \quad (3.8)$$

It follows from (3.4) that  $K \rightarrow 1$  as  $\kappa \rightarrow \lambda$ . Thus we deduce from (3.8) that  $\dim J(f_\kappa) \rightarrow \dim J(f_\lambda)$  as  $\kappa \rightarrow \lambda$ . Hence  $\lambda \mapsto \dim J(f_\lambda)$  is continuous.  $\square$

*Remark 3.1.* A celebrated result of Astala [3, Corollary 1.3] says that (3.8) can be improved to

$$\frac{1}{K} \left( \frac{1}{\dim J(f_\lambda)} - \frac{1}{2} \right) \leq \frac{1}{\dim J(f_\kappa)} - \frac{1}{2} \leq K \left( \frac{1}{\dim J(f_\lambda)} - \frac{1}{2} \right).$$

For our purposes, however, the weaker and simpler estimate (3.8) suffices.

## References

- [1] Simon Albrecht and Christopher J. Bishop, Speiser class Julia sets with dimension near one. *J. Anal. Math.* 141 (2020), no. 1, 49–98.
- [2] Magnus Aspenberg and Weiwei Cui, Hausdorff dimension of escaping sets of meromorphic functions. *Trans. Amer. Math. Soc.*, to appear.
- [3] Kari Astala, Area distortion of quasiconformal mappings. *Acta Math.* 173 (1994), no. 1, 37–60.
- [4] I. N. Baker, The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A, I. Math.* 1 (1975), no. 2, 277–283.
- [5] I. N. Baker, P. Domínguez, and M. E. Herring, Dynamics of functions meromorphic outside a small set. *Ergodic Theory Dynam. Systems* 21 (2001), no. 3, 647–672.
- [6] I. N. Baker, J. Kotus and Lü Yinian, Iterates of meromorphic functions IV: Critically finite functions. *Results Math.* 22 (1992), no. 3–4, 651–656.
- [7] Steven B. Bank and Robert P. Kaufman, On meromorphic solutions of first-order differential equations. *Comment. Math. Helv.* 51 (1976), no. 3, 289–299.
- [8] Krzysztof Barański, Hausdorff dimension and measures on Julia sets of some meromorphic maps. *Fund. Math.* 147 (1995), no. 3, 239–260.
- [9] Krzysztof Barański, Bogusława Karpińska and Anna Zdunik, Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts. *Int. Math. Res. Not. IMRN* 2009, no. 4, 615–624.
- [10] Walter Bergweiler, Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N. S.)* 29 (1993), no. 2, 151–188.
- [11] Walter Bergweiler and Janina Kotus, On the Hausdorff dimension of the escaping set of certain meromorphic functions. *Trans. Amer. Math. Soc.* 364 (2012), no. 10, 5369–5394.
- [12] Christopher J. Bishop, The order conjecture fails in  $\mathcal{S}$ . *J. Anal. Math.* 127 (2015), 283–302.

- [13] Christopher J. Bishop, Models for the Eremenko-Lyubich class. *J. Lond. Math. Soc.* (2) 92 (2015), no. 1, 202–221.
- [14] Christopher J. Bishop, Models for the Speiser class. *Proc. Lond. Math. Soc.* (3) 114 (2017), no. 5, 765–797.
- [15] Christopher J. Bishop, A transcendental Julia set of dimension 1. *Invent. Math.* 212 (2018), no. 2, 407–460.
- [16] Bodil Branner and John H. Hubbard, The iteration of cubic polynomials. Part II: patterns and parapatterns. *Acta Math.* 169 (1992), no. 3–4, 229–325.
- [17] P. Domínguez, Dynamics of transcendental meromorphic functions. *Ann. Acad. Sci. Fenn. Math.* 23 (1998), no. 1, 225–250.
- [18] P. Domínguez, A. Hernández and G. Sienna, Totally disconnected Julia set for different classes of meromorphic functions. *Conform. Geom. Dyn.* 18 (2014), 1–7.
- [19] Peter L. Duren, *Univalent Functions*. Springer-Verlag, New York, 1983.
- [20] Albert Edrei and Wolfgang H. J. Fuchs On the zeros of  $f(g(z))$  where  $f$  and  $g$  are entire functions. *J. Anal. Math.* 12 (1964), 243–255.
- [21] Adam Epstein and Lasse Rempe-Gillen, On invariance of order and the area property for finite-type entire functions. *Ann. Acad. Sci. Fenn. Math.* 40 (2015), no. 2, 573–599.
- [22] A. E. Eremenko, On the iteration of entire functions. In “Dynamical Systems and Ergodic Theory”. Banach Center Publications 23, Polish Scientific Publishers, Warsaw 1989, pp. 339–345.
- [23] A. Eremenko, Transcendental meromorphic functions with three singular values. *Illinois J. Math.* 48 (2004), no. 2, 701–709.
- [24] A. E. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier* 42 (1992), no. 4, 989–1020.
- [25] Kenneth Falconer, *Fractal Geometry. Mathematical Foundations and Applications*. John Wiley & Sons, Chichester, 1990.

- [26] Jane Hawkins, Stability of Cantor Julia sets in the space of iterated elliptic functions. In “Dynamical Systems and Random Processes”. *Contemp. Math.*, 736, Amer. Math. Soc., Providence, RI, 2019, pp. 69–95,
- [27] Jane Hawkins and Lorelei Koss, Connectivity properties of Julia sets of Weierstrass elliptic functions. *Topology Appl.* 152 (2005), no. 1–2, 107–137.
- [28] W. K. Hayman, *Meromorphic Functions*. Clarendon Press, Oxford, 1964.
- [29] Janina Kotus and Mariusz Urbański, Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. *Bull. London Math. Soc.* 35 (2003), no. 2, 269–275.
- [30] Janina Kotus and Mariusz Urbański, Fractal measures and ergodic theory of transcendental meromorphic functions. In “Transcendental Dynamics and Complex Analysis”. *London Math. Soc. Lect. Note Ser.* 348. Edited by P. J. Rippon and G. M. Stallard, Cambridge Univ. Press, Cambridge, 2008, pp. 251–316.
- [31] James K. Langley, Critical values of slowly growing meromorphic functions. *Comput. Methods Funct. Theory* 2 (2002), no. 2, 537–547.
- [32] Olli Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*. *Die Grundlehren der mathematischen Wissenschaften* 126. Springer-Verlag, New York, Heidelberg, 1973.
- [33] R. Daniel Mauldin and Mariusz Urbański, Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc.* (3) 73 (1996), no. 1, 105–154.
- [34] Volker Mayer, The size of the Julia set of meromorphic functions. *Math. Nachr.* 282 (2009), no. 8, 1189–1194.
- [35] Volker Mayer and Mariusz Urbański, The exact value of Hausdorff dimension of escaping sets of class B meromorphic functions. Preprint, arXiv: 2103.02254.

- [36] John Milnor, Local connectivity of Julia sets: expository lectures. In “The Mandelbrot Set, Theme and Variations”. London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, Cambridge, 2000, pp. 67–116.
- [37] John Milnor, Dynamics in One Complex Variable. Annals of Mathematics Studies 160. Princeton University Press, Princeton, NJ, 2006.
- [38] Michał Misiurewicz, On iterates of  $e^z$ . Ergodic Theory Dynam. Systems 1 (1981), no. 1, 103–106.
- [39] Lasse Rempe, Hyperbolic dimension and radial Julia sets of transcendental functions. Proc. Amer. Math. Soc. 137 (2009), no. 4, 1411–1420.
- [40] P. J. Rippon and G. M. Stallard, Iteration of a class of hyperbolic meromorphic functions. Proc. Amer. Math. Soc. 127 (1999), no. 11, 3251–3258.
- [41] David J. Sixsmith, Dynamics in the Eremenko-Lyubich class. Conform. Geom. Dyn. 22 (2018), 185–224.
- [42] G. M. Stallard, The Hausdorff dimension of Julia sets of meromorphic functions. J. London Math. Soc. (2) 49 (1994), no. 2, 281–295.
- [43] G. M. Stallard, The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions. II. Ergodic Theory Dynam. Systems 20 (2000), no. 3, 895–910.
- [44] G. M. Stallard, The Hausdorff dimension of Julia sets of entire functions. IV. J. London Math. Soc. (2) 61 (2000), no. 2, 471–488.
- [45] G. M. Stallard, Dimensions of Julia sets of transcendental meromorphic functions. In “Transcendental Dynamics and Complex Analysis”. London Math. Soc. Lect. Note Ser. 348. Edited by P. J. Rippon and G. M. Stallard, Cambridge Univ. Press, Cambridge, 2008, pp. 425–446.
- [46] Oswald Teichmüller, Einfache Beispiele zur Wertverteilungslehre. Deutsche Math. 7 (1944), 360–368; Gesammelte Abhandlungen – Collected Papers, Springer-Verlag, Berlin, Heidelberg, New York, 1982, pp. 728–736.

- [47] Jian-Hua Zheng, Dynamics of hyperbolic meromorphic functions. *Discrete Contin. Dyn. Syst.* 35 (2015), no. 5, 2273–2298.

W. Bergweiler: Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Straße 4, 24098 Kiel, Germany  
Email: [bergweiler@math.uni-kiel.de](mailto:bergweiler@math.uni-kiel.de)

W. Cui: Centre for Mathematical Sciences, Lund University, Box 118, 22 100 Lund, Sweden  
E-mail: [weiwei.cui@math.lth.se](mailto:weiwei.cui@math.lth.se)