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Lebesgue measure of Julia sets and escaping sets of certain entire functions

by

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Abstract. We give criteria for the escaping set and the Julia set of an entire function to have positive measure. The results are applied to Poincaré functions of semihyperbolic polynomials and to the Weierstraß σ-function.

1. Introduction and results. Let f be a non-linear entire function and let f^n denote the nth iterate of f. The *Fatou set* $F(f)$ is the set of all $z \in \mathbb{C}$ where the f^n form a normal family; its complement $J(f)$ is the *Julia set*. The *escaping set* $I(f)$ is the set of all $z \in \mathbb{C}$ such that $f^n(z) \to \infty$. By a result of Eremenko [17] we have $J(f) = \partial I(f)$. These sets play a key role in complex dynamics; see [5] and [32] for an introduction to the dynamics of transcendental entire functions.

A result of McMullen [27, Theorem 1.1] says that $J(\sin(\alpha z + \beta))$ has positive Lebesgue measure for all $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. In his proof McMullen actually showed that $I(\sin(\alpha z + \beta))$ has positive measure and then noted that $I(f) \subset J(f)$ for $f(z) = \sin(\alpha z + \beta)$. It was later shown by Eremenko and Lyubich [18, Theorem 1] that $I(f) \subset J(f)$ holds more generally for all transcendental entire functions f for which the set of critical and asymptotic values is bounded. The class of functions with the latter property, denoted by $\mathcal{B}$, is now called the *Eremenko–Lyubich class* and has received much attention in transcendental dynamics.

McMullen's result on the measure of $J(\sin(\alpha z + \beta))$ has been extended to various classes of functions in [2, 7, 34]. In this paper we give another criterion for the Julia set or escaping set of an entire function to have positive measure. Perhaps more importantly, we do so by a method different from

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those employed in the papers mentioned. Here we only note that distortion estimates, coming from Koebe's theorem or related results, do not occur in the proofs of our main results.

The *order* $\rho(f)$ of an entire function f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ denotes the maximum modulus of f. The area (i.e., the two-dimensional Lebesgue measure) of a measurable subset A of $\mathbb{C}$ is denoted by $\operatorname{area} A$. The *logarithmic area* of A is defined by

$$\operatorname{logarea} A = \int_A \frac{dx\,dy}{|z|^2}.$$

It occurs in transcendental dynamics in [12, p. 34] and [16, p. 575]; in the latter paper the term "cylindrical area" is used.

We are interested in behavior near ∞, and therefore instead of the logarithmic area of a set A we will usually consider the logarithmic area of $A \cap \varDelta$ where $\varDelta = \{z\colon |z| \geq 1\}$.

THEOREM 1.1. *Let f be an entire function of finite order. Let $\varepsilon > 0$ and suppose that*

$$\operatorname{logarea}\left\{ z \in \varDelta\colon \left|\frac{zf'(z)}{f(z)}\right| < |z|^{\rho(f)/2+\varepsilon} \text{ or } |f(z)| < (1+\varepsilon)|z| \right\} < \infty. \tag{1.1}$$

Then

$$\operatorname{logarea}(\varDelta \setminus I(f)) < \infty. \tag{1.2}$$

In particular, $\operatorname{area} I(f) > 0$.

If, in addition, $F(f)$ does not have a multiply connected component, then

$$\operatorname{logarea}\big(\varDelta \setminus (I(f) \cap J(f))\big) < \infty \tag{1.3}$$

and thus $\operatorname{area}(I(f) \cap J(f)) > 0$.

We consider the example $f(z) = \sin z$. Then $\rho(f) = 1$,

$$|f(z)| \geq \tfrac{1}{2}(e^{|\operatorname{Im} z|} - 1) \geq 2|z| \quad \text{if } |\operatorname{Im} z| \geq \log(4|z|+1)$$

and

$$\left|\frac{zf'(z)}{f(z)}\right| = |z \cot z| \geq \tfrac{1}{2}|z| \geq |z|^{3/4} \quad \text{if } |\operatorname{Im} z| \geq 1 \text{ and } |z| \geq 16.$$

It is easy to see that the set

$$\{z \in \varDelta\colon |\operatorname{Im} z| < \log(4|z|+1)\}$$

has finite logarithmic area. Thus Theorem 1.1 implies that $I(\sin z)$ has positive measure. Also, a result of Baker [3, p. 565] says that $F(f)$ does not have multiply connected components if f is bounded on a curve tending to ∞. Thus we also find that $J(\sin z)$ has positive measure.

With the same method we could also treat the functions $\sin(\alpha z + \beta)$ considered by McMullen and thus obtain another proof of his result that the Julia set of these functions has positive measure. More generally, the hypothesis of Theorem 1.1 is satisfied for example if $f(z) = P(z)\sin(\alpha z + \beta)$ with a polynomial P. Moreover, the result of Baker just mentioned holds more generally if $\log|f(z)| = \mathcal{O}(\log|z|)$ for z on some curve tending to ∞ [5, Theorem 10]. We thus find that $J(f)$ has positive area for such f. Note that f is not in the Eremenko–Lyubich class if P is non-constant.

Theorem 1.1 also applies to the functions

$$f(z) = \sum_{k=0}^{n} a_k \exp(b_k z) \tag{1.4}$$

considered in [7, 34]. Here the a_k and b_k are non-zero constants satisfying $\arg b_k < \arg b_{k+1} \le \arg b_k + \pi$ for $0 \le k \le n-1$ and $\arg b_0 \le \arg b_n - \pi$, with arguments chosen in $[0, 2\pi)$. More generally, one can assume that the a_k are polynomials that do not vanish identically.

A subset $A(f)$ of $I(f)$ called the *fast escaping set* was introduced in [8]. It also plays an important role in transcendental dynamics (see, e.g., [30, 31]). In order to define it, let $M^n(r, f)$ denote the nth iterate of $M(r, f)$ with respect to the first variable; that is,

$$M^1(r, f) = M(r, f) \quad \text{and} \quad M^n(r, f) = M(M^{n-1}(r, f), f) \quad \text{for } n \ge 2.$$

We note that there exists $R > 0$ such that $M(r, f) > r$ for $r \ge R$. With such a value of R the fast escaping set $A(f)$ is defined as the set of all $z \in \mathbb{C}$ for which there exists $L \in \mathbb{N}$ such that $|f^n(z)| \ge M^{n-L}(R, f)$ for $n > L$. The definition is independent of the value of R.

THEOREM 1.2. *Let f be an entire function of finite order. Let $\varepsilon > 0$ and suppose that*

$$\operatorname{logarea}\left\{ z \in \Delta \colon \left| \frac{zf'(z)}{f(z)} \right| < |z|^{\rho(f)/2+\varepsilon} \ \text{or}\ |f(z)| < \exp(|z|^{\varepsilon}) \right\} < \infty. \tag{1.5}$$

Then the conclusion of Theorem 1.1 *holds with $I(f)$ replaced by $A(f)$.*

The arguments used to show that the hypotheses of Theorem 1.1 are satisfied for $f(z) = \sin z$, or more generally for $f(z) = P(z)\sin(\alpha z + \beta)$ with a polynomial P and the functions given by (1.4), can easily be modified to show that the hypotheses of Theorem 1.2 hold for these functions as well.

We will deduce the above theorems from a more general result which does not involve the order. To state this result, for an entire function f and

$a \in \mathbb{C}$ denote by $n(r, a)$ the number of a-points of f in $\{z\colon |z| \le r\}$. Let

$$n(r) = \max_{a\in\mathbb{C}} n(r, a).$$

THEOREM 1.3. *Let f be a transcendental entire function satisfying*

$$\text{logarea}\left\{z \in \Delta\colon \left|\frac{zf'(z)}{f(z)}\right| < n(|z|)^{1/2+\varepsilon} \text{ or } |f(z)| < (1+\varepsilon)|z|\right\} < \infty \tag{1.6}$$

for some $\varepsilon > 0$. Then (1.2) *holds. In particular,* area $I(f) > 0$.

If, in addition, $F(f)$ does not have a multiply connected component, then (1.3) *also holds and thus* area$(I(f) \cap J(f)) > 0$.

In the results above the hypotheses concern both $|f(z)|$ and $|zf'(z)/f(z)|$. If $f \in \mathcal{B}$, then $|zf'(z)/f(z)|$ can be bounded in terms of $|f(z)|$. In fact, we have the following result which follows directly from [18, Lemma 1] (see [6, Lemma 2]).

PROPOSITION 1.1. *Let $f \in \mathcal{B}$. Then there exists $R > 0$ such that*

$$\left|\frac{zf'(z)}{f(z)}\right| \ge \frac{1}{4\pi}\log\frac{|f(z)|}{R} \qquad \text{for all } z \in \mathbb{C} \text{ with } f(z) \ne 0.$$

The following result is a simple consequence of Theorem 1.2 and Proposition 1.1.

THEOREM 1.4. *Let $f \in \mathcal{B}$ be of finite order. Suppose that*

$$\text{logarea}\{z \in \Delta\colon |f(z)| < \exp(|z|^{\rho(f)/2+\varepsilon})\} < \infty \tag{1.7}$$

for some $\varepsilon > 0$. Then

$$\text{logarea}(\Delta \setminus A(f)) < \infty.$$

We recall that by the result of Eremenko and Lyubich already mentioned we have $A(f) \subset I(f) \subset J(f)$ for $f \in \mathcal{B}$. Under the hypotheses of Theorem 1.4 we thus see that, in particular, logarea$(\Delta \setminus J(f)) < \infty$ and hence area $J(f) > 0$.

As an example which Theorems 1.2 and 1.4 apply to, we consider certain Poincaré functions. We recall the definition of these functions: Let p be a polynomial of degree $d \ge 2$ and let z_0 be a repelling fixed point of p; that is, $p(z_0) = z_0$ and $\lambda := p'(z_0)$ satisfies $|\lambda| > 1$. Then Schröder's functional equation

$$f(\lambda z) = p(f(z)) \tag{1.8}$$

has a solution f which is holomorphic in a neighborhood of 0 and satisfies $f(0) = z_0$ [26, Theorem 8.2]. It can be normalized to satisfy $f'(0) = 1$. This solution f actually extends to a transcendental entire function which is called the *Poincaré function* of p at z_0 [26, Corollary 8.1]. It is well-known [36, Chapter II, Section III.8] that $\rho(f) = \log d/\log|\lambda|$. We note that the trigonometric functions arise as Poincaré functions of Chebyshev polynomials.

The dynamics of Poincaré functions have been studied in [16, Section 3] and [25]. It is known that $f \in \mathcal{B}$ if and only if the orbit $\{p^n(c)\colon n \in \mathbb{N}\}$ is bounded for every critical point c of p ([25, Proposition 4.2] or [16, Section 3.1]). The latter condition is satisfied if and only if $J(p)$ is connected [26, Theorem 9.5].

A polynomial p is called *semihyperbolic* if there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that if $z \in J(f)$, $n \in \mathbb{N}$ and V is a component of $p^{-n}(D(z,\varepsilon))$, then the degree of the proper map $p^n\colon V \to D(z,\varepsilon)$ is at most N. Here $D(z,\varepsilon)$ denotes the open disk of radius ε around z. The concept of semihyperbolicity was introduced by Carleson, Jones and Yoccoz [14], who gave various characterizations of it.

Theorem 1.5. *Let p be a semihyperbolic polynomial without attracting periodic points and let f be a Poincaré function of p. Then*

$$\operatorname{logarea}\big(\Delta \setminus (A(f) \cap J(f))\big) < \infty.$$

In particular, $\operatorname{area}(A(f) \cap J(f)) > 0$.

The *filled Julia set* $K(p)$ of a polynomial p is defined by

$$K(p) = \{z\colon p^n(z) \nrightarrow \infty\}.$$

We always have $J(p) \subset K(p)$. Semihyperbolic polynomials have no parabolic points and no Siegel disks. The hypothesis in Theorem 1.5 that p has no attracting periodic points is thus equivalent to $J(p) = K(p)$. The following result shows that if $J(p)$ is connected, then this hypothesis is also necessary.

Theorem 1.6. *Let p be a polynomial with connected Julia set and let f be a Poincaré function of p. If* $\operatorname{area} K(p) > 0$, *then* $\operatorname{area} I(f) = 0$.

Buff and Chéritat [13] have shown that there exist polynomials p with Julia sets of positive measure. These polynomials p may be chosen to satisfy $J(p) = K(p)$. Theorem 1.6 thus also shows that the hypothesis that p be semihyperbolic cannot be omitted in Theorem 1.5.

Theorem 1.6 is a simple consequence of a result of Eremenko and Lyubich [18, Theorem 7], using the fact noted above that $f \in \mathcal{B}$ if $J(p)$ is connected. This fact also simplifies the proof of Theorem 1.5 considerably if $J(p)$ is connected. We will thus deal with this special case first, and afterwards provide the additional arguments that have to be made in the general case.

As a second example where our results apply we consider the Weierstraß σ-function. We recall the definition, using the terminology as in [1, 23]. For $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ with $\omega_2/\omega_1 \notin \mathbb{R}$ we consider the lattice

$$\Omega = \{m\omega_1 + n\omega_2\colon m, n \in \mathbb{Z}\}.$$

Then

$$\sigma(z)=\sigma(z|\omega_1,\omega_2):=z\prod_{w\in\Omega\setminus\{0\}}\left(1-\frac{z}{w}\right)\exp\left(\frac{z}{w}+\frac{1}{2}\left(\frac{z}{w}\right)^2\right). \tag{1.9}$$

The Weierstraß ζ-function and $\wp$-function are defined by

$$\zeta(z)=\frac{\sigma'(z)}{\sigma(z)}\quad\text{and}\quad\wp(z)=-\zeta'(z).$$

Moreover, $\eta_1:=2\zeta(\omega_1/2)$.

It can be assumed without loss of generality that $\tau:=\omega_2/\omega_1$ satisfies $\operatorname{Im}\tau>0$. Since $\sigma(cz|c\omega_1,c\omega_2)=c\sigma(z|\omega_1,\omega_2)$ for every $c\in\mathbb{C}\setminus\{0\}$, it suffices to consider the case that $\omega_1=1$ and thus $\tau=\omega_2$.

The Nevanlinna deficiency $\delta(0,\sigma)$ was studied by Gol'dberg [19] and Korenkov [24]; see [20, 21] as a reference for Nevanlinna theory. The result of [24] says that $\delta(0,\sigma)=0$ if and only if

$$\operatorname{Re}\left(\frac{1}{\eta_1}\right)\geq\frac{\operatorname{Im}\tau}{2\pi}. \tag{1.10}$$

We note that the terminology used in [19, 24] is different, with $\eta_1=\zeta(1/2)$, but we have converted the result to the terminology of [1, 23] introduced above.

The set of all τ satisfying (1.10) is shown in Figure 1. Since [37, p. 8]

$$\eta_1=\pi^2\left(\frac{1}{3}-2\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}\right)=\frac{\pi^2}{3}\left(1-24e^{-2\pi\tau}+\mathcal{O}(e^{-4\pi\operatorname{Im}\tau})\right)$$

as $\operatorname{Im}\tau\to\infty$, the upper boundary of this set is very close (but not equal) to the line $\operatorname{Im}\tau=6/\pi$. Consequently, the other boundary components, which are images of the upper boundary under the modular group, are close to circles.

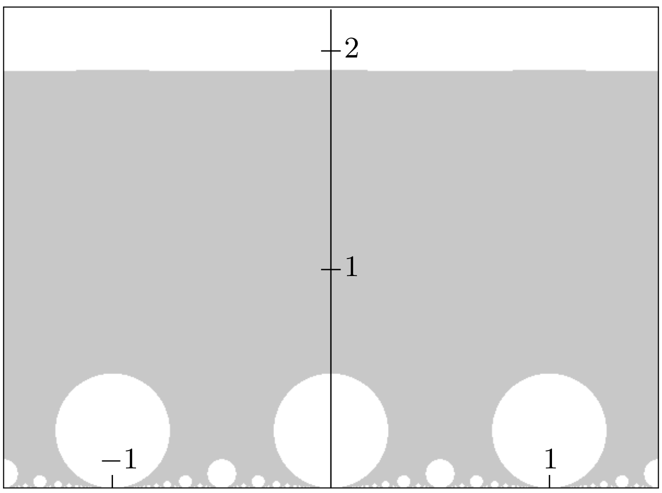

Fig. 1. The set of τ satisfying (1.10)

THEOREM 1.7. *Suppose that* (1.10) *holds. Then* $\operatorname{area}(J(\sigma)\cap A(\sigma))>0$.

The results in [19, 24] actually show that if (1.10) is not satisfied, then $\sigma(z)$ tends to 0 as $z\to\infty$ in some sector. This suggests that $\operatorname{area} I(\sigma)=0$ in this case.

2. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.3. Set

$$X=\left\{z\colon \left|\frac{zf'(z)}{f(z)}\right|\geq n(|z|)^{1/2+\varepsilon}\right\},\qquad Y=\{z\colon |f(z)|\geq(1+\varepsilon)|z|\},$$

and $W=\Delta\setminus(X\cap Y)$. Then (1.6) takes the form

$$\operatorname{logarea} W<\infty. \tag{2.1}$$

Note that points in $\mathbb{C}\setminus\{0\}$ which stay in $X\cap Y$ under iteration of f are contained in $I(f)$. In order to study the set of such points, we consider, for $k\in\mathbb{N}$, the sets

$$A_k=\{z\colon 2^k\leq n(|z|)<2^{k+1}\}.$$

For a measurable subset P of $\mathbb{C}$ we then find that

$$\begin{aligned}\operatorname{logarea}(P\cap X\cap A_k)&=\int\limits_{P\cap X\cap A_k}\frac{dx\,dy}{|z|^2}\leq\int\limits_{P\cap X\cap A_k}\frac{|f'(z)|^2}{n(|z|)^{1+2\varepsilon}|f(z)|^2}\,dx\,dy\\&\leq\frac{1}{2^{(1+2\varepsilon)k}}\int\limits_{P\cap X\cap A_k}\frac{|f'(z)|^2}{|f(z)|^2}\,dx\,dy,\end{aligned}$$

provided the integral on the right hand side exists. Taking $P=f^{-1}(S)$ for a subset S of $\mathbb{C}$ of finite logarithmic area, we find that

$$\begin{aligned}\operatorname{logarea}(f^{-1}(S)\cap X\cap A_k)&\leq\frac{1}{2^{(1+2\varepsilon)k}}\int\limits_S\operatorname{card}(f^{-1}(w)\cap X\cap A_k)\frac{du\,dv}{|w|^2}\\&\leq\frac{2^{k+1}}{2^{(1+2\varepsilon)k}}\operatorname{logarea} S=\frac{2}{2^{2\varepsilon k}}\operatorname{logarea} S.\end{aligned}$$

Let $R>1$ be large and choose $K\in\mathbb{N}$ such that $2^K\leq n(R)$. We deduce that

$$\begin{aligned}\operatorname{logarea}(f^{-1}(S)\cap X\cap\{z\colon |z|\geq R\})&\leq\sum_{k=K}^{\infty}\operatorname{logarea}(f^{-1}(S)\cap X\cap A_k)\\&\leq 2\operatorname{logarea} S\sum_{k=K}^{\infty}\frac{1}{2^{2\varepsilon k}}=\frac{2^{1-2\varepsilon K}}{1-2^{-2\varepsilon}}\operatorname{logarea} S,\end{aligned}$$

and thus

$$\operatorname{logarea}(f^{-1}(S)\cap X\cap\{z\colon |z|\geq R\})\leq\tfrac{1}{2}\operatorname{logarea} S \tag{2.2}$$

if K is sufficiently large, which can be achieved by choosing R large.

Define $S_0 = W \cup \{z\colon |z| < R\}$ and $S_k = f^{-1}(S_{k-1}) \cap X \cap \{z\colon |z| \geq R\}$ for $k \geq 1$. Then

$$\operatorname{logarea} S_k \leq \tfrac{1}{2} \operatorname{logarea} S_{k-1} \tag{2.3}$$

for $k \geq 2$ by (2.2), and for large R we also have

$$\operatorname{logarea} S_1 \leq \tfrac{1}{2} \operatorname{logarea}(S_0 \cap \Delta). \tag{2.4}$$

Note that

$$\begin{aligned} \operatorname{logarea}(S_0 \cap \Delta) &\leq \operatorname{logarea} W + \operatorname{logarea}(D(0,R) \cap \Delta) \\ &= \operatorname{logarea} W + \log R < \infty \end{aligned}$$

by (2.1). It follows from (2.3) and (2.4) that

$$\operatorname{logarea}\Big(\bigcup_{k=0}^{\infty} S_k \cap \Delta\Big) \leq 2 \operatorname{logarea}(S_0 \cap \Delta) < \infty. \tag{2.5}$$

Let now

$$T = \{z\colon f^k(z) \in X \cap Y \text{ and } |f^k(z)| \geq R \text{ for all } k \geq 0\}.$$

Here, as usual, $f^0(z) = z$ so that if $z \in T$, then in particular $z \in X \cap Y$ and $|z| \geq R$.

Suppose that $z \in \mathbb{C} \setminus T$. Then there exists $k \geq 0$ such that $f^k(z) \notin X \cap Y$ or $|f^k(z)| < R$. Thus $f^k(z) \in S_0$. Assuming k to be minimal we have $f^j(z) \in X \cap Y$ and $|f^j(z)| \geq R$ for $0 \leq j \leq k-1$. We conclude that $f^{k-1}(z) \in S_1$. Inductively we see that $f^{k-j}(z) \in S_j$. In particular, $z \in S_k$. This implies that

$$\mathbb{C} \setminus T \subset \bigcup_{k=0}^{\infty} S_k. \tag{2.6}$$

On the other hand, for $z \in T$ and $k \geq 0$ we have

$$|f^k(z)| \geq (1+\varepsilon)^k |z| \geq (1+\varepsilon)^k R, \tag{2.7}$$

by the definition of T and Y. Hence $T \subset I(f)$ and thus $\mathbb{C} \setminus I(f) \subset \mathbb{C} \setminus T$. Together with (2.5) and (2.6) this yields (1.2).

To prove the second claim we only have to show that if $T \cap F(f) \neq \emptyset$, then $F(f)$ has a multiply connected component. Our arguments for this are similar to those in [34, Theorem 3.1].

So let $z \in T \cap F(f)$. Choose $\delta > 0$ such that $D(z,\delta) \subset F(f)$. Since $f^n(z) \to \infty$ as $n \to \infty$, we may assume that $f^n(\zeta) \neq 0$ for all $\zeta \in D(z,\delta)$ and $n \in \mathbb{N}$. We consider the functions $g_n\colon D(z,\delta) \to \mathbb{C}$, $g_n(\zeta) = f^n(\zeta)/f^n(z)$.

Then $g_n(z) = 1$, and since $f^j(z) \in X$ for $0 \le j \le n-1$, we have

$$|g_n'(z)| = \left|\frac{(f^n)'(z)}{f^n(z)}\right| = \frac{1}{|f^n(z)|}\prod_{j=0}^{n-1}|f'(f^j(z))|$$
$$\ge \frac{1}{|f^n(z)|}\prod_{j=0}^{n-1}\frac{n(|f^j(z)|)^{1/2+\varepsilon}|f(f^j(z))|}{|f^j(z)|} = \frac{1}{|z|}\prod_{j=0}^{n-1} n(|f^j(z)|)^{1/2+\varepsilon}.$$

Thus $|g_n'(z)| \to \infty$ as $n \to \infty$. Hence the g_n do not form a normal family. It now follows easily from Montel's theorem that there exist arbitrarily large n such that $\partial D(0,1) \subset g_n(D(z,\delta))$ or $\partial D(0,2) \subset g_n(D(z,\delta))$. In fact, this holds for all large n. Thus we have $\partial D(0,|f^n(z)|) \subset f^n(D(z,\delta)) \subset F(f)$ or $\partial D(0,2|f^n(z)|) \subset f^n(D(z,\delta)) \subset F(f)$. Since $f^n(z) \to \infty$, this implies that $F(f)$ has a multiply connected component. ∎

Proof of Theorem 1.1. Upper bounds for $n(r)$ have been given by Hayman and Stewart [22, Theorem 5], and we follow the reasoning there. Nevanlinna's first fundamental theorem implies that there exists a constant C such that

$$\int_1^r \frac{n(t,a)}{t}\,dt \le T(r,f) + C$$

for all $a \in \mathbb{C}$ and $r > 1$, with the Nevanlinna (or Ahlfors–Shimizu) characteristic $T(r,f)$. Thus

$$n(r,a) = n(r,a)\int_r^{er}\frac{dt}{t} \le \int_r^{er}\frac{n(t,a)}{t}\,dt \le T(er,f) + C \le \log M(er,f) + C$$

for all $a \in \mathbb{C}$ and $r > 1$. Given $\delta > 0$ we thus have

$$n(r) \le \log M(er,f) + C \le r^{\rho(f)+\delta}$$

for large r. And for a given $\varepsilon > 0$ we may choose $\delta \in (0,\varepsilon]$ such that

$$n(r)^{1/2+\delta} \le r^{(\rho(f)+\delta)(1/2+\delta)} \le r^{\rho(f)/2+\varepsilon}.$$

We thus deduce from (1.1) that (1.6) holds with ε replaced by δ. The conclusion now follows from Theorem 1.3. ∎

In order to prove Theorem 1.2 we consider, for $\alpha > 0$, the function

$$E_\alpha\colon [0,\infty) \to [0,\infty), \qquad E_\alpha(x) = \exp(x^\alpha),$$

and note that there exists $x_\alpha \ge 0$ such that $E_\alpha(x) > x$ for $x > x_\alpha$, and thus $E_\alpha^k(x) \to \infty$ as $k \to \infty$ if $x > x_\alpha$. We shall use the following lemma which can be deduced from the arguments in [15, proof of Lemma 4.7], but for completeness we include the proof, following [15].

Lemma 2.1. *Let $\beta > \alpha > 0$. Then there exists $x_0 > 0$ such that*

$$E_\alpha^k(x) \geq E_\beta^{k-2}(x) \quad \textit{for } k \geq 4 \textit{ and } x \geq x_0.$$

Proof. Let $F_\alpha(x) = \alpha e^x$. Then

$$E_\alpha(\exp\exp(x)) = \exp\exp(F_\alpha(x))$$

and thus

$$E_\alpha^k(x) = \exp\exp(F_\alpha^k(\log\log x)) = \exp\exp(F_\alpha^{k-2}(\alpha x^\alpha)) \tag{2.8}$$

for $k \geq 2$. Set $c = \log(2\beta/\alpha)$. For large x we have

$$F_\alpha(x+c) = \alpha e^{x+c} = \alpha e^c e^x = 2F_\beta(x) \geq F_\beta(x) + c$$

and thus

$$F_\alpha^k(x+c) \geq F_\beta^k(x) + c. \tag{2.9}$$

For large x we also have

$$F_\alpha(\alpha x^\alpha) \geq x + c \quad \text{and} \quad F_\beta(x) \geq \beta x^\beta. \tag{2.10}$$

Combining (2.8)–(2.10) we obtain

$$\begin{aligned} E_\alpha^k(x) &= \exp\exp(F_\alpha^{k-3}(F_\alpha(\alpha x^\alpha))) \geq \exp\exp(F_\alpha^{k-3}(x+c)) \\ &\geq \exp\exp(F_\beta^{k-3}(x)+c) \geq \exp\exp(F_\beta^{k-3}(x)) \\ &= \exp\exp(F_\beta^{k-4}(F_\beta(x))) \geq \exp\exp(F_\beta^{k-4}(\beta x^\beta)) = E_\beta^{k-2}(x) \end{aligned}$$

for $k \geq 4$ and large x. ∎

Proof of Theorem 1.2. Let $E_\varepsilon(x) = \exp(x^\varepsilon)$ and, for some large $R > 0$, let $B(f)$ be the set of all $z \in \mathbb{C}$ such that

$$|f^k(z)| \geq E_\varepsilon^k(R) \tag{2.11}$$

for all $k \geq 0$. We proceed as in the proofs of Theorems 1.3 and 1.1, with the definition of Y changed to

$$Y = \{z\colon |f(z)| \geq E_\varepsilon(|z|)\},$$

however. Instead of (2.7) we now obtain (2.11) for $z \in T$ and $k \geq 0$. We deduce that (1.2), and if f has no multiply connected wandering domains also (1.3), hold with $I(f)$ replaced by $B(f)$. Thus we only have to show that $B(f) \subset A(f)$.

In order to do so we use the hypothesis that f has finite order. It implies that if $\mu > \rho(f)$ and R is sufficiently large, then $|f(z)| \leq \exp(|z|^\mu)$ for $|z| \geq R$. With $E_\mu(x) = \exp(x^\mu)$ we thus have

$$M^k(R,f) \leq E_\mu^k(R) \quad \text{for all } k \geq 0. \tag{2.12}$$

Applying Lemma 2.1 with $\alpha = \varepsilon$ and $\beta = \mu$ we deduce from (2.11) and (2.12) that if $z \in B(f)$, then $|f^k(z)| \geq M^{k-2}(R, f)$ for all $k \geq 4$, provided R has been chosen sufficiently large. It follows that $z \in A(f)$ and hence $B(f) \subset A(f)$. ∎

Proof of Theorem 1.4. Let $0 < \delta < \varepsilon$. It follows from Proposition 1.1 that if

$$\left|\frac{zf'(z)}{f(z)}\right| < |z|^{\rho(f)/2+\delta},$$

then

$$|f(z)| < R\exp(4\pi|z|^{\rho(f)/2+\delta}) \leq \exp(|z|^{\rho(f)/2+\varepsilon}),$$

if $|z|$ is sufficiently large. We conclude that if z is in the set occurring on the left hand side of (1.5), with ε replaced by δ, and if $|z|$ is sufficiently large, then z is also in the set on the left hand side of (1.7). Thus (1.5), with ε replaced by δ, is a consequence of (1.7).

It now follows from Theorem 1.2 that the conclusion of Theorem 1.1 holds with $I(f)$ replaced by $A(f)$. Moreover, since $f \in \mathcal{B}$, we deduce from the result of Baker [3, p. 565] already used after Theorem 1.1 that $F(f)$ has no multiply connected component. (To rule out multiply connected components of $F(f)$, we could alternatively use the result of Eremenko and Lyubich [18, Theorem 1] that if $f \in \mathcal{B}$, then $I(f) \subset J(f)$, together with the well-known fact that multiply connected components of $F(f)$ are in $I(f)$.) We thus conclude that (1.3) holds with $I(f)$ replaced by $A(f)$, as claimed. ∎

3. Proofs of Theorems 1.5 and 1.6. For the proof of Theorem 1.5 we shall use the following result of Peters and Smit [28, Proposition 10].

LEMMA 3.1. *Let p be a semihyperbolic polynomial. Let A be an open set containing all attracting periodic points such that $\overline{p(A)} \subset A \subset F(f)$ and let $R > 0$ be such that $|p(z)| > 2R$ for $|z| > R$. Let $U_0 = \{z\colon |z| > R\} \cup A$ and, for $n \in \mathbb{N}$, define $U_n = f^{-n}(U_0)$ and $V_n = \mathbb{C} \setminus U_n$. Then there exist $c_0 > 0$ and $\theta \in (0, 1)$ such that*

$$\operatorname{area} V_n \leq c_0\theta^n \quad \text{for all } n \in \mathbb{N}.$$

The proof of Theorem 1.5 is easier if $J(p)$ is connected, because—as already noted—this is equivalent to $f \in \mathcal{B}$ so that Theorem 1.4 can be applied. Therefore we consider this special case first, and add the arguments required for the general case afterwards.

Proof of Theorem 1.5 if $J(p)$ is connected. Let R, A, U_n and V_n be as in Lemma 3.1. Since p does not have attracting periodic points, we can take $A = \emptyset$. Hence $U_0 = \{z\colon |z| > R\}$ and thus

$$V_n = \{z\colon |p^n(z)| \leq R\}. \tag{3.1}$$

Note that if $z \in V_n$, then also $|p^k(z)| \leq R$ for $1 \leq k \leq n$.

We may assume that $R > 1$. Denote by d the degree of p. It is easy to see that there exists a positive constant c_1 such that if $|z| > R$, then $|p^n(z)| > \exp(c_1 d^n)$. For example, this follows from Böttcher's theorem [26, Theorem 9.1] which says that p is conjugate to $z \mapsto z^d$ in some neighborhood of ∞.

Let $\varepsilon > 0$. For $n \in \mathbb{N}$ we let $m = \lceil \varepsilon n \rceil$ and $W_n = V_m$. With c_0 and θ as in Lemma 3.1 and $\gamma = \theta^\varepsilon$ we then have

$$\text{area}\, W_n \leq c_0 \theta^m \leq c_0 \theta^{\varepsilon n} = c_0 \gamma^n. \tag{3.2}$$

If $z \notin W_n$, then $|p^m(z)| > R$ and hence

$$|p^n(z)| = |p^{n-m}(p^m(z))| \geq \exp(c_1 d^{n-m}) = \exp(c_1 d^{\lfloor (1-\varepsilon)n \rfloor}).$$

With $c_2 = c_1/d$ we thus have

$$|p^n(z)| \geq \exp(c_2 d^{(1-\varepsilon)n}) \quad \text{for } z \notin W_n. \tag{3.3}$$

Let now $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ be such that Schröder's functional equation (1.8) holds. As noted before Theorem 1.5, our hypotheses imply $f \in \mathcal{B}$.

Choosing $r_0 \in (0, 1]$ sufficiently small we may achieve that f is univalent in $D(0, 2r_0)$. In particular, $f'(z) \neq 0$ for $z \in A := \{\zeta\colon r_0/|\lambda| \leq |\zeta| \leq r_0\}$. With

$$S_n := f^{-1}(W_n) \cap A \tag{3.4}$$

and

$$c_3 := \frac{1}{\min_{z \in A} |f'(z)|^2}, \tag{3.5}$$

we then have

$$\text{area}\, S_n \leq c_3 \,\text{area}\, W_n. \tag{3.6}$$

For $n \in \mathbb{N}$ we define

$$A_n := \lambda^n A = \{z\colon |\lambda|^{n-1} r_0 \leq |z| \leq |\lambda|^n r_0\} \quad \text{and} \quad T_n := \lambda^n S_n.$$

For $|z| \geq r_0$ we now choose $n \in \mathbb{N}$ such that $z \in A_n$. Then z has the form $z = \lambda^n \zeta$ with $\zeta \in A$. If $z \in A_n \setminus T_n$, then $\zeta \in A \setminus S_n$ and thus $f(\zeta) \notin W_n$. Hence (3.3) yields

$$|f(z)| = |f(\lambda^n \zeta)| = |p^n(f(\zeta))| \geq \exp(c_2 d^{(1-\varepsilon)n}) \quad \text{for } z \in A_n \setminus T_n.$$

As already mentioned before Theorem 1.5, we have $\rho(f) = \log d / \log |\lambda|$ so that $d = |\lambda|^{\rho(f)}$. Noting that $|\lambda|^n \geq |z|/r_0 \geq |z|$ for $z \in A_n$, we thus get

$$d^{(1-\varepsilon)n} = |\lambda|^{(1-\varepsilon)\rho(f)n} \geq |z|^{(1-\varepsilon)\rho(f)} \quad \text{for } z \in A_n. \tag{3.7}$$

Combining the last two inequalities we find that

$$|f(z)| \geq \exp(c_2 |z|^{(1-\varepsilon)\rho(f)}) \quad \text{for } z \in A_n \setminus T_n. \tag{3.8}$$

Now

$$\operatorname{area} T_n = |\lambda|^{2n} \operatorname{area} S_n \le c_0 c_3 |\lambda|^{2n} \gamma^n$$

by (3.2) and (3.6), and thus

$$\operatorname{logarea} T_n = \int_{T_n} \frac{dx\,dy}{|z|^2} \le \frac{1}{(|\lambda|^{n-1} r_0)^2} \operatorname{area} T_n \le \frac{|\lambda|^2 c_0 c_3}{r_0^2} \gamma^n.$$

Consequently,

$$T := \bigcup_{n=1}^{\infty} T_n \tag{3.9}$$

satisfies

$$\operatorname{logarea} T < \infty. \tag{3.10}$$

On the other hand,

$$\{z \colon |z| \ge r_0 \text{ and } |f(z)| < \exp(c_2 |z|^{(1-\varepsilon)\rho(f)})\} \subset T \tag{3.11}$$

by (3.8). Thus (1.7) holds if ε is chosen such that $(1-\varepsilon)\rho(f) > \rho(f)/2 + \varepsilon$.

Note that we have not used yet the property that $J(p)$ is connected. But since we assume that this is the case, we have $f \in \mathcal{B}$. Thus (1.7) yields the conclusion in view of Theorem 1.4. ∎

To deal with the general case, we use the following result of Carleson, Jones and Yoccoz [14, Theorem 2.1], which was also crucial in the proof of Lemma 3.1 of the present paper that appeared in [28]. Here $\operatorname{diam} A$ denotes the (Euclidean) diameter of a subset A of $\mathbb{C}$.

LEMMA 3.2. *Let p be a semihyperbolic polynomial. Then there exist $\eta > 0$, $K_0 > 0$ and $\tau \in (0,1)$ such that if $z \in J(f)$, $n \in \mathbb{N}$ and V is a component of $f^{-n}(D(z,\eta))$, then*

$$\operatorname{diam} V \le K_0 \tau^n.$$

In order to rule out multiply connected wandering domains, we will use the following result of Zheng [39].

LEMMA 3.3. *Let f be a transcendental entire function with a multiply connected wandering domain U. Then there exist sequences (r_n) and (R_n) satisfying $r_n \to \infty$ and $R_n/r_n \to \infty$ such that*

$$\{z \colon r_n \le |z| \le R_n\} \subset f^n(U) \subset \{z \colon R_{n-1} \le |z| \le r_{n+1}\}$$

for large n.

The conclusion that $R_n/r_n \to \infty$ was strengthened to $R_n \ge r_n^{1+\varepsilon}$ for some $\varepsilon > 0$ in [11, Theorem 1.2], but we do not need this result here.

Proof of Theorem 1.5 in the general case. We will use the notation and results of the proof given above for the special case that $J(p)$ is connected.

In particular, the set T defined by (3.9) satisfies (3.10) and (3.11). In order to apply Theorem 1.2 it remains to find an upper bound for the size of the set of z satisfying $|zf'(z)/f(z)| < |z|^{\rho(f)/2+\varepsilon}$.

To estimate $|f'(z)|$ we note that

$$\lambda^n f'(\lambda^n \zeta) = (p^n)'(f(\zeta))f'(\zeta) \tag{3.12}$$

by (1.8). We are thus looking for an estimate of $|(p^n)'(z)|$ for $z \in \mathbb{C} \setminus W_n$. Here, as before, $W_n = V_m$ where $m = \lceil \varepsilon n \rceil$ and V_m is defined by (3.1). As before, we write $p^n(z) = p^{n-k}(p^k(z))$ so that

$$(p^n)'(z) = (p^{n-k})'(p^k(z))(p^k)'(z). \tag{3.13}$$

We will then estimate $|(p^k)'(z)|$ for $z \in \mathbb{C} \setminus V_m$, where k is chosen such that $p^k(z) \in \mathbb{C} \setminus V_0 = \{w\colon |w| > R\}$, and next estimate $|(p^{n-k})'(w)|$ for $|w| > R$.

We may assume that R in (3.1) is chosen so large that $|p'(z)| > 1$ for $|z| > R$. In particular, this implies that all critical points of p are contained in $V_0 = \overline{D(0, R)}$. Let η be as in Lemma 3.2. We may assume that η is chosen so small that if c is a critical point of p which is not contained in $J(p)$, then $\operatorname{dist}(p^k(c), J(p)) > \eta$ for all $k \geq 0$, where $\operatorname{dist}(\cdot, \cdot)$ denotes the (Euclidean) distance. This assumption can be made since $p^k(c) \to \infty$ for every critical point $c \notin J(p)$.

There exists $M \in \mathbb{N}$ such that

$$V_{M-1} \subset \left\{\zeta\colon \operatorname{dist}(\zeta, J(p)) \leq \tfrac{1}{2}\eta\right\}.$$

By the choice of η the only critical points of p that are contained in V_{M-1} are those that are already contained in $J(p)$. Together with the choice of R we thus see that the critical points of p that are not contained in $J(p)$ are contained in $V_0 \setminus V_{M-1}$.

Now $d_0 := \operatorname{dist}(V_{M-1} \setminus V_M, J(p))$ satisfies $0 < d_0 < \eta/2$. We conclude that if $w \in V_{M-1} \setminus V_M$, then $D(w, d_0) \cap J(p) = \emptyset$. For $\xi \in J(p)$ we then have $D(w, d_0) \subset D(\xi, \eta)$.

Let now $k > M$ and $z \in V_{k-1} \setminus V_k$. Then $w = p^{k-M}(z) \in V_{M-1} \setminus V_M$. Denote by U the component of $p^{-(k-M)}(D(w, d_0))$ that contains z. Then, as just noted, U is contained in a component of $p^{-(k-M)}(D(\xi, \eta))$ for some $\xi \in J(p)$ and thus Lemma 3.2 implies that

$$\operatorname{diam} U \leq K_0 \tau^{k-M}. \tag{3.14}$$

Since our choice of η implies that $D(w, d_0)$ does not intersect the orbit of any critical point, $p^{k-M}\colon U \to D(w, d_0)$ is biholomorphic. Koebe's one quarter theorem, applied to the inverse $\varphi\colon D(w, d_0) \to U$ of $p^{k-M}\colon U \to D(w, d_0)$, thus yields

$$U = \varphi(D(w, d_0)) \supset D\left(\varphi(w), \frac{1}{4}|\varphi'(w)| d_0\right) = D\left(z, \frac{d_0}{4|(p^{k-M})'(z)|}\right).$$

Hence we can deduce from (3.14) that if $k \geq M$, then

$$|(p^{k-M})'(z)| \geq \frac{c_4}{\tau^{k-M}} \quad \text{for } z \in V_{k-1} \setminus V_k, \tag{3.15}$$

with $c_4 = d_0/(2K_0)$.

Next we note that there exist $c_5, K > 0$ such that if $t > 0$ and $|z - c| > t$ for every critical point c of p, then $|p'(z)| \geq c_5 t^K$. It follows that there exist $c_6, L > 0$ such that if $1 \leq k \leq M$ and $\delta > 0$, then

$$\text{area}\{z \in V_{k-1} \setminus V_k \colon |(p^M)'(z)| \leq \delta\} \leq c_6 \delta^L. \tag{3.16}$$

In particular, this holds for $k = M$, which together with (3.15) shows that

$$\text{area}\left\{z \in V_{k-1} \setminus V_k \colon |(p^k)'(z)| \leq \frac{c_4 \delta}{\tau^{k-M}}\right\} \leq d^{k-M} \left(\frac{\tau^{k-M}}{c_4}\right)^2 c_6 \delta^L$$

for $k > M$. Since $\tau < 1$, we thus have

$$\text{area}\{z \in V_{k-1} \setminus V_k \colon |(p^k)'(z)| \leq c_4 \delta\} \leq c_7 d^k \delta^L \tag{3.17}$$

for $k > M$, with $c_7 = c_6/c_4^2$. We may assume that $c_4 \leq 1$, so that (3.17) also holds for $1 \leq k \leq M$ by (3.16).

Next, as explained after (3.13), we want to estimate $|(p^j)'(w)|$ for $|w| > R$. In order to do so, let g be the Green function of the (super)attracting basin of ∞. Then

$$g(p(z)) = dg(z).$$

This implies that $g(p^j(z)) = d^j g(p(z))$ and thus

$$|\nabla g(p^j(z))| \cdot |(p^j)'(z)| = d^j |\nabla g(z)|.$$

We have $g(z) = \log|z| + c + o(1)$ as $z \to \infty$ for some constant c. It is not difficult to show that this implies that

$$|\nabla g(z)| \sim \frac{1}{|z|} \quad \text{as } z \to \infty. \tag{3.18}$$

Hence

$$\left|\frac{(p^j)'(z)}{p^j(z)}\right| \sim d^j |\nabla g(z)| \quad \text{as } j \to \infty.$$

Using (3.18) again we deduce that there exists a positive constant c_8 such that

$$\left|\frac{(p^j)'(w)}{p^j(w)}\right| \geq \frac{c_8}{|w|} d^j \quad \text{for } |w| \geq R. \tag{3.19}$$

Recall from the proof for the special case where $J(p)$ is connected that for $n \in \mathbb{N}$ we set $m = \lceil \varepsilon n \rceil$ and $W_n = V_m$. With $\alpha = d^{-2\varepsilon/L}$ we now let

$$W_n' = \bigcup_{k=1}^{m} \{z \in V_{k-1} \setminus V_k \colon |(p^k)'(z)| \leq c_4 \alpha^n\}.$$

and deduce from (3.17) that

$$\text{area}\, W_n' \le c_7\alpha^{Ln}\sum_{k=1}^{m} d^k \le \frac{c_7 d}{d-1}\alpha^{Ln}d^m \le c_9\alpha^{Ln}d^{\varepsilon n} = c_9 d^{-\varepsilon n} \tag{3.20}$$

with $c_9 = c_7 d^2/(d-1)$.

Let now $z \in \overline{D(0,R)} \setminus (W_n \cup W_n') = V_0 \setminus (V_m \cup W_n')$. Then $z \in V_{k-1} \setminus V_k$ for some $k \in \{1,\dots,m\}$. Since $z \notin W_n'$, we have $|(p^k)'(z)| > c_4\alpha^n$. Moreover, $w := p^k(z)$ satisfies $R < |w| \le M(R,p)$. Together with (3.19) we thus find with $c_{10} = c_4 c_8/(M(R,p)d)$ that

$$\begin{aligned}\left|\frac{(p^n)'(z)}{p^n(z)}\right| &= \left|\frac{(p^{n-k})'(w)}{p^{n-k}(w)}(p^k)'(z)\right| > \frac{c_8}{|w|}d^{n-k}c_4\alpha^n \\ &\ge \frac{c_4c_8}{M(R,p)}d^{n-m}\alpha^n \ge c_{10}d^{(1-\varepsilon)n}\alpha^n = c_{10}d^{(1-\varepsilon-2\varepsilon/L)n}.\end{aligned}$$

With $\varepsilon' = \varepsilon + 2\varepsilon/L$ we have

$$\left|\frac{(p^n)'(z)}{p^n(z)}\right| \ge c_{10}d^{(1-\varepsilon')n} \qquad \text{for } z \in \overline{D(0,R)} \setminus (W_n \cup W_n'). \tag{3.21}$$

Similarly to (3.4) we consider

$$S_n' := f^{-1}(W_n \cup W_n') \cap A$$

and deduce, analogously to (3.6), that

$$\text{area}\, S_n' \le c_3(\text{area}\, W_n + \text{area}\, W_n'). \tag{3.22}$$

In analogy to the previous arguments we set $T_n' = \lambda^n S_n'$. Writing $z \in A_n$ in the form $z = \lambda^n\zeta$ with $\zeta \in A$ we have

$$\frac{zf'(z)}{f(z)} = \frac{\lambda^n\zeta f'(\lambda^n\zeta)}{f(\lambda^n\zeta)} = \frac{(p^n)'(f(\zeta))\zeta f'(\zeta)}{p^n(f(\zeta))}$$

by (3.12). Using (3.5) and (3.21) we deduce that

$$\left|\frac{zf'(z)}{f(z)}\right| \ge c_{11}d^{(1-\varepsilon')n} \qquad \text{for } z \in A_n \setminus T_n'$$

with $c_{11} = c_{10}r_0/(|\lambda|\sqrt{c_3})$. It thus follows from (3.7) that

$$\left|\frac{zf'(z)}{f(z)}\right| \ge c_{11}|z|^{(1-\varepsilon')\rho(f)} \qquad \text{for } z \in A_n \setminus T_n'.$$

In analogy to (3.9)–(3.11) we now deduce from (3.22), (3.20) and (3.2) that the set T' defined by

$$T' := \bigcup_{n=1}^{\infty} T_n'$$

satisfies

$$\text{logarea}\, T' < \infty \tag{3.23}$$

and

$$\left\{z\colon |z| \geq r_0 \text{ and } \left|\frac{zf'(z)}{f(z)}\right| < c_{10}|z|^{(1-\varepsilon')\rho(f)}\right\} \subset T'. \tag{3.24}$$

It follows from (3.23) and (3.24), together with (3.10) and (3.11), that (1.5) holds if ε and hence ε' are sufficiently small. The conclusion will thus follow from Theorem 1.2 if we can show that f does not have multiply connected wandering domains.

In order to do so, let $u_0 \in \mathbb{C}$ be such that $v_0 := f(u_0) \in J(p)$. It follows from (1.8) that

$$f(\lambda^n u_0) = p^n(f(u_0)) = p^n(v_0) \in J(p)$$

and thus $|f(\lambda^n u_0)| \leq R$ for all $n \in \mathbb{N}$. Lemma 3.3 now implies that f does not have multiply connected wandering domains. ∎

The result of Eremenko and Lyubich [18, Theorem 7] already mentioned in the introduction that we will use is the following.

Lemma 3.4. *Let $f \in \mathcal{B}$ and suppose that there exists $R > 0$ such that*

$$\liminf_{r\to\infty} \frac{\operatorname{logarea}\big(f^{-1}(D(0,R)) \cap D(0,r) \cap \Delta\big)}{\log r} > 0. \tag{3.25}$$

Then $\operatorname{area} I(f) = 0$.

Proof of Theorem 1.6. Assume that (1.8) holds and that $\operatorname{area} K(p) > 0$. Set $L = f^{-1}(K(p))$ and choose $R > 0$ such that $K(p) \subset D(0,R)$. It follows that $L \subset f^{-1}(D(0,R))$ and $\operatorname{area} L > 0$. Since $K(p)$ is invariant under p, we can deduce from (1.8) that L is invariant under the map $z \mapsto \lambda z$. Thus also $A := \operatorname{area}(L \cap \{z\colon 1 \leq |z| \leq |\lambda\}) > 0$. For $r > 1$ we choose $n \in \mathbb{N}$ with $|\lambda|^{n-1} \leq r < |\lambda|^n$. Hence

$$\begin{aligned}
\operatorname{logarea}(L \cap D(0,r) \cap \Delta) &\geq \operatorname{logarea}(L \cap D(0,|\lambda|^{n-1}) \cap \Delta)\\
&= (n-1)\operatorname{logarea}(L \cap D(0,|\lambda|) \cap \Delta)\\
&\geq (n-1)\frac{A}{|\lambda|^2} \geq \frac{n-1}{n}\,\frac{A}{|\lambda|^2 \log|\lambda|}\log r.
\end{aligned}$$

Since $L \subset f^{-1}(D(0,R))$ and since n tends to ∞ with r, we deduce that the lower limit in (3.25) is at least $A/(|\lambda|^2 \log|\lambda|)$. The conclusion now follows from Lemma 3.4. ∎

4. Proof of Theorem 1.7. It is well-known that $\rho(\sigma) = 2$. This is also an immediate consequence of the following lemma, which is a special case of the asymptotics of σ and ζ that were obtained in [38]. Here we write $w_{mn} = m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$.

LEMMA 4.1. *Let*

$$E=\bigcup_{m,n\in\mathbb{Z}} D(w_{m,n},e^{-|w_{n,m}|}) \quad \textit{and} \quad F=\bigcup_{m,n\in\mathbb{Z}} D\left(w_{m,n},\frac{1}{\sqrt{|w_{n,m}|}}\right).$$

Then

$$\log|\sigma(z)|=V(z)+\mathcal{O}(|z|) \quad \text{as } |z|\to\infty,\ z\notin E, \tag{4.1}$$

where

$$V(z)=\frac{\pi}{2\operatorname{Im}\tau}|z|^2+\operatorname{Re}\left(\left(\frac{\eta_1}{2}-\frac{\pi}{2\operatorname{Im}\tau}\right)z^2\right), \tag{4.2}$$

and

$$\zeta(z)=\eta_1 z-\frac{2\pi i}{\operatorname{Im}\tau}\operatorname{Im} z+\mathcal{O}\big(\sqrt{|z|}\big) \quad \text{as } |z|\to\infty,\ z\notin F. \tag{4.3}$$

Proof of Theorem 1.7. First we note that (1.9) is equivalent to

$$\left|\eta_1-\frac{\pi}{\operatorname{Im}\tau}\right|\le\frac{\pi}{\operatorname{Im}\tau}.$$

This means that the second term on the right hand side of (4.2) is not bigger than the first term.

Let $B=\pi/\operatorname{Im}\tau$. The last inequality says that there exist $A\in[0,B]$ and $\alpha\in(-\pi,\pi]$ such that $\eta_1-B=Ae^{i\alpha}$. With these abbreviations, (4.2) takes the form

$$V(z)=\tfrac{1}{2}(B|z|^2+\operatorname{Re}(Ae^{i\alpha}z^2)), \tag{4.4}$$

which we may also write as

$$V(re^{i\theta})=\tfrac{1}{2}(B+A\cos(\alpha+2\theta))r^2.$$

Let $\theta^{\pm}=(\pm\pi-\alpha)/2$ and

$$G=\{re^{i\theta}\colon |\theta-\theta^+|\le r^{-1/4} \text{ or } |\theta-\theta^-|\le r^{-1/4}\}.$$

Then

$$\begin{aligned} V(re^{i\theta}) &\ge \tfrac{1}{2}(B+A\cos(\pi+r^{-1/4}))r^2\ge\tfrac{1}{2}B(1-\cos(r^{-1/4}))r^2 \\ &= (1+o(1))\tfrac{1}{4}Br^{3/2} \quad \text{as } r\to\infty,\ re^{i\theta}\notin G. \end{aligned} \tag{4.5}$$

It thus follows from (4.1) that there exists a positive constant c_1 such that

$$\log|\sigma(z)|\ge c_1|z|^{3/2} \quad \text{for } z\in\Delta\setminus(E\cup G). \tag{4.6}$$

To estimate $z\sigma'(z)/\sigma(z)=z\zeta'(z)$ we note that

$$\eta_1 z-\frac{2\pi i}{\operatorname{Im}\tau}\operatorname{Im} z=(B+Ae^{i\alpha})z-2iB\operatorname{Im} z=B\overline{z}+Ae^{i\alpha}z$$

and hence

$$z\left(\eta_1 z-\frac{2\pi i}{\operatorname{Im}\tau}\operatorname{Im} z\right)=B|z|^2+Ae^{i\alpha}z^2.$$

Combining this with (4.4) we see that

$$\left|z\left(\eta_1 z - \frac{2\pi i}{\operatorname{Im}\tau}\operatorname{Im} z\right)\right| \geq \operatorname{Re}\left(z\left(\eta_1 z - \frac{2\pi i}{\operatorname{Im}\tau}\operatorname{Im} z\right)\right) = B|z|^2 + A\operatorname{Re}(e^{i\alpha}z^2) = 2V(z). \tag{4.7}$$

Together with (4.3) and (4.5) this implies that there exists a constant c_2 such that

$$\left|\frac{z\sigma'(z)}{\sigma(z)}\right| = |z\zeta(z)| \geq c_2|z|^{3/2} \quad \text{for } z \in \varDelta \setminus (F \cup G). \tag{4.8}$$

It is easy to see that $\operatorname{logarea}(\varDelta \cap (E \cup F \cup G)) < \infty$. Hence (4.6) and (4.8) say that (1.5) holds for $f = \sigma$ if $0 < \varepsilon < 1/2$. Since Lemma 3.3 implies that f has no multiply connected wandering domains, the conclusion now follows from Theorem 1.2. ∎

5. Remarks

REMARK 5.1. The main tool used by Eremenko and Lyubich [18] in their proof of Proposition 1.1 is a logarithmic change of variable which consists in considering the function $F(\zeta) = \log f(e^\zeta)$ in certain domains. With $z = e^\zeta$ we have $F'(\zeta) = zf'(z)/f(z)$. In our results we also use the expression $zf'(z)/f(z)$, even though we do not assume that $f \in \mathcal{B}$ anymore.

We mention that the quantity $zf'(z)/f(z)$ also appears in [7] and in [34]. The result in [7, Theorem 1.4] required lower bounds for $\operatorname{Re}(zf'(z)/f(z))$ while our results only assume bounds for $|zf'(z)/f(z)|$. We note, however, that (4.7) also yields lower bounds for $\operatorname{Re}(z\sigma'(z)/\sigma(z))$.

REMARK 5.2. Besides the area of $J(\sin(\alpha z+\beta))$, McMullen [27, Theorem 1.2] also considered the Hausdorff dimension of $J(\lambda e^z)$. This result and the techniques used in its proof have been the starting point of many results on the Hausdorff dimension of Julia sets; see [35] for a survey and, e.g., [4, 9, 10, 29, 33] for some more recent results.

The methods in [9, 33] also use estimates of $zf'(z)/f(z)$, but otherwise they are quite different from the ones employed here.

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