Meromorphic functions with three radially distributed values

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Dedicated to the memory of Walter K. Hayman

Abstract

We consider transcendental meromorphic functions for which the zeros, 1-points and poles are distributed on three distinct rays. We show that such functions exist if and only if the rays are equally spaced. We also obtain a normal family analogue of this result.

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1 Introduction and results

Our starting point is the following result.

Theorem A. *There is no transcendental entire function for which all zeros lie on one ray and all* 1*-points lie on a different ray.*

This was proved by Biernacki [6, p. 533] and Milloux [17] for functions of finite order; see also [3]. The restriction on the order can be omitted by a later result of Edrei [9]. This result yields that if all zeros and 1-points of an entire function f lie on finitely many rays, then f has finite order.

The following normal family analogue of Theorem A was proved in [4, Theorem 1.1]. Here $\mathbb{D}$ denotes the unit disk.

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Theorem B. *Let L_0 and L_1 be two distinct rays emanating from the origin and let $\mathcal{F}$ be the family of all functions holomorphic in $\mathbb{D}$ for which all zeros lie on L_0 and all 1-points lie on L_1. Then $\mathcal{F}$ is normal in $\mathbb{D}\backslash\{0\}$.*

The purpose of this paper is to consider analogues of these results for meromorphic functions, with poles being distributed on some further ray. First we note that there exist meromorphic functions for which zeros, 1-points and poles lie on three distinct rays.

Example 1.1. Let Ai be the Airy function and

$$f(z) = e^{\pi i/3} \frac{\mathrm{Ai}(e^{2\pi i/3} z)}{\mathrm{Ai}(e^{-2\pi i/3} z)} \tag{1.1}$$

Then all zeros of f are on the ray $\{re^{i\pi/3} \colon r > 0\}$, all poles of f are on $\{re^{-i\pi/3} \colon r > 0\}$ and all 1-points of f are on the negative real axis.

We will verify at the beginning of Section 2 that the function f defined in Example 1.1 has the properties stated there.

We note that in Example 1.1 the rays are equally spaced. If the rays are not equally spaced, then we have analogues of Theorems A and B.

Theorem 1.1. *Let L_0, L_1 and L_∞ be three distinct rays emanating from the origin. If the rays are not equally spaced, then there is no transcendental meromorphic function for which all but finitely many zeros lie on L_0, all but finitely many 1-points lie on L_1 and all but finitely many poles lie on L_∞.*

Theorem 1.2. *Let L_0, L_1 and L_∞ be three distinct rays emanating from the origin and let $0 \leq r < R \leq \infty$. Let $\mathcal{F}$ be the family of all functions meromorphic in $\{z \in \mathbb{C} \colon r < |z| < R\}$ for which all zeros are on L_0, all 1-points are on L_1 and all poles are on L_∞.*

Then $\mathcal{F}$ is normal if and only if the rays are not equally spaced.

One can deduce Theorem A from Theorem B by considering the family $\{f(rz) \colon r > 0\}$. Given any transcendental entire function f, this family is not normal at some point in $\mathbb{C}\backslash\{0\}$. This approach does not suffice to deduce Theorem 1.1 from Theorem 1.2, since there are meromorphic functions f for which the family $\{f(rz) \colon r > 0\}$ is normal in $\mathbb{C} \setminus \{0\}$. Such functions are called *normal functions* or *Julia exceptional functions*. They were studied in detail by Ostrowski [21]. Thus, in order to prove Theorem 1.1, normal functions have to be considered separately; see Proposition 3.1 below.

The Airy function satisfies the differential equation $\mathrm{Ai}''(z) = z\,\mathrm{Ai}(z)$. This implies that the function f given by (1.1) satisfies the differential equation

$$S(f)(z) = -2z,$$

where

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f'''}{f'}\right)' - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

denotes the Schwarzian derivative.

The following result says that – in some sense – all meromorphic functions for which zeros, 1-points and poles are distributed on three rays are similar to the function of Example 1.1.

Theorem 1.3. *Let L^1, L^2 and L^3 be three equally spaced rays and let f be a meromorphic function. Then there exist distinct values a_1, a_2 and a_3 such that all but finitely many a_j-points are on L^j if and only if*

$$S(f)(z) = e^{i3\theta} z R(z^3), \tag{1.2}$$

where θ is the argument of one of the rays L^j and R is a real rational function satisfying $0 < R(\infty) < \infty$.

If L is a linear fractional transformation, then $S(L \circ f) = S(f)$. One may choose L such that a_1, a_2 and a_3 are mapped to 0, 1 and ∞.

The rational functions Q for which the equation $S(f) = Q$ has a meromorphic solution f have been classified by Elfving [10, Kapitel IV]; see also [7], [15, Theorem 6.7] and [16].

An example of a rational functions R with poles such that (1.2) has a solution f for which all (and not only all but finitely many) zeros, 1-points and poles are on the rays will be given in Remark 4.2.

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2 Proof of Theorem 1.2

As we will use Example 1.1 in one direction of the proof, we begin by verifying the properties of this example.

Verification of Example 1.1. The zeros of the Airy function are all negative [20, §9.9]. This implies that all zeros of f are on $\{re^{i\pi/3}\colon r>0\}$ and all poles of f are on $\{re^{-i\pi/3}\colon r>0\}$. By [20, Equation 9.2.12] we have

$$\operatorname{Ai}(z)+e^{-2\pi i/3}\operatorname{Ai}(e^{-2\pi i/3}z)+e^{2\pi i/3}\operatorname{Ai}(e^{2\pi i/3}z)=0.$$

This implies that

$$\begin{aligned} f(z)-1&=\frac{e^{\pi i/3}\operatorname{Ai}(e^{2\pi i/3}z)-\operatorname{Ai}(e^{-2\pi i/3}z)}{\operatorname{Ai}(e^{-2\pi i/3}z)}\\ &=e^{-\pi i/3}\frac{e^{2\pi i/3}\operatorname{Ai}(e^{2\pi i/3}z)+e^{-2\pi i/3}\operatorname{Ai}(e^{-2\pi i/3}z)}{\operatorname{Ai}(e^{-2\pi i/3}z)}\\ &=-e^{-\pi i/3}\frac{\operatorname{Ai}(z)}{\operatorname{Ai}(e^{-2\pi i/3}z)}. \end{aligned}$$

Hence the 1-points of f are all negative. □

Let $\overline{D}(a,r)$ denote the closed disk of radius r around a point a. The following result was proved in [4, Theorem 1.3] and [5, Proposition 1.1].

Lemma 2.1. *Let D be a domain and let L be a straight line which divides D into two subdomains D^+ and D^-. Let $\mathcal{F}$ be a family of functions holomorphic in D which do not have zeros in D and for which all 1-points lie on L.*

Suppose that $\mathcal{F}$ is not normal at $z_0\in D\cap L$ and let (f_k) be a sequence in $\mathcal{F}$ which does not have a subsequence converging in any neighborhood of z_0. Suppose that $(f_k|_{D^+})$ converges. Then either $(f_k|_{D^+})\to 0$ and $(f_k|_{D^-})\to\infty$ or $(f_k|_{D^+})\to\infty$ and $(f_k|_{D^-})\to 0$.

Let $r>0$ with $\overline{D}(z_0,r)\subset D$. Then for sufficiently large k there exists a 1-point a_k of f_k such that $a_k\to z_0$ and if M_k is the line orthogonal to L which intersects L at a_k, then $|f_k(z)|\neq 1$ for $z\in M_k\cap\overline{D}(z_0,r)\setminus\{a_k\}$.

Lemma 2.2. *Let D be a domain and let L be a straight line. Let $\xi\in D\setminus L$ and let K be a compact subset of D. Then there exists $C>0$ such that if $f\colon D\to\mathbb{C}$ is a holomorphic function satisfying $|f(\xi)|>2$ which has no zeros in D and for which all 1-points lie on L then $\log|f(z)|\leq C\log|f(\xi)|$ for all $z\in K$.*

Proof. Without loss of generality we may assume that $L=\mathbb{R}$ and $\operatorname{Im}\xi>0$. Suppose that the conclusion is not true. Then there exists a sequence (f_k) of functions holomorphic in D which satisfy the hypotheses of the lemma and a sequence (ζ_k) in K such that

$$\frac{\log|f_k(\zeta_k)|}{\log|f_k(\xi)|} \to \infty. \tag{2.1}$$

Since the f_k have no zeros and 1-points in $D\setminus\mathbb{R}$, the sequence (f_k) is normal in $D\setminus\mathbb{R}$.

Suppose first that the sequence (f_k) is normal in D. If $|f_k(\xi)| \not\to \infty$, then there exists a subsequence of (f_k) which tends to a limit function holomorphic in D. This contradicts (2.1). Thus $|f_k(\xi)| \to \infty$ and hence $f_k|_D \to \infty$. But then for large k the functions u_k given by

$$u_k(z) := \frac{\log|f_k(z)|}{\log|f_k(\xi)|} \tag{2.2}$$

are positive harmonic functions in some connected neighborhood of K, and a contradiction to (2.1) is now obtained from Harnack's inequality.

We may thus assume that (f_k) is not normal in D. In fact, with

$$D^+ := \{z \in D\colon\ \operatorname{Im} z > 0\} \quad \text{and} \quad D^- := \{z \in D\colon\ \operatorname{Im} z < 0\}$$

we may assume (f_k) converges in D^+ but that there exists $a \in D\cap\mathbb{R}$ such that no subsequence of (f_k) converges in any neighborhood of a. It follows from Lemma 2.1 that either $(f_k|_{D^+}) \to 0$ and $(f_k|_{D^-}) \to \infty$ or $(f_k|_{D^+}) \to \infty$ and $(f_k|_{D^-}) \to 0$. The first possibility is ruled out since we assumed that $\xi \in D^+$ and $|f_k(\xi)| > 2$. Thus

$$(f_k|_{D^+}) \to \infty \quad \text{and} \quad (f_k|_{D^-}) \to 0. \tag{2.3}$$

We may assume that $\zeta_k \to \zeta_0 \in K$. Harnack's inequality implies that the functions u_k given by (2.2) are bounded on any compact subset of D^+. It follows that $\zeta_0 \in \mathbb{R}$. Without loss of generality we assume that $\zeta_0 = 0$. Choose $\varepsilon > 0$ such that $\overline{D}(0,10\varepsilon) \subset D$. We may assume that $\overline{D}(0,10\varepsilon) \subset K$.

Put

$$K_\varepsilon^+ := \{z \in K\colon\ \operatorname{Im} z \geq \varepsilon\} \quad \text{and} \quad K_\varepsilon^- := \{z \in K\colon\ \operatorname{Im} z \leq -\varepsilon\}.$$

Then by Harnack's inequality the functions u_k are bounded on K_ε^+. This means that there exists $C_1 > 1$ such that

$$|f_k(z)| \leq C_1 |f_k(\xi)| \quad \text{for } z \in K_\varepsilon^+. \tag{2.4}$$

By (2.3) we also have $|f_k(z)| \geq 1$ for $z \in K_\varepsilon^+$ and

$$|f_k(z)| \leq 1 < C_1|f_k(\xi)| \quad \text{for } z \in K_\varepsilon^-, \tag{2.5}$$

provided k is large enough.

On the other hand, for large k we have $|f_k(\zeta_k)| > C_1|f_k(\xi)|$ and $|\zeta_k| < \varepsilon$. By the maximum principle, there exists a curve α_k connecting ζ_k with the circle $\{z\colon |z| = 5\varepsilon\}$ such that

$$|f_k(z)| \geq |f_k(\zeta_k)| > C_1|f_k(\xi)| > 1 \quad \text{for } z \in \alpha_k. \tag{2.6}$$

It follows from (2.4), (2.5) and (2.6) that

$$\alpha_k \subset \overline{D}(0, 5\varepsilon) \setminus (K_\varepsilon^+ \cap K_\varepsilon^-) = \{z\colon |z| \leq 5\varepsilon,\ |\operatorname{Im} z| < \varepsilon\}.$$

It is no loss of generality to assume that α_k connects ζ_k with a point on the right arc of the boundary of the latter set; that is, α_k connects ζ_k with the arc $\{z\colon |z| = 5\varepsilon,\ |\operatorname{Im} z| < \varepsilon,\ \operatorname{Re} z > 0\}$.

It also follows from Lemma 2.1 that for large k there exists 1-points a_k and b_k of f_k satisfying $\varepsilon < a_k < 2\varepsilon$ and $3\varepsilon < b_k < 4\varepsilon$ such that if M_k and N_k are the lines orthogonal to $\mathbb{R}$ which intersect $\mathbb{R}$ in a_k and b_k, respectively, then we have $|f_k(z)| \neq 1$ for $z \in M_k \cap \overline{D}(0, 10\varepsilon) \setminus \{a_k\}$ and $z \in N_k \cap \overline{D}(0, 10\varepsilon) \setminus \{b_k\}$. This implies that $|f_k(z)| > 1$ for $z \in (M_k \cup N_k) \cap \overline{D}(0, 10\varepsilon) \cap D^+$ and $|f_k(z)| < 1$ for $z \in (M_k \cup N_k) \cap \overline{D}(0, 10\varepsilon) \cap D^-$.

The curve α_k intersects both lines M_k and N_k. Let β_k be the subcurve of α_k which begins at the last intersection point of α_k with M_k and ends at the first intersection point of α_k with N_k. Note that by (2.6) the starting point u_k of β_k lies on $M_k \cap \{z\colon 0 < \operatorname{Im} z < \varepsilon\}$ while the end point v_k lies on $N_k \cap \{z\colon 0 < \operatorname{Im} z < \varepsilon\}$.

We consider the domain G_k bounded by the arc $\{z\colon |z| = 5\varepsilon,\ \operatorname{Im} z \geq \varepsilon\}$, the horizontal line segments $\{x + i\varepsilon\colon -\sqrt{24}\varepsilon \leq x \leq a_k\}$ and $\{x + i\varepsilon\colon b_k \leq x \leq \sqrt{24}\varepsilon\}$, the vertical line segments $\{a_k + iy\colon \operatorname{Im} u_k \leq y \leq \varepsilon\}$ and $\{b_k + iy\colon \operatorname{Im} v_k \leq y \leq \varepsilon\}$, and the curve β_k; see Figure 1.

Let $\gamma := \{x - i\varepsilon\colon 2\varepsilon \leq x \leq 3\varepsilon\}$ and let H be the domain bounded by the arc $\{z\colon |z| = 5\varepsilon,\ \operatorname{Im} z \geq \varepsilon\}$, the horizontal line segments $\{x + i\varepsilon\colon -\sqrt{24}\varepsilon \leq x \leq 2\varepsilon\}$, $\{x + i\varepsilon\colon 3\varepsilon \leq x \leq \sqrt{24}\varepsilon\}$ and γ, and the vertical line segments $\{2\varepsilon + iy\colon -\varepsilon \leq y \leq \varepsilon\}$ and $\{3\varepsilon + iy\colon -\varepsilon \leq y \leq \varepsilon\}$.

For a bounded domain G and a compact subset A of ∂G, let $\omega(z, A, G)$ be the harmonic measure of A at a point $z \in G$. A standard estimate (see [5,

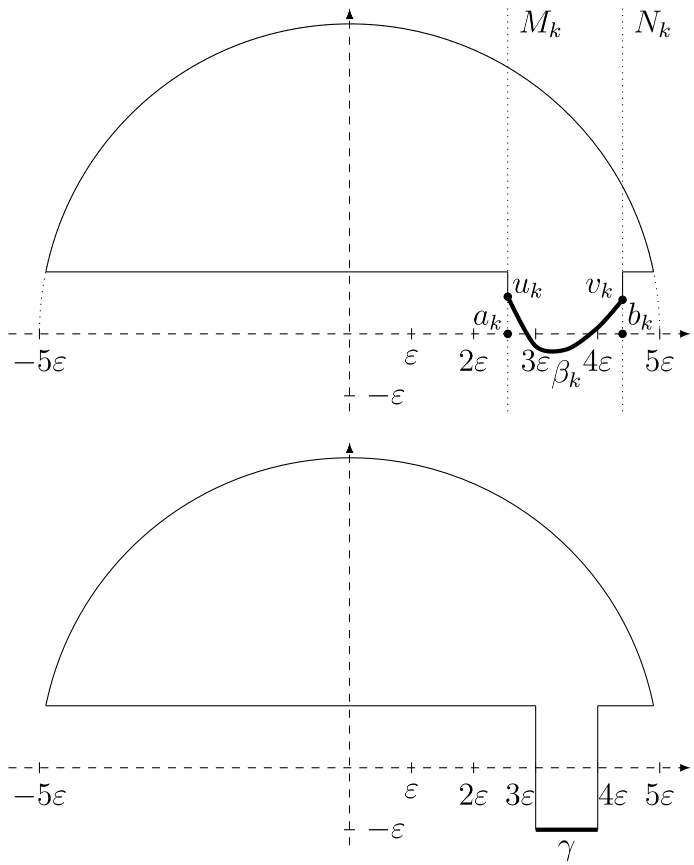

Figure 1: The domains G_k (top) and H (bottom).

Lemma 2.6]) yields that

$$\omega(2i\varepsilon, \gamma, H) \leq \omega(2i\varepsilon, \beta_k, G_k). \tag{2.7}$$

By the choice of G_k we have $|f_k(z)| \geq 1$ for $z \in \partial G$ and large k. Since f has no zeros, this implies that $|f_k(z)| > 1$ for all $z \in G$. Noting that $\beta_k \subset \alpha_k$ we deduce from (2.6) and a standard harmonic measure estimate that

$$\log |f_k(z)| \geq \omega(z, \beta_k, G_k) \log |f_k(\zeta_k)|.$$

Combining this with (2.4) and (2.7) we find that

$$\begin{aligned} \log |f_k(\xi)| + \log C_1 \geq \log |f_k(2\varepsilon i)| &\geq \omega(2\varepsilon i, \beta_k, G_k) \log |f_k(\zeta_k)| \\ &\geq \omega(2\varepsilon i, \gamma, H) \log |f_k(\zeta_k)|. \end{aligned}$$

This contradicts (2.1). □

Proof of Theorem 1.2. Suppose first the the rays L_0, L_1 and L_∞ are equally spaced. Let f be the function of Example 1.1. Then there exists $\theta \in \mathbb{R}$ such that either $g(z) := f(e^{i\theta}z)$ or $g(z) := 1/f(e^{i\theta}z)$ defines a meromorphic function g for which all zeros are on L_0, all 1-points are on L_1 and all poles are on L_∞. Thus for each $k \in \mathbb{N}$ the function g_k defined by $g_k(z) = g(kz)$ is contained in $\mathcal{F}$. It is easy to see that $\{g_k \colon k \in \mathbb{N}\}$ is not normal on any point on one of the rays L_0, L_1 and L_∞. Thus $\mathcal{F}$ is not normal there.

Suppose now that $\mathcal{F}$ is not normal. The rays L_0, L_1 and L_∞ divide $A := \{z \colon r < |z| < R\}$ into three sectors which we denote by S_0, S_1 and S_∞. Here S_0 is the sector "opposite" to L_0; that is, the sector bounded by L_1 and L_∞. Similarly, S_1 and S_∞ are the sectors opposite to L_1 and L_∞, respectively.

By Montel's theorem, $\mathcal{F}$ is normal in $S_0 \cup S_1 \cup S_\infty = A \setminus (L_0 \cup L_1 \cup L_\infty)$. Thus our assumption that $\mathcal{F}$ is not normal implies that $\mathcal{F}$ is not normal at some point in $A \cap (L_0 \cup L_1 \cup L_\infty)$. Without loss of generality we may assume that $\mathcal{F}$ is not normal at some point $z_1 \in A \cap L_1$. Let (f_k) be a sequence in $\mathcal{F}$ which does not have a subsequence converging in any neighborhood of z_1. Since $\mathcal{F}$ is normal in S_0 and S_∞, we may assume that (f_k) converges in S_0 and S_∞. Lemma 2.1 implies that either $(f_k|_{S_0}) \to 0$ and $(f_k|_{S_\infty}) \to \infty$ or $(f_k|_{S_0}) \to \infty$ and $(f_k|_{S_\infty}) \to 0$.

This implies that (f_k) is not normal at some point of $A \cap (L_0 \cup L_\infty)$. Assuming without loss of generality that (f_k) is not normal at some point $z_0 \in A \cap L_0$, we deduce from Lemma 2.1, applied to $1 - f_k$ instead of f_k, that either $(f_k|_{S_1}) \to 1$ and $(f_k|_{S_\infty}) \to \infty$ or $(f_k|_{S_1}) \to \infty$ and $(f_k|_{S_\infty}) \to 1$. The latter possibility contradicts our previous finding that either $(f_k|_{S_\infty}) \to \infty$ or $(f_k|_{S_\infty}) \to 0$. Altogether we thus have $(f_k|_{S_0}) \to 0$, $(f_k|_{S_1}) \to 1$ and $(f_k|_{S_\infty}) \to \infty$; that is,

$$f_k(z) \to \begin{cases} 0 & \text{for } z \in S_0, \\ 1 & \text{for } z \in S_1, \\ \infty & \text{for } z \in S_\infty. \end{cases} \tag{2.8}$$

Let now $\xi \in S_\infty$. Then $|f_k(\xi)| \to \infty$ as $k \to \infty$. Hence we may assume that $|f_k(\xi)| > 2$ for all $k \in \mathbb{N}$. Lemma 2.2 yields that the functions u_k defined by

$$u_k(z) := \frac{\log |f_k(z)|}{\log |f_k(\xi)|} \tag{2.9}$$

are locally bounded in $T_1 := S_0 \cup S_\infty \cup (L_1 \cap A)$. Note that the u_k are also harmonic in T_1. Passing to a subsequence if necessary we may thus assume that there exists a function u harmonic in T_1 such that

$$u_k(z) \to u(z) \quad \text{for } z \in T_1. \tag{2.10}$$

Similarly, put

$$v_k(z) := \frac{\log|f_k(z)-1|}{\log|f_k(\xi)|}. \tag{2.11}$$

Then the v_k are harmonic in $T_0 := S_1 \cup S_\infty \cup (L_0 \cap A)$. Since $\log|f_k(\xi)| \sim \log|f_k(\xi)-1|$ we see that the v_k are locally bounded in T_0. Passing to a subsequence if necessary we thus find that there exists a function v harmonic in T_0 such that

$$v_k(z) \to v(z) \quad \text{for } z \in T_0. \tag{2.12}$$

Moreover,

$$u(z) = v(z) \quad \text{for } z \in T_0 \cap T_1 = S_\infty. \tag{2.13}$$

We now consider the functions w_k defined by

$$w_k(z) := u_k(z) - v_k(z) = \frac{1}{\log|f_k(\xi)|} \cdot \log\left|\frac{f_k(z)}{f_k(z)-1}\right|. \tag{2.14}$$

These functions w_k are harmonic in $T_\infty := S_0 \cup S_1 \cup (L_\infty \cap A)$. We have $w_k \to u$ in S_0 and $w_k \to -v$ in S_1. It follows that there is a function w harmonic in T_∞ such that

$$w_k(z) \to w(z) \quad \text{for } z \in T_\infty \tag{2.15}$$

and

$$w(z) = \begin{cases} u(z) & \text{for } z \in S_0, \\ -v(z) & \text{for } z \in S_1. \end{cases} \tag{2.16}$$

Let now S'_∞ and S''_∞ be the two preimages of S_∞ under $z \mapsto z^2$. Analogously we define S'_0, S''_0, S'_0 and S''_0. We may assume that these sectors are arranged in the cyclic order $S'_\infty, S'_0, S'_1, S''_\infty, S''_0, S''_1$.

We now define a function $h\colon A \to \mathbb{R}$ as follows:

$$h(z) = \begin{cases} v(z^2) = u(z^2) & \text{for } z \in S'_\infty, \\ u(z^2) = w(z^2) & \text{for } z \in S'_0, \\ w(z^2) = -v(z^2) & \text{for } z \in S'_1, \\ -v(z^2) = -u(z^2) & \text{for } z \in S''_\infty, \\ -u(z^2) = -w(z^2) & \text{for } z \in S''_0, \\ -w(z^2) = v(z^2) & \text{for } z \in S''_1. \end{cases} \tag{2.17}$$

Here the two expressions used in the definition are equal by (2.13) and (2.16).

It follows from (2.8) that $u(z) \geq 0$ for $z \in S_\infty$ while $u(z) \leq 0$ for $z \in S_0$. Since $u(\xi) = 1$ we see that u is non-constant and thus $u(z) > 0$ for $z \in S_\infty$ while $u(z) < 0$ for $z \in S_0$. Analogously we see that $v(z) > 0$ for $z \in S_\infty$ while $v(z) < 0$ for $z \in S_1$. This implies that

$$h(z) \begin{cases} > 0 & \text{for } z \in S'_\infty \cup S'_1 \cup S''_0, \\ < 0 & \text{for } z \in S'_0 \cup S''_\infty \cup S''_1. \end{cases} \tag{2.18}$$

Let L be any ray separating two of the sectors S'_∞, S'_0, S'_1, S''_∞, S''_0 and S''_1. Thus L is one of the preimages of L_0, L_1 or L_∞ under $z \mapsto z^2$. Let σ_L be the reflection in L. The reflection principle for harmonic functions implies that $h(\sigma_L(z)) = -h(z)$. This implies that all sectors S'_∞, S'_0, S'_1, S''_∞, S''_0 and S''_1 have the same opening angle. It follows that S_0, S_1 or S_∞ all have opening angle $2\pi/3$. Thus the rays L_0, L_1 or L_∞ are equally spaced. □

3 Proof of Theorem 1.1

As mentioned in the introduction, normal functions cannot be dealt with by Theorem 1.2. The results of Ostrowski [21] already mentioned imply in particular that normal functions have order 0. The following result actually covers functions of order less than 1.

Proposition 3.1. *Let L_0, L_1 and L_∞ be three distinct rays emanating from the origin. Then there is no transcendental meromorphic function of order less than 1 for which all but finitely many zeros lie on L_0, all but finitely many 1-points lie on L_1 and all but finitely many poles lie on L_∞.*

To prove this proposition, we will use the following lemma.

Lemma 3.1. *Let $a, b, p, q \in \mathbb{C} \setminus \{0\}$ and suppose that 1, p and q are distinct. Then there exist $\delta \in (0, \pi)$ such that for arbitrarily large $n \in \mathbb{N}$ the points 1, ap^n and bq^n lie in a sector opening angle δ.*

To prove Lemma 3.1, we will use several other lemmas.

Lemma 3.2. *Let $A, B \in \partial\mathbb{D}$ such that $\operatorname{Re}(A+B) > 0$. Then 1, A and B lie on an arc of $\partial\mathbb{D}$ of length at most $\arccos(\operatorname{Re}(A+B)-1)$.*

Proof. The hypothesis implies that $A \neq -1$ and $B \neq -1$. We may assume that $\operatorname{Im} B \geq 0$, since otherwise we can pass to the complex conjugates of A and B. We may thus write $A = e^{i\alpha}$ and $B = e^{i\beta}$ with $\alpha \in (-\pi, \pi)$ and $\beta \in [0, \pi)$. Then

$$\operatorname{Re}(A+B) = \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\beta-\alpha}{2}. \tag{3.1}$$

Suppose first that $\alpha < 0$. Then $-\pi < \alpha+\beta < \pi$ and thus $\cos((\alpha+\beta)/2) > 0$. Hence $\cos((\beta-\alpha)/2) > 0$ so that $0 \leq |\alpha+\beta| < \beta-\alpha < \pi$. Hence

$$\operatorname{Re}(A+B) \geq 2\cos^2\frac{\beta-\alpha}{2} = 1 + \cos(\beta-\alpha).$$

As the arc on $\partial\mathbb{D}$ which connects A with B and contains 1 has length $\beta-\alpha$, the conclusion follows.

Suppose now that $\alpha \geq 0$. We may suppose that $\alpha \leq \beta$. Then there is an arc on $\partial\mathbb{D}$ of length β which contains 1, A and B. Now (3.1) yields that

$$\operatorname{Re}(A+B) \geq 2\cos^2\frac{\alpha+\beta}{2} = 1 + \cos(\alpha+\beta).$$

Thus $\beta \leq \alpha+\beta \leq \arccos(\operatorname{Re}(A+B)-1)$, and again the conclusion follows. $\square$

For a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$, we define the *average*

$$\operatorname{av}((z_n)_{n\in\mathbb{N}}) := \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} z_k,$$

provided that the limit exists. For $c \in \mathbb{C}$ and $\xi \in \partial\mathbb{D} \setminus \{1\}$ we have

$$\operatorname{av}((c\xi^n)_{n\in\mathbb{N}}) = \lim_{n\to\infty}\frac{1}{n}c\sum_{k=1}^{n}\xi^n = \lim_{n\to\infty}\frac{1}{n}c\,\frac{\xi-\xi^{n+1}}{1-\xi} = 0.$$

Taking the real part yields for $c = e^{i\gamma}$ and $\xi = e^{i\tau}$ that

$$\operatorname{av}((\cos(\gamma + n\tau))_{n\in\mathbb{N}}) = \begin{cases} 0 & \text{if } \tau \not\equiv 0 \pmod{2\pi}, \\ \cos\gamma & \text{if } \tau \equiv 0 \pmod{2\pi}. \end{cases} \tag{3.2}$$

We will use the following lemma.

Lemma 3.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded real sequence satisfying* $\operatorname{av}((x_n)) = 0$. *If* $\operatorname{av}((x_n^2))$ *exists, then*

$$\limsup_{n\to\infty} x_n \cdot \limsup_{n\to\infty} |x_n| \geq \operatorname{av}((x_n^2)).$$

Proof. Let $\alpha, \beta > 0$ and suppose that $x_n \leq \alpha$ and $|x_n| \leq \beta$ for all large $n \in \mathbb{N}$, say for $n \geq N$. Then $(x_n - \alpha)^2 \leq -(\alpha+\beta)(x_n - \alpha)$ for all $n \geq N$ and thus

$$\frac{1}{n}\sum_{k=N}^{n} x_k^2 - 2\alpha\frac{1}{n}\sum_{k=N}^{n} x_k + \alpha^2\frac{n-N+1}{n} = \frac{1}{n}\sum_{k=N}^{n}(x_k - \alpha)^2$$
$$\leq -(\alpha+\beta)\frac{1}{n}\sum_{k=N}^{n}(x_k - \alpha) = (\alpha+\beta)\alpha\frac{n-N+1}{n} - (\alpha+\beta)\frac{1}{n}\sum_{k=N}^{n} x_k.$$

It follows that $\operatorname{av}((x_n^2)) + \alpha^2 \leq (\alpha+\beta)\alpha$ and hence that

$$\operatorname{av}((x_n^2)) \leq \alpha\beta,$$

from which the conclusion follows. □

Proof of Lemma 3.1. Since the conclusion depends only on the arguments of a, b, p and q, we may assume that $|a| = |b| = |p| = |q| = 1$. We write $a = e^{i\alpha}$, $b = e^{i\beta}$, $p = e^{i\phi}$ and $q = e^{i\psi}$, with $\alpha, \beta, \phi, \psi \in (-\pi, \pi]$. Since 1, p and q are distinct we have

$$\phi \neq 0, \quad \psi \neq 0 \quad \text{and} \quad \phi \neq \psi. \tag{3.3}$$

We will apply Lemma 3.3 with

$$x_n := \operatorname{Re}(ap^n + bq^n) = \cos(\alpha + n\phi) + \cos(\beta + n\psi). \tag{3.4}$$

It follows from (3.2) that $\operatorname{av}((x_n)) = 0$. We have

$$x_n^2 = \cos^2(\alpha + n\phi) + \cos^2(\beta + n\psi) + 2\cos(\alpha + n\phi)\cos(\beta + n\psi)$$
$$= 1 + \frac{1}{2}\cos(2\alpha + 2n\phi) + \frac{1}{2}\cos(2\beta + 2n\psi) + \cos(\alpha + \beta + n(\phi + \psi))$$
$$+ \cos(\alpha - \beta + n(\phi - \psi))$$

Suppose first that $2\phi \not\equiv 0 \pmod{2\pi}$ and $2\psi \not\equiv 0 \pmod{2\pi}$. Equivalently, $\phi \neq \pi$ and $\psi \neq \pi$. It follows from (3.2) and (3.3) that

$$\mathrm{av}((x_n^2)) = \begin{cases} 1 & \text{if } \phi \neq -\psi, \\ 1 + \cos(\alpha + \beta) & \text{if } \phi = -\psi. \end{cases}$$

Thus $\mathrm{av}((x_n^2)) > 0$ unless $\phi = -\psi$ and $\alpha + \beta \equiv \pi \pmod{2\pi}$. Postponing this exceptional case, and noting that $\limsup_{n\to\infty} |x_n| \leq 2$ by (3.4), we deduce from Lemma 3.3 that there exist arbitrarily large $n \in \mathbb{N}$ such that $x_n \geq \mathrm{av}((x_n^2))/4$. Lemma 3.2 implies that for such n the points 1, ap^n and bq^n are contained in an arc of length $\arccos(\mathrm{av}((x_n^2))/4 - 1)$. In this case we may thus take $\delta = \arccos(\mathrm{av}((x_n^2))/4 - 1)$.

Suppose now that $2\phi \equiv 0 \pmod{2\pi}$. Then $\phi = \pi$ and thus $\phi \neq -\psi$. Hence

$$\mathrm{av}((x_n^2)) = 1 + \frac{1}{2}\cos(2\alpha) > 0,$$

and the conclusion follows as before. The case that $2\psi \equiv 0 \pmod{2\pi}$ and thus $\psi = \pi$ is analogous.

It remains to consider the case that $\phi = -\psi$ and $\alpha + \beta \equiv \pi \pmod{2\pi}$. Then $q = \overline{p}$ and $b = -\overline{a}$. Thus ap^n and bq^n are symmetric with respect to the imaginary axis so that $\mathrm{Im}(ap^n)$ and $\mathrm{Im}(bq^n)$ have the same sign. If δ with the properties claimed does not exist, we thus must have

$$\min\{|ap^n + 1|, |bq^n + 1|\} = \min\{|ap^n + 1|, |ap^n - 1|\} \to 0.$$

Thus the only accumulation points of the sequence (ap^n) are ± 1. This implies that $p = \pm 1$ and $q = \mp 1$, contradicting the hypothesis that 1, p and q are distinct. □

Lemma 3.4. *Let F, G and H be transcendental entire functions for which the arguments of the Taylor coefficients tend to* 0*. Let $p, q, r \in \partial\mathbb{D}$ be distinct. Then $F(pz)$, $G(qz)$ and $H(rz)$ are linearly independent.*

Proof. Let

$$F(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \quad G(z) = \sum_{n=0}^{\infty} \beta_n z^n \quad \text{and} \quad H(z) = \sum_{n=0}^{\infty} \gamma_n z^n.$$

The hypothesis says that

$$\arg \alpha_n \to 0, \quad \arg \beta_n \to 0 \quad \text{and} \quad \arg \gamma_n \to 0 \tag{3.5}$$

as $n \to \infty$. Suppose now that

$$aF(pz) + bG(qz) + cH(rz) = 0,$$

with $a, b, c \in \mathbb{C}$. If $c = 0$, then we easily obtain $a = b = 0$. Thus suppose that $c \neq 0$. Without loss of generality we may assume that $c = 1$. We may also assume that $r = 1$. It follows that

$$\alpha_n a p^n + \beta_n b q^n + \gamma_n = 0 \tag{3.6}$$

for all $n \geq 0$. Lemma 3.1 implies that there exists arbitrarily large n such that the arguments of ap^n, bq^n and 1 lie in an interval of length at most δ. It thus follows from (3.5) that the arguments of $\alpha_n a p^n$, $\beta_n b q^n$ and γ_n lie in an interval of length less than π. This contradicts (3.6). □

Lemma 3.5. *Let F be an entire function of the form*

$$F(z) = P(z) \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right),$$

where (x_k) is a sequence of positive numbers tending to ∞ and where P is a polynomial with positive leading coefficient. Then the arguments of the coefficients of F tend to 0

Proof. Let

$$\prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right) = \sum_{n=0}^{\infty} a_n z^n, \quad P(z) = \sum_{n=0}^{d} b_n z^n \quad \text{and} \quad F(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then

$$c_n = \sum_{k=0}^{d} b_k a_{n-k}. \tag{3.7}$$

It is well-known and easy to prove that $a_n > 0$ and $a_n^2 > a_{n-1} a_{n+1}$ for all $n \in \mathbb{N}$. (More generally, the sequence (a_n) is totally positive; see [1].) Thus a_{n+1}/a_n is decreasing. Since F is entire, this implies that $a_{n+1}/a_n \to 0$.

Dividing (3.7) by a_{n-d} we find that

$$\frac{c_n}{a_{n-d}} = b_d + \sum_{k=0}^{d-1} b_k \frac{a_{n-k}}{a_{n-d}} \to b_d.$$

as $n \to \infty$. Since $a_{n-d} > 0$ and $b_d > 0$ we conclude that $\arg c_n \to 0$. □

Proof of Proposition 3.1. Let f be a transcendental meromorphic function of order less than 1 for which all but finitely many zeros lie on L_0, all but finitely many 1-points lie on L_1 and all but finitely many poles lie on L_∞. Without loss of generality we may assume that $f(0) \in \mathbb{C} \setminus \{0\}$. Let Π_0, Π_1 and Π_∞ be the canonical products of the zeros, 1-points and poles. Then

$$f = f(0)\frac{\Pi_0}{\Pi_\infty} \quad \text{and} \quad f - 1 = C\frac{\Pi_1}{\Pi_\infty}$$

for some constant C. It follows that

$$f(0)\Pi_0 = C\Pi_1 + \Pi_\infty. \tag{3.8}$$

Let $-\overline{p}$, $-\overline{q}$ and $-\overline{r}$ be the points where the rays L_0, L_1 and L_∞ intersect $\partial\mathbb{D}$. Then Π_0 can be written in the form $\Pi_0(z) = aF(pz)$ where F satisfies the hypothesis of Lemma 3.5 and $a \in \mathbb{C} \setminus \{0\}$. Similarly, $\Pi_1(z) = bG(pz)$ and $\Pi_\infty(z) = cH(pz)$ for entire functions G and H satisfying the hypothesis of Lemma 3.5, and $b, c \in \mathbb{C} \setminus \{0\}$. Equation (3.8) says that $F(pz)$, $G(qz)$ and $H(rz)$ are linearly dependent. This contradicts Lemma 3.4. □

Proof of Theorem 1.1. Let f be a transcendental meromorphic function for which all but finitely many zeros lie on L_0, all but finitely many 1-points lie on L_1 and all but finitely many poles lie on L_∞. Proposition (3.1) implies that f has order at least 1. The results of Ostrowski [21] already quoted yield that the family $\{f(rz) \colon r > 0\}$ is not normal in $\mathbb{C} \setminus \{0\}$. The conclusion now follows from Theorem 1.1. □

4 Proof of Theorem 1.3

We will need a version of Proposition 3.1 for functions which are meromorphic in $\mathbb{C} \setminus \{0\}$. We note that the concept of the order of a meromorphic function makes sense also for functions defined in a neighborhood of an isolated singularity. If f is meromorphic in $\mathbb{C} \setminus \{0\}$ and has order less than 1 near both singularities 0 and ∞, and if (a_j) is the sequence of zeros of f, then f can be written in the form

$$f(z) = c \prod_{|a_j| \le 1} \left(1 - \frac{z}{a_j}\right) \cdot \prod_{|a_j| < 1} \left(1 - \frac{a_j}{z}\right)$$

with a constant c. Replacing Π_0, Π_1 and Π_∞ in the proof of Proposition 3.1 by such products, and taking Laurent series instead of Taylor series in Lemma 3.4, we obtain the following result.

Proposition 4.1. *Let L_0, L_1 and L_∞ be three distinct rays emanating from the origin. Let f be meromorphic in $\mathbb{C} \setminus \{0\}$ and suppose that f has order less than 1 near both singularities 0 and ∞. Suppose also that all zeros lie on L_0, all 1-points lie on L_1 and all poles lie on L_∞. Then f is rational.*

Let $g\colon [r_0, \infty) \to \mathbb{R}$ be a positive increasing function and $\lambda \geq 0$. A sequence (r_k) tending to ∞ is called a sequence of Pólya peaks (of the first kind) of order λ for g if given $\varepsilon > 0$, we have

$$g(tr_k) \leq (1+\varepsilon)t^\lambda g(r_k) \quad \text{for } \varepsilon \leq t \leq \frac{1}{\varepsilon} \tag{4.1}$$

for all large k. If instead of (4.1) we have

$$g(tr_k) \geq (1-\varepsilon)t^\lambda g(r_k) \quad \text{for } \varepsilon \leq t \leq \frac{1}{\varepsilon}$$

for all large k, then (r_k) is called a sequence of Pólya peaks of the second kind (of order λ for g).

Put

$$\rho^* := \sup \left\{ p \in \mathbb{R} \colon \limsup_{r,t\to\infty} \frac{g(tr)}{t^p g(r)} = \infty \right\}$$

and

$$\rho_* := \inf \left\{ p \in \mathbb{R} \colon \liminf_{r,t\to\infty} \frac{g(tr)}{t^p g(r)} = 0 \right\}. \tag{4.2}$$

Then

$$0 \leq \rho_* \leq \liminf_{r\to\infty} \frac{\log g(r)}{\log r} \leq \limsup_{r\to\infty} \frac{\log g(r)}{\log r} \leq \rho^* \leq \infty. \tag{4.3}$$

The upper and lower limit in (4.3) are called the order and lower order of g. For a meromorphic function f the order and lower order are obtained by taking $g(r)$ as the Nevanlinna characteristic $T(r, f)$.

The following result is due to Drasin and Shea [8].

Lemma 4.1. *Let $g\colon [r_0, \infty) \to \mathbb{R}$ be a positive increasing function and $\lambda \geq 0$. Then the following are equivalent:*

(*a*) $\rho_* \leq \lambda \leq \rho^*$.

(*b*) *g has Pólya peaks of the first kind of order* λ.

(*c*) *g has Pólya peaks of the second kind of order* λ.

We will also use the following standard result about positive harmonic functions.

Lemma 4.2. *Let u be a positive harmonic function in the right half-plane which extends continuously to $i\mathbb{R}\setminus\{0\}$, with $u(iy) = 0$ for $y \in \mathbb{R}\setminus\{0\}$. Then u has the form $u(z) = \operatorname{Re}(az + b/z)$ with $a, b \geq 0$.*

To prove this result, we note that [2, Theorem 7.26] yields that u has the form $a \operatorname{Re} z + P(z)$ where P is a Poisson integral for the right half-plane. Applying [2, Theorem 7.19] to $u(z) - a \operatorname{Re} z$ shows that P has the form $P(z) = \operatorname{Re}(b/z)$.

Proposition 4.1 and Lemmas 4.1 and 4.2 will be used to prove that the Schwarzian $S(f)$ is rational. In order to prove that $S(f)$ is not only rational, but has the form (1.2), we need results of Elfving [10] and Nevanlinna [18] concerning meromorphic functions with rational Schwarzian derivative. These results were proved by Nevanlinna for the case of a polynomial Schwarzian derivative and extended to rational Schwarzian derivatives by Elfving.

The first result we need is the following.

Lemma 4.3. *Let Q be a rational function satisfying $Q(z) \sim az^d$ as $z \to \infty$, with $d \in \mathbb{N}$ and $a \in \mathbb{C}\setminus\{0\}$. Let f be a meromorphic function satisfying*

$$S(f) = Q. \tag{4.4}$$

Then f has order $(d+2)/2$.

We will see that in our case the order of f is $3/2$ so that $d = 1$. Thus $Q(z) \sim az$ as $z \to \infty$. It is no loss of generality to assume that $a < 0$. The asymptotics of f are then described by the following result.

Lemma 4.4. *For $j \in \{1, 2, 3\}$, let $L^j = \{re^{i(2j-1)\pi/3} \colon r > 0\}$. The rays L^j divide $\mathbb{C}$ into three sectors. Let V_j be the sector opposite to L^j.*

Let $c > 0$ and let Q be a rational function satisfying

$$Q(z) \sim -cz \quad \text{as } z \to \infty.$$

Let f be a meromorphic function satisfying (4.4).

Then there exist distinct values $a_1, a_2, a_3 \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ *such that* $f(z) \to a_j$ *as* $|z| \to \infty$ *in any closed subsector of* V_j. *These values* a_j *are logarithmic singularities, and* f *has no other asymptotic values.*

For each $j \in \{1, 2, 3\}$, *the function* f *has infinitely any* a_j*-points, and given* $\varepsilon > 0$, *all but finitely many* a_j*-points are contained in the sector of opening angle* ε *bisected by* L^j.

Moreover, a meromorphic function F *satisfies* $S(F) = Q$ *if and only if* F *is of the form* $F = L \circ f$ *with a linear fractional transformation* L.

Replacing f by $L \circ f$ with a linear fractional transformation L we can replace the values a_1, a_2 and a_3 by three other distinct values, in particular by the values 0, 1 and ∞.

The following result is due to Gundersen [12, Theorem 3]. Here a meromorphic function f is called real if $f(x) \in \mathbb{R} \cup \{\infty\}$ for all $x \in \mathbb{R}$. Otherwise it is called nonreal.

Lemma 4.5. *Let* A *be a nonreal polynomial of degree* n, *put*

$$F(z) = \frac{A(z) - \overline{A(\overline{z})}}{2i},$$

and let p *denote the number of distinct real zeros of* F. *Let* w *be a nontrivial solution of*

$$w'' + Aw = 0. \tag{4.5}$$

Then the number k *of real zeros of* w *is finite and we have* $k \leq p + 1$. *In particular,* $k \leq n + 1$.

If A is a polynomial, then every solution w of (4.5) is entire. It follows from Lemma 4.5 that if there is a solution of (4.5) which has infinitely many real zeros, then A is real.

If w_1 and w_2 are linearly independent solutions of (4.5), then $f := w_1/w_2$ satisfies

$$S(f) = 2A. \tag{4.6}$$

Conversely, every solution f of (4.6) is a quotient of two linearly independent solutions of (4.5). Thus we find that if f satisfies (4.6) for some polynomial A and if f has infinitely many real zeros, then A is real. Since $S(L \circ f) = S(f)$ for every linear fractional transformation L we see that if a meromorphic function f satisfying (4.6) has infinitely many real a-points for some $a \in \widehat{\mathbb{C}}$, then A is real.

It turns out that this remains valid for rational functions A.

Lemma 4.6. *Let Q be a rational function and let f be a meromorphic functions satisfying $S(f) = Q$. If f has infinitely many real a-points for some $a \in \widehat{\mathbb{C}}$, then Q is real.*

As explained above, this result follows from Lemma 4.5 if Q is a polynomial. However, the proof extends to the case that Q is rational. We note that in order to prove Lemma 4.6 for rational Q it does not suffice to extend Lemma 4.5 to the case that A is rational and w is meromorphic, since for rational A the solutions of (4.5) may be multi-valued, but the quotient of two multi-valued solutions may be single-valued. However, the proof of Lemma 4.5 given in [12] also extends to multi-valued functions.

The proof of the following lemma uses Lommel's method to prove that the zeros of Bessel functions are real; see [22, p. 482].

Lemma 4.7. *Let $r > 0$, $\gamma > -2$ and $0 < \alpha < \pi$ with $(2+\gamma)\alpha < \pi$. Let u and A be holomorphic in $S := \{z \colon |z| > r, |\arg z| < \alpha\}$ and suppose that $u'' + Au = 0$. Suppose also that both u and A are real on the real axis and that there exists $c > 0$ such that*

$$A(z) \sim cz^\gamma \quad \text{as } z \to \infty,\ z \in S. \tag{4.7}$$

Then there exists $x_1 > r$ such that all zeros of u in $\{z \colon |\arg(z - x_1)| < \alpha\}$ are real.

Proof. A classical result of Kneser [14, no. 6] implies that u has arbitrarily large positive zeros. (Kneser's result says that this is the case if there exists $\delta > 0$ such that $x^2 A(x) \geq 1/4 + \delta$ for all large x.)

It follows from (4.7) that $\arg A(z) = \gamma \arg z + o(1)$ as $z \to \infty$. Since $\arg A(x) = \gamma \arg x = 0$ for $x > r$ this actually implies that

$$\arg A(z) = (\gamma + o(1) \arg z \quad \text{as } z \to \infty,\ z \in S. \tag{4.8}$$

If x_1 is large and $|\arg z| < \alpha$, then $|x_1 + z|$ is also large. Thus (4.8) yields that

$$\arg\big(z^2 A(z + x_1)\big) = 2 \arg z + (\gamma + o(1) \arg(z + x_1)$$

as $x_1 \to \infty$. Since $|\arg(z + x_1)| < |\arg z|$, it now follows from the hypotheses $\gamma > -2$ and $(2+\gamma)\alpha < \pi$ that

$$\operatorname{Im}\big(z^2 A(z + x_1)\big) > 0 \quad \text{for } z \in S \text{ with } \operatorname{Im} z > 0, \tag{4.9}$$

provided x_1 is sufficiently large.

Put $v(z) := u(x_1+z)$ and $B(z) := A(x_1+z)$, with a large zero x_1 of u. Then $v''+Bv=0$ and $v(0)=0$. Let $a,b\in S-x_1=\{z-x_1\colon z\in S\}$. Then

$$\begin{aligned}\frac{d}{dt}\left(a\,v'(at)v(bt)-b\,v(at)v'(bt)\right)&=a^2v''(at)v(bt)-b^2v(at)v''(bt)\\&=\big(a^2B(at)-b^2B(bt)\big)\,v(at)v(bt).\end{aligned}\tag{4.10}$$

Let now $\xi\in S_1:=\{z\colon|\arg(z-x_1)|<\alpha\}$ be a non-real zero of u. Then $\overline{\xi}$ is also a zero of u. We may assume that $\operatorname{Im}\xi>0$. With $a:=\xi-x_1\in S$ and $b:=\overline{\xi}-x_1\in S$ we have $v(a)=v(b)=0$. It follows from (4.10) that

$$\begin{aligned}0&=\int_0^1\big(a^2B(at)-b^2B(bt)\big)\,v(at)v(bt)dt\\&=2i\int_0^1\operatorname{Im}\big(a^2B(at)\big)\,|v(at)|^2dt.\end{aligned}\tag{4.11}$$

By (4.9) we have

$$\operatorname{Im}\big(a^2B(at)\big)=\frac{1}{t^2}\operatorname{Im}\big((ta)^2A(at+x_1)\big)>0.$$

This contradicts (4.11). □

Remark 4.1. Considering $u(-z)$ instead of $u(z)$ we see that Lemma 4.7 remains valid if we put $S:=\{z\colon|z|>r,\,|\arg z-\pi|\le\alpha\}$ and assume that there exists $c>0$ such that $A(z)\sim-cz^\gamma$ as $z\to\infty$ in S.

Proof of Theorem 1.3. We know already from Theorem 1.1 that the rays L_0, L_1 and L_∞ are equally spaced. Also, by Proposition 3.1, the order of f is at least 1.

For a sequence (r_k) tending to ∞, we consider the sequence (f_k) defined by $f_k(z)=f(r_kz)$. We will proceed as in the proof of Theorem 1.2, but this time S_0 will be the sector in $\mathbb{C}$ which is opposite to L_0, and not its intersection with the annulus A. Similarly, S_1 and S_∞ are sectors in $\mathbb{C}$, and so are the sectors T_a, S'_a and S''_a with $a\in\{0,1,\infty\}$. For example, $T_1:=S_0\cup S_\infty\cup L_1\setminus\{0\}$.

As the rays L_0, L_1 and L_∞ are equally spaced, the sectors S_0, S_1 and S_∞ have opening angles $2\pi/3$.

We will consider the functions $n(r):=n(r,1)+n(r,1)+n(r,\infty)$ and

$$N(r):=\int_0^r(n(t)-n(0))\frac{dt}{t}+n(0)\log r$$

First we show that if (f_k) is normal, then (r_k) has a subsequence which is a sequence of Pólya peaks of order 0 for $N(r)$. Thus suppose that (f_k) is normal in $\mathbb{C} \setminus \{0\}$. Passing to a subsequence if necessary, we may assume that (f_k) converges, say $f_k(z) \to \phi(z)$ in $\mathbb{C} \setminus \{0\}$. Proposition 4.1 yields that ϕ is rational. Let d denote the degree of ϕ and let $0 < \varepsilon < 1$. For large k we then have

$$n(r_k) \leq n(r_k/\varepsilon) \leq n(\varepsilon r_k) + 3d.$$

This implies that

$$N(r_k) \geq \int_{\varepsilon r_k}^{r_k} n(t)\frac{dt}{t} \geq n(\varepsilon r_k) \int_{\varepsilon r_k}^{r_k} \frac{dt}{t} = n(\varepsilon r_k) \log \frac{1}{\varepsilon} \geq (n(r_k) - 3d) \log \frac{1}{\varepsilon}.$$

As ε can be chosen arbitrarily small, this yields that

$$\lim_{k\to\infty} \frac{n(r_k)}{N(r_k)} = 0. \tag{4.12}$$

For $\varepsilon r_k \leq r \leq r_k/\varepsilon$ we have

$$\left| N(r) - N(r_k) - n(r_k) \log \frac{r}{r_k} \right| = \left| \int_{r_k}^{r} (n(r) - n(r_k)) \frac{dt}{t} \right| \leq 3d \left| \log \frac{r}{r_k} \right|$$

and thus

$$\left| \frac{N(r)}{N(r_k)} - 1 \right| \leq \frac{3d}{N(r_k)} \left| \log \frac{r}{r_k} \right| + \frac{n(r_k)}{N(r_k)} \left| \log \frac{r}{r_k} \right| \leq \frac{3d + n(r_k)}{N(r_k)} \log \frac{1}{\varepsilon}.$$

Using (4.12) we see that (r_k) is a sequence of Pólya peaks for $N(r)$ of order 0 of both the first and second kind.

Since f has order at least 1, Nevanlinna's second fundamental theorem (see [11] or [13]) implies that $N(r)$ has order at least 1. Lemma 4.1 yields that there is a sequence (r_k) for which no subsequence is a sequence of Pólya peaks for $N(r)$ of order 0. Hence there is a sequence (r_k) such that (f_k) is not normal in $\mathbb{C} \setminus \{0\}$.

Let (r_k) be such a sequence. We proceed as in the proof of Theorem 1.2 and define u_k, v_k and w_k by (2.9), (2.11) and (2.14). Passing to a subsequence of (r_k) if necessary we find as in the proof of Theorem 1.2 that these sequences converge in the appropriate sectors; that is, we have (2.10), (2.12) and (2.15). With h defined by (2.17) we find again that (2.18) holds.

Lemma 4.2 yields that u has the form $u(z) = \text{Re}(e^{i\tau}(az^{3/2}+b/z^{3/2}))$ where $a, b, \tau \in \mathbb{R}$ with $a, b \geq 0$. Since

$$u_k(\xi/r_k) = \frac{\log|f_k(\xi/r_k)|}{\log|f_k(\xi/r_k)|} = \frac{\log|f(\xi)|}{\log|f(r_k\xi)|} \to 0$$

we deduce that $b = 0$. This implies that h has the form

$$h(z) = \text{Re}(cz^3) \tag{4.13}$$

for some $c \in \mathbb{C} \setminus \{0\}$.

It follows from (2.10) and (4.13) there exists a sequence (c_k) in $\mathbb{C}$ such that

$$\log f(r_k z) \sim c_k z^{3/2} \quad \text{for } z \in T_1. \tag{4.14}$$

This implies that there is a sequence (ε_k) tending to 0 such that for suitable sequences (a_k) and (b_k) we have

$$N(r,1) - a_k \sim b_k r^{3/2} \quad \text{for } \varepsilon_k r_k \leq r \leq \frac{r_k}{\varepsilon_k}.$$

The same reasoning can be made for zeros and poles and this yields that

$$N(r) - A_k \sim B_k r^{3/2} \quad \text{for } \varepsilon_k r_k \leq r \leq \frac{r_k}{\varepsilon_k} \tag{4.15}$$

for suitable $A_k, B_k \in \mathbb{R}$ with $B_k > 0$. Let ρ_* be defined by (4.2), with $g(r)$ replaced by $N(r)$. It follows from (4.15) that $\rho_* \leq 3/2$. Lemma 4.1 implies in particular that $N(r)$ has Pólya peaks of some finite order. We claim that $N(r)$ cannot have Pólya peaks of any positive order other than 3/2.

Indeed, let (r_k) be a sequence of Pólya peaks (of the first kind) for $N(r)$ of order $\lambda > 0$. Then $\{f(r_k z)\colon k \in \mathbb{N}\}$ is not normal, since otherwise – as proved above – (r_k) would be a sequence of Pólya peaks for $N(r)$ of order 0. Thus we may assume that (4.15) holds.

Let $M > 1 > \varepsilon > 0$. For large k we then have

$$N(\varepsilon r_k) \leq (1+\varepsilon)\varepsilon^\lambda N(r_k),$$

which together with (4.15) yields that

$$\begin{aligned}(1-\varepsilon)B_k\varepsilon^{3/2}r_k^{3/2} &\leq N(\varepsilon r_k) - A_k \leq (1+\varepsilon)\varepsilon^\lambda N(r_k) - A_k \\ &\leq (1+\varepsilon)\varepsilon^\lambda\left(A_k + (1+\varepsilon)B_k r_k^{3/2}\right) - A_k.\end{aligned}$$

Hence

$$\left(1-(1+\varepsilon)\varepsilon^{\lambda}\right) A_k \leq \left((1+\varepsilon)^2\varepsilon^{\lambda}-(1-\varepsilon)\varepsilon^{3/2}\right) B_k r_k^{3/2}.$$

Similarly,

$$\left(1-(1+\varepsilon)M^{\lambda}\right) A_k \leq \left((1+\varepsilon)^2 M^{\lambda}-(1-\varepsilon)M^{3/2}\right) B_k r_k^{3/2}.$$

The last two inequalities imply that

$$\frac{(1+\varepsilon)^2\varepsilon^{\lambda}-(1-\varepsilon)\varepsilon^{3/2}}{1-(1+\varepsilon)\varepsilon^{\lambda}} \geq \frac{A_k}{B_k r_k^{3/2}} \geq \frac{(1-\varepsilon)M^{3/2}-(1+\varepsilon)^2 M^{\lambda}}{(1+\varepsilon)M^{\lambda}-1}.$$

Suppose now that $\lambda < 3/2$. Then for small ε the left hand side is less than 1, while for large M the right hand side is greater than 1. This is a contradiction. This implies that there are no Pólya peaks of order less than 3/2.

The same arguments arguments can be made for Pólya peaks of the second kind. This yields there are no Pólya peaks of the second kind of order greater than 3/2. Lemma 4.1 yields that $\rho^* = \rho_* = 3/2$, meaning that all Pólya peaks of the first or second kind have order 3/2.

Overall we see that if (r_k) tends to ∞, then (f_k) is not normal. The argument also shows that $N(r)$ and hence f have order 3/2.

Next we show that f has only finitely many critical points; that is, f' has only finitely many zeros and f has only finitely many multiple poles. Suppose that f has infinitely many critical points. Then one of the sectors T_0, T_1 and T_∞ contains a closed subsector which contains infinitely many critical points. Without loss of generality we may assume that this holds for T_1; say (z_k) is a sequence of critical points contained in a closed subsector T_1' of T_1 such that $r_k := |z_k| \to \infty$. As the sequence (f_k) is not normal, we may assume that (4.14) holds. Differentiating we obtain

$$\frac{r_k f'(r_k z)}{f(r_k z)} \sim \frac{3}{2} c_k z^{1/2} \quad \text{for } z \in T_1.$$

This contradicts the assumption that T_1' contains a critical point of modulus r_k. Hence f has only finitely many critical points. This implies that the Schwarzian $S(f)$ has only finitely many poles so that $N(r, S(f)) = \mathcal{O}(\log r)$.

Since f has finite order, the lemma on the logarithmic derivative [11, Section 3.1] yields that $m(r, S(f)) = \mathcal{O}(\log r)$. It follows that

$$T(r,f) = N(r,f) + m(r,f) = \mathcal{O}(\log r)$$

so that $S(f)$ is rational.

Let $Q := S(f)$. Since f has order $3/2$, Lemma 4.3 yields that there exists $a \in \mathbb{C} \setminus \{0\}$ such that $Q(z) \sim az$ as $z \to \infty$. With out loss of generality we may assume that a is negative, say $a = -c$ with $c > 0$.

Lemma 4.4 implies that the set $\{L^1, L^2, L^3\}$ of rays considered there coincides with the set $\{L_0, L_1, L_\infty\}$. As L^2 is the negative real axis, Lemma 4.6 implies that Q is real.

Let $\omega = e^{2\pi i/3}$ and put $f_1(z) := f(\omega z)$. Then $S(f_1)(z) = \omega^2 Q(\omega z)$. Lemma 4.6 implies that $\omega^2 Q(\omega z)$ is also real.

Writing

$$Q(z) = -cz + \sum_{j=-\infty}^{0} c_j z^j$$

we have

$$\omega^2 Q(\omega z) = -cz + \sum_{j=-\infty}^{0} c_j \omega^{2+j} z^j$$

It follows that both c_j and $c_j \omega^{2+j}$ are real for all $j \leq 0$. This implies that $c_j = 0$ if $c \neq 1 \pmod 3$. Hence Q has the form $Q(z) = -zR(z^3)$ where $R(\infty) = c > 0$.

It remains to prove the converse direction. Thus suppose that (1.2) has a meromorphic solution. Then, as remarked after Lemma 4.4, the equation (1.2) also has a meromorphic solution f with the asymptotic values 0, 1 and ∞.

Without loss of generality we may assume that $L_1 = L^2 = (-\infty, 0]$. Thus the 1-points of f are near the negative real axis. The functions $\overline{f(\overline{z})}$ and $1/f(z)$ have the same asymptotic values in the sectors V_j as f. Since both functions have Schwarzian derivative Q, and thus by Lemma 4.4 differ only by linear fractional transformation, this yields that they are actually equal; that is,

$$\frac{1}{f(z)} = \overline{f(\overline{z})}. \tag{4.16}$$

It follows from (4.16) that the 1-points of f are symmetric with respect to the real axis.

We may write $f = w_1/w_2$ where the w_j satisfy $w_j'' + Aw_j = 0$ with $A = Q/2$. We have $f = 1$ if and only if $w := w_1 - w_2 = 0$. Thus the zeros of w are also symmetric with respect to the real axis. This implies that $\overline{w(\overline{z})} = cw(z)$ where $c = e^{i\gamma}$ for some $\gamma \in \mathbb{R}$. Thus $u := e^{i\gamma/2} w$ is real on the real axis. Choosing $\alpha < \pi/3$ we deduce from Lemma 4.7 and Remark 4.1 all but finitely many zeros of u are negative.

It follows that all but finitely many 1-points are contained in the negative real axis L^2. The proof that the other two rays L^1 and L^3 contain all but finitely many zeros and poles follows with the same argument. $\square$

Remark 4.2. The main objective of the papers of Nevanlinna [18] and Elfving [10] cited above was to study Riemann surfaces with finitely many branch points. They showed that such surfaces correspond to meromorphic functions with rational Schwarzian derivative.

Elfving described such surfaces (and functions) in terms of *line complexes* (also called *Speiser graphs*). We do not give the definition of a line complex here, but refer to [11, Section 7.4] and [19, Section XI.2]. Two line complexes are sketched in Figure 2. The left one was also considered by Elfving [10,

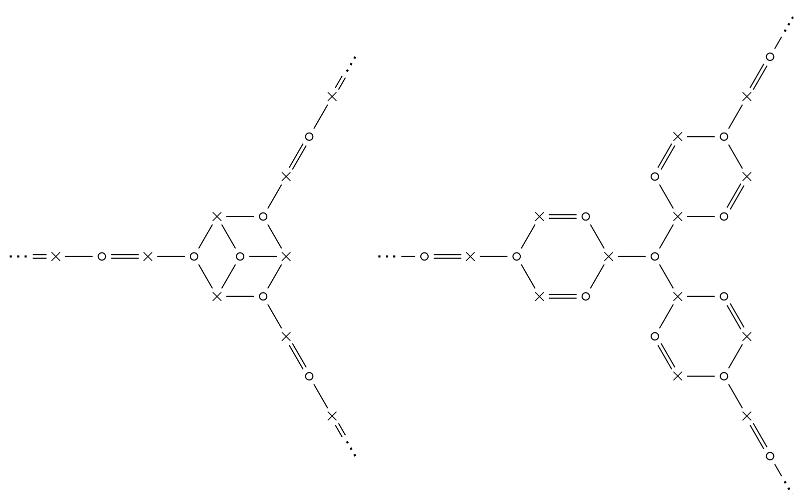

Figure 2: Two line complexes.

no. 2, Abb. 3]. The function corresponding to this line complex has three logarithmic singularities and three critical points, and the critical values corresponding to these three critical points coincide with the three logarithmic singularities.

Elfving [10, no. 47] considered how symmetry of the line complex is reflected in the function; see also [18, no. 42]. For the line complex given in Figure 2, and the associated meromorphic function f, it follows [10, p. 59] that $S(f)$ has the form (1.2) with a rational function R satisfying $R(\infty) \in \mathbb{C}\backslash\{0\}$. In addition, the mirror symmetry of the line complex implies that R is real.

As f has only three (simple) critical points, $S(f)$ has three (double) poles. Thus R has only one (double) pole p and hence the form

$$R(z) = -c + \frac{a}{z-p} + \frac{b}{(z-p)^2}. \tag{4.17}$$

We recall that Elfving [10, Kapitel IV] determined for which rational functions Q the equation $S(f) = Q$ has a meromorphic solution f. It can be deduced from his result that if R is given by (4.17), then (1.2) has a meromorphic solution if and only if $b = -27p/2$ and $c = (4a^2 + 36a + 45)/72p$.

We may assume that $-c = R(\infty) < 0$ and that f has logarithmic singularities over 0, 1 and ∞, with the 1-points close to the negative real axis, corresponding to the branch of the line complex which extends to the left. The simple 1-points then correspond to the double edges of the line complex on this branch, and there is one double 1-point corresponding to the diamond at the end of this branch. Since 1-points are symmetric with respect to the real axis, it follows that all 1-points must lie on the negative real axis.

Thus there are rational functions R with poles such that (1.2) has a solution f for which all (and not only all but finitely many) zeros, 1-points and poles lie on three rays.

For the right line complex in Figure 2 the situation is different. Assume again that 1-points are distributed along the negative real axis, corresponding to the branch of the line complex which extends to the left. The center of the hexagon on this branch corresponds to a negative 1-point. However, there are also further 1-points corresponding to double edges of the hexagons on the other branches. So it may happen that not all but only all but finitely many zeros, 1-points and poles lie on the rays.

Putting more than one hexagon on the branches stretching to ∞, or replacing the hexagons by $(4n+2)$-gons for some $n > 1$, we find that the rational function R in (1.2) may have arbitrarily high degree.

Remark 4.3. In the proof of Theorem 1.3, we have used Lemma 4.7 to prove that zeros, 1-points and poles are on the respective rays. Alternatively, we could have used the symmetry of the associated line complex, similarly to the reasoning in Remark 4.2.

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