Non-escaping points of Zorich maps

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Abstract

We extend results about the dimension of the radial Julia set of certain exponential functions to quasiregular Zorich maps in higher dimensions. Our results improve on previous estimates of the dimension also in the special case of exponential functions.

1 Introduction

The *Julia set* $J(f)$ of an entire function f is the set where the iterates f^n of f do not form a normal family and the *escaping set* $I(f)$ consists of all points which tend to infinity under iteration of f. These sets play a fundamental role in the iteration theory of entire functions. A result of Eremenko [5] says that $J(f) = \partial I(f)$. We refer to [1, 14] for an introduction to the iteration theory of entire functions.

We consider the exponential family consisting of the functions $E_\lambda(z) := \lambda e^z$ with $\lambda \in \mathbb{C} \setminus \{0\}$. If $0 < \lambda < 1/e$, then E_λ has an attracting fixed point. Devaney and Krych [4] showed that then $J(E_\lambda)$ is equal to the complement of the attracting basin of this fixed point and $J(E_\lambda)$ consists of uncountably many pairwise disjoint curves (called *hairs*) which connect a finite point (called the *endpoint* of the hair) with ∞. Let C_λ be the set of endpoints of the hairs that form $J(E_\lambda)$. The results of Devaney and Krych also yield that $J(E_\lambda) \setminus C_\lambda \subset I(E_\lambda)$.

McMullen [12] showed that $\dim J(E_\lambda) = 2$. Here and in the following $\dim X$ denotes the Hausdorff dimension of a set X. In fact, McMullen showed that $\dim I(E_\lambda) = 2$ and $I(E_\lambda) \subset J(E_\lambda)$. Karpińska [10] obtained the surprising result that $\dim J(E_\lambda) \setminus C_\lambda = 1$.

A 3-dimensional analogue of the results of Devaney and Krych, McMullen and Karpińska was obtained in [2]. Here the exponential function was replaced by a quasiregular map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ introduced by Zorich [18, p. 400].

As noted in [11, § 8.1], Zorich maps exist in $\mathbb{R}^d$ for all $d \geq 2$, and this can be used (see [2, Remark 9] and [3]) to obtain a d-dimensional analogue of the above results.

To define a Zorich map, following [8, §6.5.4], we fix $\rho > 0$ and consider the cube

$$Q := \left\{x \in \mathbb{R}^{d-1} \colon \|x\|_\infty \leq \rho\right\} = [-\rho, \rho]^{d-1}$$

and the upper hemisphere

$$U := \left\{x \in \mathbb{R}^d \colon \|x\|_2 = 1, x_d \geq 0\right\}.$$

Let $h \colon Q \to U$ be a bi-Lipschitz map and define

$$F \colon Q \times \mathbb{R} \to \mathbb{R}^d, \; F(x_1, \dots, x_d) = e^{x_d} h(x_1, \dots, x_{d-1}). \tag{1.1}$$

The map F is then extended to a map $F \colon \mathbb{R}^d \to \mathbb{R}^d$ by repeated reflection at hyperplanes.

The main result of [2] says that if $a \in \mathbb{R}$ is sufficiently large, then the map

$$f_a \colon \mathbb{R}^d \to \mathbb{R}^d, \; f_a(x) = F(x) - (0, \dots, 0, a), \tag{1.2}$$

has an attracting fixed point ξ_a such that the complement of the attracting basin of ξ_a consists of hairs, the set of endpoints of the hairs has dimension d, but the union of the hairs without the endpoints has dimension 1.

The purpose of this paper is to extend some further results about the exponential family to the higher dimensional setting. Let $J_{\text{bd}}(E_\lambda)$ be the set of all $z \in J(E_\lambda)$ for which the orbit $\{E_\lambda^n(z) \colon n \in \mathbb{N}\}$ is bounded. Karpińska [9, Theorem 2] also showed that $\dim J_{\text{bd}}(E_\lambda) > 1$ for all $\lambda \in (0, 1/e)$ and

$$1 + \frac{1}{\log\log(1/\lambda)} < \dim J_{\text{bd}}(E_\lambda) < 1 + \frac{1}{\log\log\log(1/\lambda)} \tag{1.3}$$

if λ is sufficiently small.

Urbański and Zdunik [15] considered the set $J_{\text{r}}(E_\lambda) := J(E_\lambda) \setminus I(E_\lambda)$. We note that, in general, the notation $J_{\text{r}}(f)$ is used for the *radial Julia set* of an entire function f, but for the functions E_λ with $0 < \lambda < 1/e$ this agrees with the above definition; see [13] for a discussion of radial Julia sets. Clearly $J_{\text{r}}(E_\lambda) \supset J_{\text{bd}}(E_\lambda)$. Urbański and Zdunik proved [15, Theorem 4.5] that $\dim J_{\text{r}}(E_\lambda) < 2$ for $0 < \lambda < 1/e$ and [15, Theorem 7.2]

$$\lim_{\lambda \to 0} \dim J_{\text{r}}(E_\lambda) = 1. \tag{1.4}$$

They also showed that the function $\lambda \mapsto \dim J_{\rm r}(E_\lambda)$ is continuous [15, Theorem 4.7] in the interval $(0,1/e)$ and in fact real-analytic [16, Theorem 9.3]. The function $\lambda \mapsto \dim J_{\rm r}(E_\lambda)$, and in particular its behavior as $\lambda \to 1/e$, was further studied in [7, 17].

We consider the corresponding sets for the Zorich maps. Denoting by ξ_a the attracting fixed point of f_a we thus put

$$J_{\rm bd}(f_a) = \left\{x \in \mathbb{R}^d \colon f_a^n(x) \not\to \xi_a \text{ and } (f_a^n(x)) \text{ is bounded}\right\}$$

and

$$J_{\rm r}(f_a) = \left\{x \in \mathbb{R}^d \colon f_a^n(x) \not\to \xi_a \text{ and } f_a^n(x) \not\to \infty\right\}.$$

Theorem 1.1. *Let $0 < \eta < \frac{1}{2}$. Then*

$$d - 1 + \eta \frac{\log\log a}{\log a} < \dim J_{\rm bd}(f_a) \le \dim J_{\rm r}(f_a) \le d - 1 + \frac{\log\log a}{\log a}$$

for sufficiently large a.

In order to compare this result with (1.3) we note that for $d = 2$, $\rho = \pi/2$,

$$h\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^2 = \mathbb{C},\ h(x) = (\sin x, \cos x) = \sin x + i \cos x = ie^{-ix},$$

and $z = (x, y) = x + iy$ the Zorich map F takes the form

$$F(z) = F(x, y) = e^y h(x) = ie^{y-ix} = ie^{-iz}.$$

Hence for $a > 0$ and $\lambda = e^{-a}$ we have

$$f_a(z) = F(z) - ia = i(e^{-iz} - a) = (L \circ E_\lambda \circ L^{-1})(z)$$

with $L(z) = i(z - a)$. Thus f_a is conjugate to E_λ. Since $a = \log(1/\lambda)$ we see that Theorem 1.1 not only extends the results for the functions E_λ to higher dimensions, but also improves (1.3) and (1.4) to

$$1 + \eta \frac{\log\log\log(1/\lambda)}{\log\log(1/\lambda)} < \dim J_{\rm bd}(E_\lambda) \le \dim J_{\rm r}(E_\lambda) \le 1 + \frac{\log\log\log(1/\lambda)}{\log\log(1/\lambda)}.$$

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2 Preliminaries

We collect some results about Zorich maps which can be found in [2, 3]. As mentioned above, in [2] only the case $d = 3$ is treated, but the changes to handle the general case are minor. We also note that in [2, 3] only the case $\rho = 1$ is considered. However, it was noted already in [2, Remark 1] that one may replace the unit cube by a cube of other sidelength and in fact by a rectangular box. The reason that we do not restrict to the case $\rho = 1$ is that this way the exponential map is (conjugate to) a special Zorich map, as described in the introduction.

We will use $|x|$ for the Euclidean norm of a point $x \in \mathbb{R}^d$; that is, we write $|x| = \|x\|_2$. For $c \in \mathbb{R}$ we define the half-space

$$H_{>c} := \left\{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_d > c\right\}.$$

The half-spaces $H_{<c}$, $H_{\geq c}$ and $H_{\leq c}$ and the hyperplane $H_{=c}$ are defined analogously.

First we note that the derivative $DF(x)$ exists almost everywhere and that if $DF(x_1, \dots, x_{d-1}, 0)$ exists, then

$$DF(x_1, \dots, x_d) = e^{x_d} DF(x_1, \dots, x_{d-1}, 0). \tag{2.1}$$

This implies that there exist $\alpha, m, M \in \mathbb{R}$ with $0 < \alpha < 1$ and $m < M$ such that

$$|DF(x)| := \sup_{|h|=1} |DF(x)(h)| \leq \alpha \quad \text{a.e. for } x \in H_{\leq m}$$

while

$$\ell(DF(x)) := \inf_{|h|=1} |DF(x)(h)| \geq \frac{1}{\alpha} \quad \text{a.e. for } x \in H_{\geq M}. \tag{2.2}$$

It was shown in [2] that if

$$a \geq e^M - m, \tag{2.3}$$

then f_a has an attracting fixed point $\xi_a \in H_{\leq m}$ and the properties mentioned in the introduction hold; that is, the complement of the basin of attraction of ξ_a consists of hairs, the set of endpoints of the hairs has dimension d, and the union of the hairs without endpoints has dimension 1.

The following result can be considered as an analogue of the result of Karpińska [9, Theorem 2] that $\dim J_{\text{bd}}(E_\lambda) > 1$ for $0 < \lambda < 1/e$.

Theorem 2.1. *If a satisfies* (2.3)*, then* $\dim J_{\text{bd}}(f_a) > d - 1$.

Remark 2.2. Urbański and Zdunik [15, Corollary 7.3] showed that (1.4) implies that $\dim J_{\mathrm{r}}(E_\lambda) < 2$ whenever $0 < \lambda < 1/e$ (and in fact whenever E_λ has an attracting fixed point). The proof uses that if two functions in the exponential family both have an attracting fixed point, then they are quasiconformally conjugate. This argument is not available in the higher-dimensional setting. We do not know whether $\dim J_{\mathrm{r}}(f_a) < d$ whenever a satisfies (2.3).

For $r = (r_1, \dots, r_{d-1}) \in \mathbb{Z}^{d-1}$ we put

$$P(r) := \left\{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} \colon |x_j - 2\rho r_j| < \rho \text{ for } 1 \le j \le d-1\right\}$$

so that $P(0)$ is the interior of Q. Let

$$S := \left\{ r \in \mathbb{Z}^{d-1} \colon \sum_{j=1}^{d-1} r_j \text{ is even} \right\}.$$

Then F maps $P(r) \times \mathbb{R}$ onto $H_{>0}$ if $r \in S$ and onto $H_{<0}$ if $r \in \mathbb{Z}^{d-1} \setminus S$. Thus f_a maps $P(r) \times \mathbb{R}$ onto $H_{>-a}$ if $r \in S$ and onto $H_{<-a}$ if $r \in \mathbb{Z}^{d-1} \setminus S$. For $r \in S$ we put

$$T(r) = P(r) \times (M, \infty).$$

A short computation (see [2, (2.1)] or [3, (2.2)]) shows that $f_a(T(r)) \supset H_{\ge M}$. Thus there exists a branch $\Lambda^r \colon H_{\ge M} \to T(r)$ of the inverse function of f_a. With $\Lambda := \Lambda^{(0,\dots,0)}$ we have

$$\Lambda^{(r_1,\dots,r_{d-1})}(x) = \Lambda(x) + (2\rho r_1, \dots, 2\rho r_{d-1}, 0)$$

for all $x \in H_{\ge M}$ and all $r \in S$. We have

$$D\Lambda(x) = Df_a(\Lambda(x))^{-1} = DF(\Lambda(x))^{-1} \quad \text{a.e. for } x \in H_{\ge M}. \tag{2.4}$$

It thus follows from (2.2) that $|D\Lambda(x)| \le \alpha$ a.e. for $x \in H_{\ge M}$. This implies that

$$|\Lambda(x) - \Lambda(y)| \le \alpha |x - y| \quad \text{for } x, y \in H_{\ge M}.$$

Noting that $Df_a(x) = DF(x)$ we deduce from (2.1) that there exist positive constants c_1 and c_2 such that

$$c_1 e^{x_d} \le \ell(Df_a(x)) \le |Df_a(x)| \le c_2 e^{x_d} \quad \text{a.e.} \tag{2.5}$$

It was shown in [2, 3] that there exists positive constants c_3 and c_4 such that

$$\frac{c_3}{|x|} \leq \ell(D\Lambda(x)) \leq |D\Lambda(x)| \leq \frac{c_4}{|x|} \quad \text{a.e. for } x \in H_{\geq M} \tag{2.6}$$

and this was used to prove that

$$|\Lambda(x) - \Lambda(y)| \leq c_4 \pi \frac{|x-y|}{\min\{|x|, |y|\}}. \tag{2.7}$$

We have to consider how the bounds for $\ell(D\Lambda(x))$ and $|D\Lambda(x)|$ in (2.6) depend on a. We will write $\overline{a} = (0, \ldots, 0, a)$ so that $f_a(x) = F(x) - \overline{a}$.

Lemma 2.3. *There exist constants c_3 and c_4 depending only on F such that*

$$\frac{c_3}{|x + \overline{a}|} \leq \ell(D\Lambda(x)) \leq |D\Lambda(x)| \leq \frac{c_4}{|x + \overline{a}|} \quad \text{a.e. for } x \in H_{\geq M}.$$

Proof. By (2.4), (2.5), (1.1) and (1.2) we have

$$\begin{aligned} |D\Lambda(x)| &\leq \frac{1}{\ell(DF(\Lambda(x)))} \leq \frac{1}{c_1 \exp(\Lambda_d(x))} \\ &= \frac{1}{c_1 |F(\Lambda(x))|} = \frac{1}{c_1 |f_a(\Lambda(x)) + \overline{a}|} = \frac{1}{c_1 |x + \overline{a}|}. \end{aligned}$$

The proof of the lower bound for $\ell(D\Lambda(x))$ is similar. □

Lemma 2.3 implies that (2.7) can be improved to

$$|\Lambda(x) - \Lambda(y)| \leq c_4 \pi \frac{|x-y|}{\min\{|x + \overline{a}|, |y + \overline{a}|\}} \tag{2.8}$$

for $x, y \in H_{\geq M}$.

3 Proof of the upper bound in Theorem 1.1

For $r \in S$ and $A \subset \mathbb{R}^d$, we will use the notation

$$\begin{aligned} A^r &:= (2\rho r_1, \ldots, 2\rho r_{d-1}, 0) + A \\ &= \{(2\rho r_1 + x_1, \ldots, 2\rho r_{d-1} + x_{d-1}, x_d) \colon x \in A\}. \end{aligned}$$

We also write $B(x, R)$ for the closed ball of radius R around a point $x \in \mathbb{R}^d$.

Lemma 3.1. *Let $r, s \in S$ and let $A \subset T(s)$ be bounded. Then*

$$\operatorname{diam} \Lambda(A^r) \le c_4 \pi \frac{\operatorname{diam} A}{\sqrt{\rho^2 |r+s|^2 + a^2}}.$$

Proof. Since $A^r \subset T(r+s)$ we find that if $x = (x_1, \dots, x_d) \in A^r$, then $|x_j| \ge \max\{2\rho|r_j + s_j| - \rho, 0\} \ge \rho |r_j + s_j|$ for $1 \le j \le d-1$ while $x_d \ge M$. Thus

$$\begin{aligned} |x + \overline{a}| &= \sqrt{\sum_{j=1}^{d-1} x_j^2 + (x_d + a)^2} \\ &\ge \sqrt{\sum_{j=1}^{d-1} \rho^2 (r_j + s_j)^2 + a^2} = \sqrt{\rho^2 |r+s|^2 + a^2}. \end{aligned}$$

The conclusion now follows from (2.8). □

Lemma 3.2. *There exist positive constants c_5 and c_6 depending only on d such that if $d - 1 < t \le d$ and $N \ge b \ge \sqrt{6(d-1)}$, then*

$$c_5 \frac{b^{d-1-t}}{t-d+1} \left(1 - \left(\frac{N}{b} \right)^{d-1-t} \right) \le \sum_{\substack{r \in S \\ |r| \le N}} \frac{1}{(|r|^2 + b^2)^{t/2}} \le c_6 \frac{b^{d-1-t}}{t-d+1}. \quad (3.1)$$

Moreover,

$$\sum_{\substack{r \in S \\ |r| \le N}} \frac{1}{(|r|^2 + b^2)^{(d-1)/2}} \ge c_5 \log \frac{N}{b} \quad (3.2)$$

Proof. For $r = (r_1, \dots, r_{d-1}) \in \mathbb{Z}^{d-1}$ let Q_r be the cube with vertices at the points $(2r_1 + e_1, \dots, 2r_{d-1} + e_{d-1})$, where $e_j \in \{-1, 1\}$ for all j. For $x \in Q_r$ and $b \ge \sqrt{d-1}/7$ we then have

$$|x|^2 \le \sum_{j=1}^{d-1} (2|r_j| + 1)^2 \le \sum_{j=1}^{d-1} (8 r_j^2 + 1) = 8|r|^2 + d - 1 \le 8|r|^2 + 7b^2$$

and thus $|x|^2 + b^2 \le 8|r|^2 + 8b^2$. Hence

$$\int_{Q_r} \frac{dx_1 \dots dx_{d-1}}{(|x|^2 + b^2)^{t/2}} \ge \int_{Q_r} \frac{dx_1 \dots dx_{d-1}}{(8|r|^2 + 8b^2)^{t/2}} = \frac{2^{d-1} 8^{-t/2}}{(|r|^2 + b^2)^{t/2}} = \frac{2^{d-1-3t/2}}{(|r|^2 + b^2)^{t/2}}.$$

For $|r| \le N$ we have $Q_r \subset B(0, N + 2\sqrt{d-1}) \subset B(0, N+d)$ and thus

$$\begin{aligned} \sum_{|r|\le N} \frac{1}{(|r|^2+b^2)^{t/2}} &\le 2^{3t/2-d+1} \sum_{|r|\le N} \int_{Q_r} \frac{dx_1 \dots dx_{d-1}}{(|x|^2+b^2)^{t/2}} \\ &\le 2^{3t/2-d+1} \int_{B(0,N+d)} \frac{dx_1 \dots dx_{d-1}}{(|x|^2+b^2)^{t/2}} \\ &= 2^{3t/2-d+1} \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{N+d} \frac{u^{d-2}}{(u^2+b^2)^{t/2}} du. \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned} \int_0^{N+d} \frac{u^{d-2}}{(u^2+b^2)^{t/2}} du &= b^{d-1-t} \int_0^{(N+d)/b} \frac{v^{d-2}}{(v^2+1)^{t/2}} dv \\ &\le b^{d-1-t} \left(1 + \int_1^\infty v^{d-2-t} dv \right) \\ &= b^{d-1-t} \left(1 + \frac{1}{t-d+1} \right) \le 2 \frac{b^{d-1-t}}{t-d+1}. \end{aligned} \tag{3.4}$$

The right inequality in (3.1) now follows from (3.3) and (3.4).

To prove the left inequality, we proceed similarly. For $r \in S$ let P_r be the cube with vertices at the points $(2r_1 + 3e_1, \dots, 2r_{d-1} + 3e_{d-1})$, where $e_j \in \{-1, 1\}$ for all j. Noting that $(2y-3)^2 \ge y^2 - 3$ for $y \in \mathbb{R}$ we find that if $x \in P_r$ and $b \ge \sqrt{6(d-1)}$, then

$$|x|^2 \ge \sum_{j=1}^{d-1} (2|r_j| - 3)^2 \ge \sum_{j=1}^{d-1} (|r_j|^2 - 3) = |r|^2 - 3(d-1) \ge |r|^2 - \frac{1}{2} b^2$$

and thus $|x|^2 + b^2 \ge \frac{1}{2}|r|^2 + \frac{1}{2}b^2$. Hence

$$\int_{P_r} \frac{dx_1 \dots dx_{d-1}}{(|x|^2+b^2)^{t/2}} \le \int_{P_r} \frac{dx_1 \dots dx_{d-1}}{(\frac{1}{2}|r|^2 + \frac{1}{2}b^2)^{t/2}} = \frac{6^{d-1} 2^{t/2}}{(|r|^2+b^2)^{t/2}}.$$

Instead of (3.3) we now obtain

$$\begin{aligned} \sum_{|r|\le N} \frac{1}{(|r|^2+b^2)^{t/2}} &\ge 6^{1-d} 2^{-t/2} \sum_{|r|\le N} \int_{P_r} \frac{dx_1 \dots dx_{d-1}}{(|x|^2+b^2)^{t/2}} \\ &\ge 6^{1-d} 2^{-t/2} \int_{B(0,N)} \frac{dx_1 \dots dx_{d-1}}{(|x|^2+b^2)^{t/2}} \\ &= 6^{1-d} 2^{-t/2} \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{N} \frac{u^{d-2}}{(u^2+b^2)^{t/2}} du \end{aligned} \tag{3.5}$$

and instead of (3.4) we have

$$\begin{aligned}\int_0^N \frac{u^{d-2}}{(u^2+b^2)^{t/2}}du &= b^{d-1-t}\int_0^{N/b} \frac{v^{d-2}}{(v^2+1)^{t/2}}dv \\ &\geq b^{d-1-t}2^{-t/2}\int_1^{N/b} v^{d-2-t}dv \\ &= \frac{b^{d-1-t}}{t-d+1}2^{-t/2}\left(1-\left(\frac{N}{b}\right)^{d-1-t}\right).\end{aligned} \tag{3.6}$$

The left inequality in (3.1) now follows from (3.5) and (3.6).

Finally, to prove (3.2) we only have to note that for $t = d-1$ we obtain

$$\begin{aligned}\int_0^N \frac{u^{d-2}}{(u^2+b^2)^{(d-1)/2}}du &= \int_0^{N/b} \frac{v^{d-2}}{(v^2+1)^{(d-1)/2}}dv \\ &\geq 2^{(1-d)/2}\int_1^{N/b} \frac{dv}{v} = 2^{(1-d)/2}\log\frac{N}{b}\end{aligned}$$

instead of (3.6). □

Lemma 3.3. *There exists a constant c_7 such that if $d-1 < t \leq d$, if $s \in S$ and if $A \subset T(s)$ is bounded, then*

$$\sum_{r\in S}(\operatorname{diam}\Lambda(A^r))^t \leq c_7\frac{a^{d-1-t}}{t-d+1}(\operatorname{diam} A)^t.$$

Proof. Without loss of generality we may assume that $s = 0$. Applying Lemma 3.1 we obtain

$$\begin{aligned}\sum_{r\in S}(\operatorname{diam}\Lambda(A^r))^t &\leq \sum_{r\in S}\frac{(c_4\pi\operatorname{diam} A)^t}{(\rho^2|r|^2+a^2)^{t/2}} \\ &\leq \left(\frac{c_4\pi\operatorname{diam} A}{\rho}\right)^t\sum_{r\in S}\frac{1}{(|r|^2+(a/\rho)^2)^{t/2}}.\end{aligned}$$

Now Lemma 3.2 yields that

$$\begin{aligned}\sum_{r\in S}(\operatorname{diam}\Lambda(A^r))^t &\leq \left(\frac{c_4\pi}{\rho}\right)^t\frac{c_6}{t-d+1}\left(\frac{a}{\rho}\right)^{d-1-t}(\operatorname{diam} A)^t \\ &= \frac{c_6(c_4\pi)^t}{\rho^{d-1}}\frac{a^{d-1-t}}{t-d+1}(\operatorname{diam} A)^t,\end{aligned}$$

from which the conclusion follows. □

Proof of the upper bound in Theorem 1.1. Let $J(f_a) := \{x \colon f_a^n(x) \not\to \xi_a\}$ be the complement of the attracting basin of ξ_a. For $R > M$ we consider the set

$$K(R) := \left\{ x \in J(f_a) \cap B(0,R) \colon \liminf_{k\to\infty} |f_a^k(x)| \leq R \right\}.$$

Let c_7 be the constant from Lemma 3.3. We show that if a and t are such that $\tau := c_7 a^{d-1-t}/(t-d+1) < 1$, then $\dim K(R) \leq t$. This implies that $\dim J_{\mathrm{r}}(f_a) \leq t$, since $J_{\mathrm{r}}(f_a) = \bigcup_{n\in\mathbb{N}} K(R_n)$ for any sequence (R_n) which tends to ∞. The conclusion follows from this, since for $t = d-1+\log\log a/\log a$ we have $a^{d-1-t}/(t-d+1) = 1/\log\log a$.

For $s \in S$, $A \subset T(s)$ and $n \in \mathbb{N}$, let $X_n(A)$ denote the set of all components of $f^{-n}(A)$ which are contained in $T(0)$. If $U \in X_n(A)$, then $f(U)$ has the form $f(U) = V^r$ for some $V \in X_{n-1}(A)$ and some $r \in S$. Equivalently, $U = \Lambda(V^r)$. In turn, if $V \in X_{n-1}(A)$ and $r \in S$, then $\Lambda(V^r) \in X_n(A)$. Together with Lemma 3.3 we thus find that

$$\sum_{U\in X_n(A)} (\operatorname{diam} U)^t = \sum_{V\in X_{n-1}(A)} \sum_{r\in S} (\operatorname{diam}\Lambda(V^r))^t \leq \tau \sum_{V\in X_{n-1}(A)} (\operatorname{diam} V)^t.$$

Induction yields that

$$\begin{aligned} \sum_{U\in X_n(A)} (\operatorname{diam} U)^t &\leq \tau^{n-1} \sum_{V\in X_1(A)} (\operatorname{diam} V)^t \\ &= \tau^{n-1}(\operatorname{diam}\Lambda(A))^t \leq \tau^{n-1}(\operatorname{diam} A)^t. \end{aligned} \tag{3.7}$$

We will apply this for $A^s := H_{\leq R} \cap T(s)$. There exists $N \in \mathbb{N}$ such that

$$J(f_a) \cap B(0,R) \subset \bigcup_{|s|\leq N} A^s.$$

Next we put

$$Y_n := \bigcup_{|s|\leq N} \bigcup_{m\geq n} X_m(A^s) \quad \text{and} \quad Z_n := \{U^s \colon |s| \leq N,\ U \in Y_n\}.$$

Then Y_n contains all points $x \in T(0)$ for which there exists $m \geq n$ and $s \in S$ with $|s| \leq N$ such that $f_a^m(x) \in A^s$. Hence Z_n contains all $x \in \bigcup_{|s|\leq N} T(s)$ for which there exists $m \geq n$ such that $f_a^m(x) \in \bigcup_{|s|\leq N} A^s$. In particular, Z_n contains all $x \in J(f_a) \cap B(0,R)$ for which there exists $m \geq n$ such that $|f_a^m(x)| \leq R$. Thus $K(R) \subset Z_n$ for all $n \in \mathbb{N}$.

Let L be the cardinality of $\{s \in S\colon |s| \le N\}$. Then

$$\begin{aligned}\sum_{U \in Z_n} (\operatorname{diam} U)^t = L \sum_{U \in Y_n} (\operatorname{diam} U)^t &= L \sum_{|s| \le N} \sum_{m \ge n} \sum_{U \in X_m(A^s)} (\operatorname{diam} U)^t \\ &\le L \sum_{|s| \le N} \sum_{m \ge n} \tau^{m-1} (\operatorname{diam} A^s)^t = L^2 \sum_{m \ge n} \tau^{m-1} \left(\operatorname{diam} A^0\right)^t \\ &= L^2 \left(\operatorname{diam} A^0\right)^t \frac{\tau^{n-1}}{1-\tau}\end{aligned}$$

by (3.7). Since the right hand side tends to 0 as $n \to \infty$, we deduce that $\dim K(R) \le t$. □

4 Proof of the lower bounds

The proof is based on the theory of iterated functions systems. In particular, we will use the following result [6, Proposition 9.7].

Lemma 4.1. *Let $S_1, \dots, S_m$ be contractions on a closed subset K of $\mathbb{R}^d$ such that there exists $b_1, \dots, b_m \in (0,1)$ with*

$$b_j|x-y| \le |S_j(x) - S_j(y)| \quad \text{for } x, y \in K \text{ and } 1 \le j \le m.$$

Suppose $K_0 \subset K$ is compact with

$$K_0 = \bigcup_{j=1}^{m} S_j(K_0)$$

and $S_j(K_0) \cap S_k(K_0) = \emptyset$ for $j \ne k$. Let $t > 0$ with

$$\sum_{j=1}^{m} b_j^t = 1. \tag{4.1}$$

Then $\dim K_0 \ge t$.

Since the left hand side of (4.1) is a decreasing function of t, it follows that if

$$\sum_{j=1}^{m} b_j^t > 1,$$

then $\dim K_0 > t$.

Proof of Theorem 2.1 and the lower bound in Theorem 1.1. Let $N \in \mathbb{N}$ with $N \geq a/\rho$ and put $R = 8\rho N$ and $K = B(-\overline{a}, R) \cap H_{\geq M}$. For $r \in S$ we have $\Lambda^r(K) \subset A^r := H_{\leq \log R} \cap T(r)$. Put $L = a + \log R = a + \log(8\rho N)$. For $y \in A^r$ we have

$$|y+\overline{a}|^2 \leq \sum_{j=1}^{d-1}(2|r_j|+1)^2\rho^2+L^2 \leq 8\rho^2|r|^2+(d-1)\rho^2+L^2 \leq 8(\rho^2|r|^2+L^2) \quad (4.2)$$

if N and hence L are large. Since $a \leq \rho N$ we have $L \leq \rho N+\log(8\rho N) \leq 2\rho N$ if N and L are large. Thus we see that if $|r| \leq N$ and $y \in A_r$, then

$$|y + \overline{a}|^2 \leq 8(\rho^2 N^2 + 4\rho^2 N^2) = 40\rho^2 N^2 \leq R^2.$$

Thus $A^r \subset K$ if $|r| \leq N$, provided N is sufficiently large.

We want to apply Lemma 4.1 with contractions S_j of the form $S_j = \Lambda^r \circ \Lambda^s$ where $|r|, |s| \leq N$.

It follows from Lemma 2.3 that $\ell(D\Lambda^r(x)) = \ell(D\Lambda(x)) \geq c_3/R$ for $x \in K$ and $r \in S$. By (4.2) we also have

$$\ell(D\Lambda^s(y)) = \ell(D\Lambda(y)) \geq \frac{c_3}{|y + \overline{a}|} \geq \frac{c_3}{2\sqrt{2}\sqrt{\rho^2|r|^2 + L^2}} \quad \text{for } y \in A^r.$$

Hence

$$\ell(D(\Lambda^s \circ \Lambda^r)(y)) \geq \ell(D\Lambda^s(\Lambda^r(x))) \cdot \ell(D\Lambda^r(x)) \geq \frac{c_3^2}{2\sqrt{2}R\sqrt{\rho^2|r|^2 + L^2}}$$

for $x \in K$. It follows that

$$|(\Lambda^s \circ \Lambda^r)(x) - (\Lambda^s \circ \Lambda^r)(y)| \geq b_{r,s}|x - y|$$

with

$$b_{r,s} = \frac{c_3^2}{2\sqrt{2}R\sqrt{\rho^2|r|^2 + L^2}},$$

for $|r|, |s| \leq N$ and $x, y \in K$. The limit set of the iterated function system generated by the $\Lambda_{r,s}$ is contained in $J_{\mathrm{bd}}(f_a)$. It thus follows from Lemma 4.1 and the remark following it that $\dim J_{\mathrm{bd}}(f_a) > t$ if

$$\sum_{|r|\leq N} \sum_{|s|\leq N} b_{r,s}^t > 1. \quad (4.3)$$

Let the cube P_r be defined as in the proof of Lemma 3.2. Then each P_r has volume 6^{d-1} and the union of all P_r for which $|r| \leq N$ covers $B(0,N)$ and thus has volume at least $\pi^{(d-1)/2}N^{d-1}/\Gamma((d+1)/2)$. Thus the set of all $s \in S$ for which $|s| \leq N$ has at least $\lfloor 6^{1-d}\pi^{(d-1)/2}N^{d-1}/\Gamma((d+1)/2)\rfloor$ elements. Recalling that $R = 8\rho N$ we deduce that there exist a constant c_8 such that with $b := L/\rho$ we have

$$\begin{aligned}\sum_{|r|\leq N}\sum_{|s|\leq N} b_{r,s}^t &\geq \left\lfloor \frac{6^{1-d}\pi^{(d-1)/2}N^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)}\right\rfloor \frac{c_3^{2t}}{(2\sqrt{2}R)^t}\sum_{|r|\leq N}\frac{1}{(\rho^2|r|^2+L^2)^{t/2}} \\ &\geq c_8 N^{d-1-t}\sum_{|r|\leq N}\frac{1}{(|r|^2+b^2)^{t/2}},\end{aligned} \tag{4.4}$$

for $d-1 \leq t \leq d$.

Suppose first that $t = d-1$. Using (3.2) we find with $c_9 = c_8 c_5$ that

$$\sum_{|r|\leq N}\sum_{|s|\leq N} b_{r,s}^{d-1} \geq c_9 \log\frac{N}{b} = c_9 \log\frac{N\rho}{a+\log(8\rho N)}. \tag{4.5}$$

The right hand side of (4.5) tends to ∞ as $N \to \infty$. In particular, it is greater than 1 for large N so that (4.3) holds for $t = d-1$. Thus $\dim J_{\text{bd}}(f_a) > d-1$. This proves Theorem 2.1.

Now we consider the behavior as $a \to \infty$ and assume that $t > d-1$. Let $0 < \delta < 1/(2\eta) - 1$. We may choose N such that $N \sim a^{1+\delta}$ as $a \to \infty$. Then $L \sim a$ and hence $b = L/\rho \sim a/\rho$. Thus for large a we have $Nb \leq a^{2+2\delta}$ and $N/b \geq a^{\delta/2}$. For $t = d-1+\eta\log\log a/\log a$ we deduce from (4.4) and Lemma 3.2 that

$$\begin{aligned}\sum_{|r|\leq N}\sum_{|s|\leq N} b_{r,s}^t &\geq c_9\frac{(Nb)^{d-1-t}}{t-d+1}\left(1-\left(\frac{N}{b}\right)^{d-1-t}\right) \\ &\geq c_9\frac{\left(a^{2+2\delta}\right)^{d-1-t}}{t-d+1}\left(1-\left(a^{\delta/2}\right)^{d-1-t}\right) \\ &= \frac{c_9}{\eta}\frac{(\log a)^{1-(2+2\delta)\eta}}{\log\log a}\left(1-(\log a)^{-\delta\eta/2}\right).\end{aligned} \tag{4.6}$$

By the choice of δ, we have $(2+2\delta)\eta < 1$ and thus the right hand side of (4.6) tends to ∞ as $a \to \infty$. Hence (4.3) holds for $t = d-1+\eta\log\log a/\log a$ and large a. This proves the lower bound in Theorem 1.1. □

References

[1] Walter Bergweiler, Iteration of meromorphic functions. Bull. Amer. Math. Soc. (N.S.) 29 (1993), no. 2, 151–188.

[2] Walter Bergweiler, Karpińska's paradox in dimension 3. Duke Math. J. 154 (2010), no. 3, 599–630.

[3] Patrick Comdühr, On the differentiability of hairs for Zorich maps. Ergodic Theory Dynam. Systems 39 (2019), no. 7, 1824–1842.

[4] Robert L. Devaney and Michał Krych, Dynamics of $\exp(z)$. Ergodic Theory Dynam. Systems 4 (1984), no. 1, 35–52.

[5] A. È. Erëmenko, On the iteration of entire functions. In "Dynamical systems and ergodic theory (Warsaw, 1986)", Banach Center Publications 23. Polish Scientific Publishers, Warsaw 1989, pp. 339–345.

[6] Kenneth Falconer, Fractal geometry. Mathematical foundations and applications. John Wiley & Sons, Chichester, 1990.

[7] Guillaume Havard, Mariusz Urbański and Michel Zinsmeister, Variations of Hausdorff dimension in the exponential family. Ann. Acad. Sci. Fenn. Math. 35 (2010), no. 2, 351–378.

[8] Tadeusz Iwaniec and Gaven Martin, Geometric function theory and nonlinear analysis. Oxford Mathematical Monographs. Oxford University Press, New York, 2001.

[9] Bogusława Karpińska, Area and Hausdorff dimension of the set of accessible points of the Julia sets of λe^z and $\lambda \sin z$. Fund. Math. 159 (1999), no. 3, 269–287.

[10] Bogusława Karpińska, Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$. C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), no. 11, 1039–1044.

[11] Olli Martio and Uri Srebro, Periodic quasimeromorphic mappings in R^n. J. Anal. Math. 28 (1975), no. 1, 20–40.

[12] Curt McMullen, Area and Hausdorff dimension of Julia sets of entire functions. Trans. Amer. Math. Soc. 300 (1987), no. 1, 329–342.

[13] Lasse Rempe, Hyperbolic dimension and radial Julia sets of transcendental functions. Proc. Amer. Math. Soc. 137 (2009), no. 4, 1411–1420.

[14] Dierk Schleicher, Dynamics of entire functions. In "Holomorphic dynamical systems", Lecture Notes Math. 1998, Springer, Berlin, 2010, pp. 295–339.

[15] Mariusz Urbański and Anna Zdunik, The finer geometry and dynamics of the hyperbolic exponential family. Michigan Math. J. 51 (2003), no. 2, 227–250.

[16] Mariusz Urbański and Anna Zdunik, Real analyticity of Hausdorff dimension of finer Julia sets of exponential family. Ergodic Theory Dynam. Systems 24 (2004), no. 1, 279–315.

[17] Mariusz Urbański and Anna Zdunik, The parabolic map $f_{1/e}(z) = \frac{1}{e}e^z$. Indag. Math. (N.S.) 15 (2004), no. 3, 419–433.

[18] V. A. Zorich, A theorem of M. A. Lavrent'ev on quasiconformal space maps. Math. USSR, Sb. 3 (1967), no. 3, 389–403; translation from Mat. Sb. (N.S.) 74 (116), (1967), 417–433.

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