

# Radially distributed values and normal families, II

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## Abstract

We consider the family of all functions holomorphic in the unit disk for which the zeros lie on one ray while the 1-points lie on two different rays. We prove that for certain configurations of the rays this family is normal outside the origin.

## 1 Introduction and results

There is an extensive literature on entire functions whose zeros and 1-points are distributed on finitely many rays. One of the first results of this type is the following theorem of Biernacki [5, p. 533] and Milloux [11].

**Theorem A.** *There is no transcendental entire function for which all zeros lie on one ray and all 1-points lie on a different ray.*

Biernacki and Milloux proved this under the additional hypothesis that the function considered has finite order, but by a later result of Edrei [6] this is always the case if all zeros and 1-points lie on finitely many rays.

A thorough discussion of the cases in which an entire function can have its zeros on one system of rays and its 1-points on another system of rays, intersecting the first one only at 0, was given in [4]. Special attention was paid to the case where the zeros are on one ray while the 1-points are on two rays. For this case the following result was obtained [4, Theorem 2].

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**Theorem B.** *Let  $f$  be a transcendental entire function whose zeros lie on a ray  $L_0$  and whose 1-points lie on two rays  $L_1$  and  $L_{-1}$ , each of which is distinct from  $L_0$ . Suppose that the numbers of zeros and 1-points are infinite. Then  $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$ .*

The hypothesis that  $f$  has infinitely many zeros excludes the example  $f(z) = e^z$  in which case we have  $\angle(L_1, L_{-1}) = \pi$ , and  $L_0$  can be taken arbitrarily. Without this hypothesis we have the following result.

**Theorem B'.** *Let  $f$  be a transcendental entire function whose zeros lie on a ray  $L_0$  and whose 1-points lie on two rays  $L_1$  and  $L_{-1}$ , each of which is distinct from  $L_0$ . Then  $\angle(L_1, L_{-1}) = \pi$  or  $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$ .*

Bloch's heuristic principle says that the family of all functions holomorphic in some domain which have a certain property is likely to be normal if there does not exist a non-constant entire function with this property. More generally, properties which are satisfied only by "few" entire functions often lead to normality. We refer to [2], [14] and [16] for a thorough discussion of Bloch's principle.

The following normal family analogue of Theorem A was proved in [3, Theorem 1.1]. Here  $\mathbb{D}$  denotes the unit disk.

**Theorem C.** *Let  $L_0$  and  $L_1$  be two distinct rays emanating from the origin and let  $\mathcal{F}$  be the family of all functions holomorphic in  $\mathbb{D}$  for which all zeros lie on  $L_0$  and all 1-points lie on  $L_1$ . Then  $\mathcal{F}$  is normal in  $\mathbb{D} \setminus \{0\}$ .*

The purpose of this paper is to prove a normal family analogue of Theorem B'.

**Theorem 1.1.** *Let  $L_0$ ,  $L_1$  and  $L_{-1}$  be three distinct rays emanating from the origin and let  $\mathcal{F}$  be the family of all functions holomorphic in  $\mathbb{D}$  for which all zeros lie on  $L_0$  and all 1-points lie on  $L_1 \cup L_{-1}$ . Assume that neither  $\angle(L_{-1}, L_1) = \pi$  nor  $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$ . Then  $\mathcal{F}$  is normal in  $\mathbb{D} \setminus \{0\}$ .*

It was shown in [4, Theorem 3] that if  $\alpha$  is of the form  $\alpha = 2\pi/n$  with  $n \in \mathbb{N}$ ,  $n \geq 5$ , then there exist rays  $L_0$  and  $L_{\pm 1}$  with  $\angle(L_0, L_1) = \angle(L_0, L_{-1}) = \alpha$  and an entire function  $f$  with all zeros on  $L_0$  and all 1-points on  $L_1$  and  $L_{-1}$ . In [7] such an entire function  $f$  was constructed for every  $\alpha \in (0, \pi/3]$ .

The functions constructed in [4, 7] have the property that  $f(re^{i\theta}) \rightarrow 0$  as  $r \rightarrow \infty$  for  $|\theta| < \alpha$  while  $f(re^{i\theta}) \rightarrow \infty$  as  $r \rightarrow \infty$  for  $\alpha < |\theta| \leq \pi$ . Considering

the family  $\{f(kz)\}_{k \in \mathbb{N}}$  we see that the conclusion of Theorem 1.1 does not hold if  $\angle(L_0, L_1) = \angle(L_0, L_{-1}) \in (0, \pi/3] \cup \{2\pi/5\}$ . The example  $\{e^{kz}\}_{k \in \mathbb{N}}$  shows that it does not hold if  $\angle(L_{-1}, L_1) = \pi$ .

The question whether the conclusion of Theorem 1.1 holds if  $\angle(L_0, L_1) = \angle(L_0, L_{-1}) \in (\pi/3, \pi/2) \setminus \{2\pi/5\}$  remains open.

We note that Theorem B' follows from Theorem 1.1. To see this we only have to note that if  $f$  is a transcendental entire function and  $(z_k)$  is a sequence tending to  $\infty$  such that  $|f(z_k)| \leq 1$  for all  $k \in \mathbb{N}$ , then  $\{f(2|z_k|z)\}_{k \in \mathbb{N}}$  is not normal at some point of modulus  $\frac{1}{2}$ ; see the remark after Theorem 1.1 in [4].

A key tool in the theory of normal families is Zalcman's lemma [15]; see Lemma 2.1 below. An extension of this result (Lemma 2.2 below) was also crucial in the proof of Theorem C in [3]. In fact, this extension was used to prove the following result [3, Theorem 1.3] from which Theorem C can be deduced.

**Theorem D.** *Let  $D$  be a domain and let  $L$  be a straight line which divides  $D$  into two subdomains  $D^+$  and  $D^-$ . Let  $\mathcal{F}$  be a family of functions holomorphic in  $D$  which do not have zeros in  $D$  and for which all 1-points lie on  $L$ .*

*Suppose that  $\mathcal{F}$  is not normal at  $z_0 \in D \cap L$  and let  $(f_k)$  be a sequence in  $\mathcal{F}$  which does not have a subsequence converging in any neighborhood of  $z_0$ . Suppose that  $(f_k|_{D^+})$  converges. Then either  $f_k|_{D^+} \rightarrow 0$  and  $f_k|_{D^-} \rightarrow \infty$  or  $f_k|_{D^+} \rightarrow \infty$  and  $f_k|_{D^-} \rightarrow 0$ .*

Note that  $\mathcal{F}$  is normal in  $D^+$  by Montel's theorem. So it is no restriction to assume that  $(f_k|_{D^+})$  converges, since this can be achieved by passing to a subsequence.

Theorem D will also play an important role in the proof of Theorem 1.1. However, we will also need the following addendum to Theorem D. Here and in the following  $D(a, r)$  and  $\overline{D}(a, r)$  denote the open and closed disk of radius  $r$  centered at a point  $a \in \mathbb{C}$ .

**Proposition 1.1.** *Let  $D, L, \mathcal{F}, z_0$  and  $(f_k)$  be as in Theorem D. Let  $r > 0$  with  $\overline{D}(z_0, r) \subset D$ . Then for sufficiently large  $k$  there exists a 1-point  $a_k$  of  $f_k$  such that  $a_k \rightarrow z_0$  and if  $M_k$  is the line orthogonal to  $L$  which intersects  $L$  at  $a_k$ , then  $|f_k(z)| \neq 1$  for  $z \in M_k \cap \overline{D}(z_0, r) \setminus \{a_k\}$ .*

For large  $k$  this yields that  $|f_k(z)| > 1$  for  $z \in M_k \cap D^+ \cap \overline{D}(z_0, r)$  and  $|f_k(z)| < 1$  for  $z \in M_k \cap D^- \cap \overline{D}(z_0, r)$ , or vice versa.

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## 2 Preliminaries

The lemma of Zalcman already mentioned in the introduction is the following.

**Lemma 2.1. (Zalcman's Lemma)** *Let  $\mathcal{F}$  be a family of functions meromorphic in a domain  $D$  in  $\mathbb{C}$ . Then  $\mathcal{F}$  is not normal at a point  $z_0 \in D$  if and only if there exist*

- (i) *points  $z_k \in D$  with  $z_k \rightarrow z_0$ ,*
- (ii) *positive numbers  $\rho_k$  with  $\rho_k \rightarrow 0$ ,*
- (iii) *functions  $f_k \in \mathcal{F}$*

*such that*

$$f_k(z_k + \rho_k z) \rightarrow g(z)$$

*locally uniformly in  $\mathbb{C}$  with respect to the spherical metric, where  $g$  is a non-constant meromorphic function in  $\mathbb{C}$ .*

In the proof (see also [1, Section 4] or [16, p. 217f] besides [15]) one considers the spherical derivative

$$g_k^\#(z) = \frac{|g_k'(z)|}{1 + |g_k(z)|^2}$$

of the function  $g_k$  defined by

$$g_k(z) = f_k(z_k + \rho_k z) \tag{2.1}$$

and shows that for suitably chosen  $f_k$ ,  $z_k$ ,  $\rho_k$  and  $R_k$  with  $R_k \rightarrow \infty$  we have  $g_k^\#(0) = 1$  as well as

$$g_k^\#(z) \leq 1 + o(1) \quad \text{for } |z| \leq R_k \text{ as } k \rightarrow \infty.$$

Marty's theorem then implies that  $(g_k)$  has a locally convergent subsequence.

The following addendum to Lemma 2.1 was proved in [3, Lemma 2.2].

**Lemma 2.2.** *Let  $t_0 > 0$  and  $\varphi: [t_0, \infty) \rightarrow (0, \infty)$  be a non-decreasing function such that  $\varphi(t)/t \rightarrow 0$  as  $t \rightarrow \infty$  and*

$$\int_{t_0}^{\infty} \frac{dt}{t\varphi(t)} < \infty.$$

Then one may choose  $z_k$ ,  $\varrho_k$  and  $f_k$  in Zalcman's Lemma 2.1 such that

$$R_k := \frac{1}{\varrho_k \varphi(1/\varrho_k)} \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

and the functions  $g_k$  given by (2.1) are defined in the disks  $D(0, R_k)$  and satisfy

$$g_k^\#(z) \leq 1 + \frac{|z|}{R_k} \quad \text{for } |z| < R_k. \quad (2.2)$$

The next lemma is standard [12, Proposition 1.10].

**Lemma 2.3.** *Let  $\Omega$  be a convex domain and let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic. If  $\operatorname{Re} f'(z) > 0$  for all  $z \in \Omega$ , then  $f$  is univalent.*

The following result can be found in [8, p. 112].

**Lemma 2.4.** *Let  $a \in \mathbb{C}$ ,  $r > 0$  and let  $f: D(a, r) \rightarrow \mathbb{C}$  be univalent. Then*

$$\left| \arg \left( \frac{f(z) - f(a)}{f'(a)(z - a)} \right) \right| \leq \log \frac{1 + \frac{|z - a|}{r}}{1 - \frac{|z - a|}{r}}$$

for  $z \in D(a, r)$ .

The result is stated in [8] for the special case that  $a = 0$ ,  $r = 1$ ,  $f(0) = 0$  and  $f'(0) = 1$ , but the version given above follows directly from this special case.

*Proof of Proposition 1.1.* We recall some arguments of the proof of Theorem D in [3] and then describe the additional arguments that have to be made.

As in [3] we may assume that  $L = \mathbb{R}$  and we use Zalcman's Lemma 2.1 as well as Lemma 2.2, applied with  $\varphi(t) = (\log t)^2$ , to obtain a sequence  $(z_k)$  tending to  $z_0$  and a sequence  $(\rho_k)$  tending to 0 such that

$$R_k := \frac{1}{\varrho_k (\log \varrho_k)^2} \rightarrow \infty,$$

and the function  $g_k$  given by (2.1) is defined in the disk  $D(0, R_k)$  and satisfies (2.2) and  $g_k^\#(0) = 1$ .

As in [3, Proof of Theorem 1.3] we find a sequence  $(b_k)$  of 1-points of  $g$  such that

$$g_k(z) = \exp(c_k(z - b_k) + \delta_k(z)),$$

where (see [3, (3.4) and (3.5)])

$$|c_k + 2i| \leq \frac{C}{R_k} \quad \text{or} \quad |c_k - 2i| \leq \frac{C}{R_k} \quad (2.3)$$

with some constant  $C$  and (see [3, (2.22)])

$$|\delta_k(z)| \leq 2^7 \frac{|z - b_k|^2}{R_k} \quad \text{for } |z - b_k| \leq \frac{1}{16} R_k. \quad (2.4)$$

Without loss of generality we may assume that the first alternative holds in (2.3).

We put

$$h_k(z) = c_k(z - b_k) + \delta_k(z)$$

so that  $g_k(z) = \exp h_k(z)$ . We will show that  $h_k$  is univalent in  $D(b_k, 2s_k)$  where  $s_k = 2^{-11} R_k$ . In order to do so we note that for  $|z - b_k| \leq 2s_k$  we have

$$|\delta'_k(z)| = \frac{1}{2\pi} \left| \int_{|\zeta - b_k| = 4s_k} \frac{\delta_k(\zeta)}{(z - \zeta)^2} d\zeta \right| \leq 4s_k \frac{1}{(2s_k)^2} \max_{|\zeta - b_k| = 4s_k} |\delta_k(\zeta)|.$$

Since  $4s_k = 2^{-9} R_k < R_k/16$  we may apply (2.4) to estimate the maximum on the right hand side and obtain

$$|\delta'_k(z)| \leq \frac{1}{s_k} 2^7 \frac{(4s_k)^2}{R_k} = 1 \quad \text{for } |z - b_k| \leq 2s_k.$$

Thus, since we assumed that the first alternative holds in (2.3),

$$\operatorname{Re}(ih'_k(z)) = \operatorname{Re}(ic_k + i\delta'_k(z)) \geq 2 - \frac{C}{R_k} - 1 > 0$$

for  $z \in D(b_k, 2s_k)$  if  $k$  is sufficiently large. Lemma 2.3 implies that  $ih_k$  and hence  $h_k$  are univalent in this disk. Since  $h_k(b_k) = 0$  and, by (2.4),  $\delta'_k(b_k) = 0$  and thus  $h'_k(b_k) = c_k$ , Lemma 2.4 now yields that if  $z \in \overline{D}(b_k, s_k)$ , then

$$\left| \arg \left( \frac{h_k(z)}{c_k(z - b_k)} \right) \right| \leq \log 3.$$

For  $t \in \mathbb{R}$  with  $0 < |t| \leq s_k$  we thus have

$$\left| \arg \left( \frac{h_k(b_k + it)}{ic_k t} \right) \right| \leq \log 3.$$

Since we assumed that the first alternative holds in (2.3), this implies for large  $k$  that

$$|\arg(h_k(b_k + it))| \leq \log 3 + \arcsin \left( \frac{C}{2R_k} \right) < \frac{1}{2}\pi \quad \text{for } 0 < t \leq s_k$$

while

$$|\arg(h_k(b_k + it)) - \pi| < \frac{1}{2}\pi \quad \text{for } -s_k \leq t < 0.$$

Hence

$$\operatorname{Re}(h_k(b_k + it)) \begin{cases} > 0 & \text{if } 0 < t \leq s_k, \\ < 0 & \text{if } -s_k \leq t < 0, \end{cases}$$

so that

$$|g_k(b_k + it)| = \exp(\operatorname{Re}(h_k(b_k + it))) \begin{cases} > 1 & \text{if } 0 < t \leq s_k, \\ < 1 & \text{if } -s_k \leq t < 0. \end{cases} \quad (2.5)$$

As in [3, (3.2), (3.6) and (3.7)] we put

$$a_k = z_k + \rho_k b_k, \quad u_k = b_k + is_k = b_k + i2^{-11}R_k \quad \text{and} \quad \alpha_k = z_k + \rho_k u_k.$$

By (2.1) and (2.5) we have  $|f_k(z)| > 1$  for  $z$  in the line segment  $(a_k, \alpha_k]$ . Choose  $d > r$  such that  $\overline{D}(z_0, d) \in D$ . We put  $\beta_k = z_k + id$ . Then  $\beta_k \in D^+$  for large  $k$  and as in [3] we can use Landau's theorem to show that we also have  $|f_k(z)| > 1$  for  $z \in [\alpha_k, \beta_k]$ . Altogether thus  $|f_k(z)| > 1$  for  $z \in (a_k, \beta_k]$  and hence for  $z \in M_k \cap D^+ \cap \overline{D}(z_0, r)$  and large  $k$ . Analogously,  $|f_k(z)| < 1$  for  $z \in M_k \cap D^- \cap \overline{D}(z_0, r)$  and large  $k$ .  $\square$

**Lemma 2.5.** *Let  $0 < \alpha < \pi$  and  $\alpha < \beta < 2\pi - \alpha$ . Let  $u: \mathbb{D} \rightarrow [-\infty, \infty)$  be a subharmonic function which is harmonic in  $\mathbb{D} \setminus \{re^{i\beta}: 0 \leq r < 1\}$ . Suppose that  $u(z) > 0$  for  $|\arg z| < \alpha$  while  $u(z) \leq 0$  for  $\alpha \leq |\arg z| \leq \pi$ . Then  $\alpha \geq \pi/2$ . Moreover, if  $\alpha > \pi/2$ , then  $\beta = \pi$ . In addition, if  $u$  is harmonic in  $\mathbb{D} \setminus \{0\}$ , then  $\alpha = \pi/2$ .*

*Proof.* Let  $\gamma = 2\alpha/\pi$  and define  $v: \{z \in \mathbb{D}: \operatorname{Re} z \geq 0\} \rightarrow [-\infty, \infty)$  by  $v(z) = u(z^\gamma)$ . Then  $v(z) > 0$  for  $\operatorname{Re} z > 0$  while  $v(z) \leq 0$  for  $\operatorname{Re} z = 0$ . In fact,  $v(z) = 0$  for  $\operatorname{Re} z = 0$  by upper semicontinuity. We have  $v = \operatorname{Re} f$  for some function  $f$  holomorphic in  $\{z \in \mathbb{D}: \operatorname{Re} z > 0\}$ . By the Schwarz reflection principle  $f$  extends to a function holomorphic in  $\mathbb{D}$ . Hence  $f$  has a power series expansion  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  convergent in  $\mathbb{D}$ . With  $\delta = 1/\gamma$  we thus have

$$u(z) = v(z^{1/\gamma}) = \operatorname{Re} f(z^\delta) = \operatorname{Re} \left( \sum_{k=0}^{\infty} a_k z^{k\delta} \right)$$

for  $z \in \mathbb{D} \setminus \{re^{i\beta}: 0 \leq r < 1\}$ , meaning that

$$u(re^{i\theta}) = \operatorname{Re} \left( \sum_{k=0}^{\infty} a_k r^\delta e^{ik\delta\theta} \right)$$

for  $0 < r < 1$  and  $\beta - 2\pi < \theta < \beta$ .

Since  $\operatorname{Re} f(z) = v(z) > 0$  for  $\operatorname{Re} z > 0$  and  $\operatorname{Re} f(z) = 0$  for  $\operatorname{Re} z = 0$  we find that  $\operatorname{Re} a_0 = 0$  and  $a_1 > 0$ . It follows that

$$u(re^{i\theta}) = a_1 r^\delta \cos(\delta\theta) + \mathcal{O}(r^{2\delta})$$

as  $r \rightarrow 0$ , uniformly for  $\beta - 2\pi < \theta < \beta$ . We may assume that  $\beta \geq \pi$ . The condition that  $u(re^{i\theta}) \leq 0$  for  $\alpha \leq \theta < \beta$  then implies that  $\delta\pi \leq \delta\beta \leq 3\pi/2$  so that  $\delta \leq 3/2$ . Suppose that  $\delta \neq 1$ . Since  $u$  is subharmonic and a connected set containing more than one point is non-thin at every point of its closure [13, Theorem 3.8.3], we have  $u(re^{i\beta}) = u(re^{i(\beta-2\pi)})$  and thus  $\cos(\delta\beta) = \cos(\delta(\beta-2\pi))$ . This yields that  $\beta = \pi$ . Since  $u$  is subharmonic, we also have

$$\begin{aligned} 0 = u(0) &\leq \frac{1}{2\pi} \int_{\beta-2\pi}^{\beta} u(re^{i\theta}) d\theta \\ &= a_1 r^\delta \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\delta\theta) d\theta + \mathcal{O}(r^{2\delta}) \\ &= a_1 r^\delta \frac{1}{\delta\pi} \sin(\delta\pi) + \mathcal{O}(r^{2\delta}). \end{aligned}$$

Hence  $\sin(\delta\pi) \geq 0$ . Since  $\delta \leq 3/2$  and since we assumed that  $\delta \neq 1$  this implies that  $\delta < 1$ . Overall thus  $\delta \leq 1$  so that  $\alpha = \gamma\pi/2 = \pi/(2\delta) \geq \pi/2$ , and if  $\alpha > \pi/2$  so that  $\delta < 1$ , then  $\beta = \pi$ . Finally,  $u$  can be harmonic only if  $\delta = 1$ , which means that  $\alpha = \pi/2$ .  $\square$



For a bounded domain  $G$ , a point  $z \in G$  and a compact subset  $A$  of  $\partial G$  let  $\omega(z, A, G)$  denote the harmonic measure of  $A$  at a point  $z \in G$ ; see, e.g., [13, §4.3]. It is the solution of the Dirichlet problem for the characteristic function  $\chi_A$  of  $A$  on the boundary of  $G$ . Thus

$$\omega(z, A, G) = \sup_u u(z), \quad (2.6)$$

where the supremum is taken over all functions  $u$  subharmonic in  $G$  which satisfy  $\limsup_{z \rightarrow \zeta} u(z) \leq \chi_A(\zeta)$  for all  $\zeta \in \partial G$ .

**Lemma 2.6.** *Let  $G$  and  $H$  be bounded domains and let  $A \subset \partial G$  and  $B \subset \partial H$  be compact. If  $G \subset H$  and  $A \supset \partial G \cap (H \cup B)$ , then  $\omega(z, A, G) \geq \omega(z, B, H)$  for all  $z \in G$ .*

*Proof.* Let  $\zeta \in \partial G \setminus A$ . Then  $\zeta \in \partial G \setminus (H \cup B)$  and thus  $\zeta \in \partial H \setminus B$ . Hence  $\lim_{z \rightarrow \zeta} \omega(z, B, H) = 0$ . We conclude that  $\limsup_{z \rightarrow \zeta} \omega(z, B, H) \leq \chi_A(\zeta)$  for all  $\zeta \in \partial G$ . Since  $u(z) = \omega(z, B, H)$  is an admissible choice in (2.6), the conclusion follows.  $\square$

### 3 Proof of Theorem 1.1

Without loss of generality we may assume that  $L_1$  and  $L_{-1}$  are symmetric with respect to the real axis and that  $L_1$  is in the upper half-plane. Thus  $L_{\pm 1} = \{re^{\pm i\alpha} : r \geq 0\}$  for some  $\alpha \in (0, \pi)$ . We may also assume that  $L_0 = \{re^{i\beta} : r \geq 0\}$  where  $\alpha < \beta < 2\pi - \alpha$ . We define

$$\begin{aligned} S &= \{re^{i\theta} : 0 < r < 1, |\theta| < \alpha\}, \\ S^+ &= \{re^{i\theta} : 0 < r < 1, \alpha < \theta < \beta\}, \\ S^- &= \{re^{i\theta} : 0 < r < 1, \beta < \theta < 2\pi - \alpha\}. \end{aligned}$$

By Montel's theorem,  $\mathcal{F}$  is normal in  $\mathbb{D} \setminus (L_1 \cup L_0 \cup L_{-1})$ . Thus we only have to prove that  $\mathcal{F}$  is normal on  $\mathbb{D} \cap L_j \setminus \{0\}$  for  $j \in \{0, \pm 1\}$ .

First we prove that  $\mathcal{F}$  is normal on  $\mathbb{D} \cap L_0 \setminus \{0\}$ . In order to do so, suppose that  $\mathcal{F}$  is not normal at some point  $z_0 \in L_0 \setminus \{0\}$ . Applying Theorem D to the family  $\{1 - f : f \in \mathcal{F}\}$  we see that there exists a sequence  $(f_k)$  in  $\mathcal{F}$  such that either  $f_k|_{S^+} \rightarrow 1$  and  $f_k|_{S^-} \rightarrow \infty$  or  $f_k|_{S^+} \rightarrow \infty$  and  $f_k|_{S^-} \rightarrow 1$ . Without loss of generality we may assume that the first alternative holds. If  $(f_k)$  is not normal at some  $z_1 \in L_1 \setminus \{0\}$ , then – again by Theorem D –

there exists a subsequence of  $(f_k)$  which tends to 0 or to  $\infty$  in  $S^+$ . This is incompatible with our previous assumption that  $f_k|_{S^+} \rightarrow 1$ . Hence  $(f_k)$  is normal on  $\mathbb{D} \cap L_1 \setminus \{0\}$ . We conclude that  $(f_k)$  is normal in  $S^+ \cup S \cup L_1 \setminus \{0\}$  and hence that  $f_k|_{S^+ \cup S \cup L_1 \setminus \{0\}} \rightarrow 1$ . In particular,  $f_k|_S \rightarrow 1$ . On the other hand,  $f_k|_{S^-} \rightarrow \infty$ . Hence  $(f_k)$  is not normal at any point of  $L_{-1}$ . Since  $f_k|_{S^-} \rightarrow \infty$  we can now deduce from Theorem D that  $f_k|_S \rightarrow 0$ . This contradicts our previous finding that  $f_k|_S \rightarrow 1$ . Thus  $\mathcal{F}$  is normal on  $L_0 \setminus \{0\}$ . Putting  $T = S^+ \cup S^- \cup L_0 \setminus \{0\}$  we conclude that  $\mathcal{F}$  is normal in  $T$ .

Suppose now that  $\mathcal{F}$  is not normal at some point  $z_0 \in \mathbb{D} \setminus \{0\}$ . It follows that  $z_0 \in (L_1 \cup L_{-1}) \setminus \{0\}$ . Without loss of generality we may assume that  $z_0 \in L_1 \setminus \{0\}$ . Theorem D implies that there exists a sequence  $(f_k)$  in  $\mathcal{F}$  such that either  $f_k|_S \rightarrow \infty$  and  $f_k|_T \rightarrow 0$  or  $f_k|_S \rightarrow 0$  and  $f_k|_T \rightarrow \infty$ . In particular we see that the sequence  $(f_k)$  is not normal at any point of  $L_1 \cup L_{-1}$ . We begin by considering the case that the first of the two above possibilities holds; that is,  $f_k|_S \rightarrow \infty$  and  $f_k|_T \rightarrow 0$ .

We define  $u_k: \mathbb{D} \rightarrow [-\infty, \infty)$ ,

$$u_k(z) = \frac{\log |f_k(z)|}{\log |f_k(\frac{1}{2})|}.$$

We will prove that the sequence  $(u_k)$  is locally bounded in  $\mathbb{D}$ . Once this is known, we can deduce (see, for example, [9, Theorems 4.1.8, 4.1.9] or [10, Theorems 3.2.12, 3.2.13]) that some subsequence of  $(u_k)$  converges to a limit function  $u$  which is subharmonic in  $\mathbb{D}$  and harmonic in  $\mathbb{D} \setminus L_0$ . Moreover,  $u(z) > 0$  for  $z \in S$  while  $u(z) \leq 0$  for  $z \in \mathbb{D} \setminus S$ .

Lemma 2.5 now implies that  $\alpha \geq \pi/2$  and that  $\beta = \pi$  if  $\alpha > \pi/2$ . The conclusion then follows since if  $\alpha = \pi/2$ , then  $\angle(L_{-1}, L_1) = 2\alpha = \pi$ , while if  $\alpha > \pi/2$  and thus  $\beta = \pi$ , then  $\angle(L_0, L_1) = \beta - \alpha = \pi - \alpha < \pi/2$  and  $\angle(L_0, L_{-1}) = 2\pi - \alpha - \beta = \pi - \alpha = \angle(L_0, L_1)$ .

In order to prove that  $(u_k)$  is locally bounded, let  $0 < \varepsilon < 1/8$ . Proposition 1.1 yields that, for sufficiently large  $k$ , there exist simple closed curves  $\Gamma_k$  in  $\{z: 1 - \varepsilon/2 < |z| < 1 - \varepsilon/4\}$  and  $\gamma_k$  in  $\{z: \varepsilon/2 < |z| < \varepsilon\}$  such that  $|f_k(z)| > 1$  for  $z \in (\Gamma_k \cup \gamma_k) \cap S$  while  $|f_k(z)| < 1$  for  $z \in (\Gamma_k \cup \gamma_k) \cap T$ . Moreover, both  $\Gamma_k$  and  $\gamma_k$  surround 0 and they intersect  $L_1$  and  $L_{-1}$  only once, at 1-points of  $f_k$ . In fact, these curves can be constructed by taking small segments orthogonal to  $L_1$  and  $L_{-1}$ , and connecting the endpoints of these segments within the intersection of  $S$  and  $T$  with the corresponding annuli.

Let  $D_k$  be the domain between  $\gamma_k$  and  $\Gamma_k$  and let  $X_k$  be the set of all

$z \in \overline{D_k}$  for which  $|f_k(z)| = 1$ . Then both  $X_k \cap \Gamma_k$  and  $X_k \cap \gamma_k$  consist of two 1-points of  $f_k$ . Let  $U_k$  be the component of  $D_k \setminus X_k$  which contains  $\frac{1}{2}$ . Next, for large  $k$  we have  $|f_k(z)| < 1$  for  $z \in L_0$  with  $\varepsilon/2 \leq |z| \leq 1 - \varepsilon/4$  and hence in particular for  $z \in L_0 \cap D_k$ . Let  $V_k$  be the component of  $D_k \setminus X_k$  which contains  $L_0 \cap D_k$ . Then, for large  $k$ , we have  $|f_k(z)| > 1$  for  $z \in U_k$  while  $|f_k(z)| < 1$  for  $z \in V_k$ .

We claim that  $D_k \setminus X_k = U_k \cup V_k$ . Indeed, let  $W$  be a component of  $D_k \setminus X_k$  which is different from  $U_k$  and  $V_k$ . Since  $(\Gamma_k \cup \gamma_k) \cap S \subset \partial U_k$  and  $(\Gamma_k \cup \gamma_k) \cap T \subset \partial V_k$  we have  $\partial W \subset X_k$  for large  $k$ . By the maximum principle, we thus have  $|f_k(z)| < 1$  for  $z \in W$ . The minimum principle now yields that  $W$  contains a zero of  $f_k$ . Hence  $W$  and thus  $\partial W$  intersect  $L_0 \cap \overline{D_k}$ , which is a contradiction for large  $k$ , since  $\partial W \subset X_k$  and thus  $|f_k(z)| = 1$  for  $z \in \partial W$ , but  $f_k|_{L_0 \cap \overline{D_k}} \rightarrow 0$ . Thus  $D_k \setminus X_k = U_k \cup V_k$  as claimed. We also conclude that  $X_k$  consists of two analytic curves  $\sigma_{1,k}$  and  $\sigma_{-1,k}$ , which are close to the rays  $L_1$  and  $L_{-1}$ .

We now prove that  $(u_k)$  is bounded in  $\overline{D}(0, 1 - \varepsilon)$ . In order to do so we choose  $c_k \in \partial D(0, 1 - \varepsilon)$  such that

$$u_k(c_k) = \max_{|z|=1-\varepsilon} u_k(z).$$

Clearly,  $c_k \in U_k$  for large  $k$ . For  $j \in \{1, 2, 3, 4\}$ , we put  $r_j = 1 - \varepsilon j/4$ . Thus  $|c_k| = 1 - \varepsilon = r_4$ . Similar to the curve  $\Gamma_k$  in  $\{z: r_2 < |z| < r_1\}$  there exists a closed curve  $\Gamma'_k$  in  $\{z: r_4 < |z| < r_3\}$  which surrounds 0 such that  $|f_k(z)| > 1$  for  $z \in \Gamma'_k \cap S$  while  $|f_k(z)| < 1$  for  $z \in \Gamma'_k \cap T$ . Thus  $\Gamma'_k \cap S \subset U_k$  and  $\Gamma'_k \cap T \subset V_k$ .

By the maximum principle, there exists a curve  $\xi_k$  in  $U_k$  which connects  $c_k$  with  $\partial \mathbb{D}$  and on which  $u_k$  is bigger than  $u_k(c_k)$ . Let  $\tau_k$  be a part of  $\xi_k$  which connects  $\partial D(0, r_3)$  with  $\partial D(0, r_2)$  and, except for its endpoints, is contained in  $\{z: r_3 < |z| < r_2\}$ ; see Figure 1. Then  $|u_k(z)| \geq u_k(c_k)$  for  $z \in \tau_k$ . Let  $e_{k,j}$  be the endpoint of  $\tau_k$  on  $\partial D(0, r_j)$ , for  $j \in \{2, 3\}$ . Without loss of generality we may assume that the distance of  $e_{k,3}$  to  $L_{-1}$  is less than or equal to the distance to  $L_1$ , which means that  $\text{Im } e_{k,3} \leq 0$ .

We define a domain  $G_k$  as follows; cf. Figure 1. If  $\tau_k$  does not intersect the segment  $\{re^{i(\alpha-\varepsilon)}: r_3 \leq r \leq r_2\}$ , let  $G_k$  be the domain bounded by the segments  $\{re^{-i(\alpha-\varepsilon)}: \frac{1}{4} \leq r \leq r_3\}$  and  $\{re^{i(\alpha-\varepsilon)}: \frac{1}{4} \leq r \leq r_2\}$ , the arc  $\{\frac{1}{4}e^{i\theta}: |\theta| \leq \alpha - \varepsilon\}$ , the arc of  $\partial D(0, r_3)$  that connects  $e_{k,3}$  and  $r_3e^{-i(\alpha-\varepsilon)}$  in  $\{r_3e^{i\theta}: |\theta| \leq \alpha + \varepsilon\}$ , the arc of  $\partial D(0, r_2)$  that connects  $e_{k,2}$  and  $r_2e^{i(\alpha-\varepsilon)}$  in  $\{r_2e^{i\theta}: |\theta| \leq \alpha + \varepsilon\}$ , and the curve  $\tau_k$ .

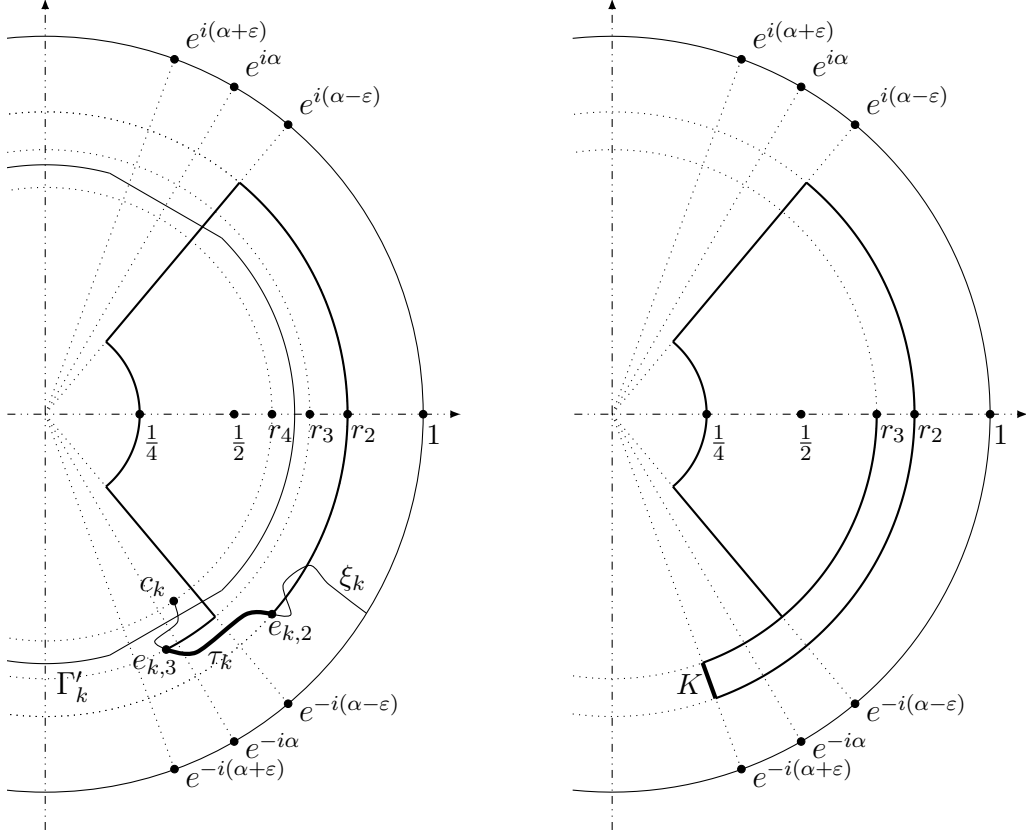


Figure 1: The curves  $\xi_k$ ,  $\tau_k$  and  $\Gamma'_k$  and the domains  $G_k$  (left) and  $H$  (right).

If  $\tau_k$  intersects the segment  $\{re^{i(\alpha-\epsilon)}: r_3 \leq r \leq r_2\}$ , let  $d_k$  denote the first point of intersection so that the part  $\tau'_k$  of  $\tau_k$  which is between  $e_{k,3}$  and  $d_k$  is contained in  $\{re^{i\theta}: r_3 < r < r_2, -\alpha - \epsilon < \theta < \alpha - \epsilon\}$ . We then define  $G_k$  as the domain bounded by the the curve  $\tau'_k$ , the segment  $\{re^{i(\alpha-\epsilon)}: \frac{1}{4} \leq r \leq |d_k|\}$  and – as before – the arc  $\{\frac{1}{4}e^{i\theta}: |\theta| \leq \alpha - \epsilon\}$ , the segment  $\{re^{-i(\alpha-\epsilon)}: \frac{1}{4} \leq r \leq r_3\}$  and the arc of  $\partial D(0, r_3)$  that connects  $e_{k,3}$  and  $r_3e^{-i(\alpha-\epsilon)}$  in  $\{r_3e^{i\theta}: |\theta| \leq \alpha + \epsilon\}$ .

We claim that  $G_k \subset U_k$  for large  $k$ . In order to prove this it suffices to prove that  $\partial G_k \subset U_k$ . We restrict to the case that  $\tau_k$  does not intersect the segment  $\{re^{i(\alpha-\epsilon)}: r_3 \leq r \leq r_2\}$ , since the other case is similar. First we note that the segments  $\{re^{-i(\alpha-\epsilon)}: \frac{1}{4} \leq r \leq r_3\}$  and  $\{re^{i(\alpha-\epsilon)}: \frac{1}{4} \leq r \leq r_2\}$  as well as the arc  $\{\frac{1}{4}e^{i\theta}: |\theta| \leq \alpha - \epsilon\}$  are clearly in  $U_k$  for large  $k$ , since  $f_k|_S \rightarrow \infty$  as  $k \rightarrow \infty$ . Since  $\xi_k$  is in  $U_k$  and  $\tau_k$  is a subcurve of  $\xi_k$ , the curve  $\tau_k$  is also

in  $U_k$ .

It remains to show that the arc of  $\partial D(0, r_3)$  that connects  $e_{k,3}$  and  $r_3 e^{-i(\alpha-\varepsilon)}$  is in  $U_k$ . If this is not the case, then this arc must intersect  $\partial U_k$  and thus must intersect the curve  $\sigma_{-1,k}$ , which constitutes the part of  $\partial U_k$  that is near  $L_{-1}$ . Since  $\xi_k$  is in  $U_k$  this means that  $\sigma_{-1,k}$  must also intersect  $\Gamma'_k$ , at a point between the intersections of  $\Gamma'_k$  with  $\xi_k$  and with the positive real axis. But this part of  $\Gamma'_k$  is in  $U_k$ , since  $\Gamma'_k \cap S \subset U_k$  and  $\Gamma'_k \cap T \subset V_k$ . Hence  $\sigma_{-1,k}$  does not intersect the arc connecting  $e_{k,3}$  and  $r_3 e^{-i(\alpha-\varepsilon)}$  and thus this arc is in  $U_k$ . Similarly, we see that the arc of  $\partial D(0, r_2)$  that connects  $e_{k,2}$  and  $r_2 e^{i(\alpha-\varepsilon)}$  is in  $U_k$ . Altogether thus  $G_k \subset U_k$  for large  $k$ .

Let

$$H = \{r e^{i\theta} : \frac{1}{4} < r < r_3, |\theta| < \alpha - \varepsilon\} \cup \{r_3 e^{i\theta} : 0 < \theta < \alpha - \varepsilon\} \\ \cup \{r e^{i\theta} : r_3 < r < r_2, -\alpha - \varepsilon < \theta < \alpha - \varepsilon\}$$

and let  $K = \{r e^{i(-\alpha-\varepsilon)} : r_3 \leq r \leq r_2\} \subset \partial H$ ; see Figure 1. Then  $G_k \subset H$ .

It follows from Lemma 2.6 and the configuration of the domains  $G_k$  and  $H$  that  $\omega(z, K, H) \leq \omega(z, \tau_k, G_k)$  for  $z \in G_k$ . In particular,  $\omega(\frac{1}{2}, K, H) \leq \omega(\frac{1}{2}, \tau_k, G_k)$ . On the other hand, since  $G_k \subset U_k$  and thus  $u_k(z) \geq 0$  for  $z \in \partial G_k$  while  $u_k(z) \geq u_k(c_k)$  for  $z \in \tau_k$  it follows that  $u_k(z) \geq u_k(c_k) \omega(z, \tau_k, G_k)$  for  $z \in G_k$ . Altogether we thus have

$$1 = u_k(\frac{1}{2}) \geq u_k(c_k) \omega(\frac{1}{2}, \tau_k, G_k) \geq u_k(c_k) \omega(\frac{1}{2}, K, H).$$

It follows that

$$\max_{|z|=1-\varepsilon} u_k(z) = u_k(c_k) \leq \frac{1}{\omega(\frac{1}{2}, K, H)}$$

so that  $(u_k)$  is bounded in  $\overline{D}(0, 1 - \varepsilon)$ . Since  $\varepsilon > 0$  can be taken arbitrarily small, we conclude that  $(u_k)$  is locally bounded in  $\mathbb{D}$ . This completes the proof in the case that  $f_k|_S \rightarrow \infty$  and  $f_k|_T \rightarrow 0$ .

It remains to consider the case that  $f_k|_S \rightarrow 0$  and  $f_k|_T \rightarrow \infty$ . Since  $L_0 \setminus \{0\} \subset T$  we conclude that if  $\varepsilon > 0$ , then, for large  $k$ , the function  $f_k$  has no zeros in  $\{z : \varepsilon < |z| < 1 - \varepsilon\}$ . Thus  $u_k$  is harmonic there. As before we see that the sequence  $(u_k)$  is locally bounded so that some subsequence of it converges to a limit  $u$  which is subharmonic in  $\mathbb{D}$ . But now  $u$  is actually harmonic in  $\mathbb{D} \setminus \{0\}$ . The conclusion follows again from Lemma 2.5 which yields that  $\alpha = \pi/2$  and hence  $\angle(L_{-1}, L_1) = 2\alpha = \pi$ .  $\square$

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