Maximum Modulus, Characteristic, and Area on the Sphere

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ABSTRACT Let f be an entire function, $M(r)$ its maximum modulus, $T(r)$ its Ahlfors-Shimizu (or Nevanlinna) characteristic, and $\pi A(r)$ the area of the image of $|z| \leq r$ on the Riemann sphere, counted according to multiplicity. It is proved that if the order of f is not less than $\frac{1}{2}$, then $\liminf \log M(r)/A(r) \leq \pi$ as $r \to \infty$. More generally, if $T(r) \leq \gamma(r)$ for some differentiable and increasing function $\gamma(r)$, then $\liminf \log M(r)/r\gamma'(r) \leq \pi$. If $\log \gamma(r)$ is convex in $\log r$, then we even have $\limsup \log M(r)/r\gamma'(r) \leq \pi$ for r outside an exceptional set of logarithmic density zero. Moreover, if $\log \gamma(r)$ is convex in $\log r$ and if $T(r) \leq \gamma(r)$ for arbitrarily large r, then $\liminf \log M(r)/\Psi(T(r)) \leq \pi$, where $\Psi(t) = \gamma'(\gamma^{-1}(t))\, \gamma^{-1}(t)$. Examples show that these results are sharp.

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1 Introduction and Results

Let f be an entire function, $M(r)$ its maximum modulus, and

$$A(r) = \frac{1}{\pi} \int_{|z| \leq r} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} dx dy.$$

Then $\pi A(r)$ is the area of the image of $|z| \leq r$ on the Riemann sphere, counted according to multiplicity. The Ahlfors-Shimizu characteristic $T(r)$ is defined by

$$T(r) = \int_0^r \frac{A(t)}{t} dt.$$

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In this paper, we shall compare the growth of $T(r)$, $A(r)$, and $\log M(r)$. Occasionally, we shall also use Nevanlinna's form of the characteristic function which we denote by $T_N(r)$. We assume some familiarity with these and other concepts of Nevanlinna theory (e. g. [8, 9, 14]). The order and the lower order of f are denoted by ρ and λ, respectively.

A well-known result which was conjectured by Paley and proved by Govorov [7] says that if $\frac{1}{2} \leq \lambda < \infty$, then

$$\liminf_{r\to\infty} \frac{\log M(r)}{T(r)} \leq \pi\lambda. \tag{1}$$

A slightly more general statement is that $\log M(r) \leq (1+o(1))\pi\mu T(r)$ as $r \to \infty$ through a sequence of Pólya peaks of order μ for $T(r)$, provided $\mu \geq \frac{1}{2}$. This latter remark does not seem to have been stated explicitly, but it may be considered as known. For instance, Edrei [6], Drasin and Shea [5], and Miles and Shea [12] have emphasized that certain results in value distribution theory hold if the limit is taken through a sequence of Pólya peaks, and this applies to the present case, too. It is not difficult to prove that if (r_j) is a sequence of Pólya peaks of order μ for $T(r)$, then $A(r_j) \sim \mu T(r_j)$. (Since I could not find this statement in the literature, a proof is given at the beginning of §5.) It follows that

$$\liminf_{r\to\infty} \frac{\log M(r)}{A(r)} \leq \pi, \tag{2}$$

if there exists a sequence of Pólya peaks for $T(r)$ of order μ for some $\mu \geq \frac{1}{2}$. This condition is clearly satisfied if $\rho \geq \frac{1}{2}$ and $\lambda < \infty$, but it also holds for certain (but not all) functions of infinite lower order (cf. [5]).

One of the purposes of this paper is to prove that (2) holds for all entire functions satisfying $\rho \geq \frac{1}{2}$. This is a consequence of the following result.

THEOREM 1 *Let f be an entire function of infinite order and let γ be an increasing function such that $T(r) \leq \gamma(r)$ for all large r.*
(a) *If γ is differentiable, then*

$$\liminf_{r\to\infty} \frac{\log M(r)}{r\gamma'(r)} \leq \pi. \tag{3}$$

(b) *If $\log \gamma(r)$ is convex in $\log r$, then*

$$\limsup_{\substack{r\to\infty \\ r\notin F}} \frac{\log M(r)}{r\gamma'(r)} \leq \pi \tag{4}$$

for some set F of logarithmic density zero.

Choosing $\gamma(r) = T(r)$ in (3) we obtain the following.

COROLLARY 1 *If $\frac{1}{2} \leq \rho \leq \infty$, then (2) holds.*

If $\log T(r)$ is convex in $\log r$ (and $\rho \geq \frac{1}{2}$), then we even have $\log M(r) \leq (1+o(1))\pi A(r)$ for r outside an exceptional set of logarithmic density zero. This does not hold in general. A proof of this assertion is sketched in the second part of §5.

Next we look for bounds for $\log M(r)$ in terms of $T(r)$. Using the well-known inequality (e. g. [8, 9, 14])

$$\log^+ M(r) \leq \frac{R+r}{R-r} T_N(R) = \frac{R+r}{R-r}(T(R) + O(1)) \quad (0 < r < R) \tag{5}$$

and Nevanlinna's refinement [13] of a growth lemma of Borel we find that

$$\liminf_{r\to\infty} \frac{\log M(r)}{\Psi(T(r))} = 0, \tag{6}$$

if $\Psi(t)$ is defined and positive for $t \geq t_0$, if $\Psi(t)/t$ is increasing for $t \geq t_0$, and if

$$\int_{t_0}^{\infty} \frac{dt}{\Psi(t)} < \infty. \tag{7}$$

Defining

$$\phi(x) = \frac{\Psi(e^x)}{e^x}, \tag{8}$$

we may express this result also in the form

$$\liminf_{r\to\infty} \frac{\log M(r)}{T(r)\phi(\log T(r))} = 0, \tag{9}$$

if $\phi(x)$ is positive and increasing for $x \geq x_0$ and if

$$\int_{x_0}^{\infty} \frac{dx}{\phi(x)} < \infty.$$

These results can be found in papers by Chuang [3], Marchenko and Shcherba [11], and Dai, Drasin, and Li [4]. In [11] and [4], it is also shown that the results are best possible in some sense. Moreover, it is proved in [4] that $\log M(r) = o(\Psi(T(r)))$ on a set of logarithmic density one and the case that f is meromorphic is also considered (cf. the last remark in §5). We note that the special case of (6) and (9) where $\Psi(t) = t(\log t)^K$, that is, $\phi(x) = x^K$, for some $K > 1$ can be found in Hayman's book [8, p. 20].

A better bound for $\log M(r)$ in terms of $T(r)$ is given by (1) if f is of finite lower order. We consider the case that f has infinite lower order, but we assume that f satisfies some other growth restriction.

THEOREM 2 *Suppose that* $\log\gamma(r)$ *is increasing and convex in* $\log r$ *and define*

$$\Psi(t) = \frac{1}{\frac{d}{dt}\log\gamma^{-1}(t)}, \tag{10}$$

that is, $\Psi(t) = \gamma'(\gamma^{-1}(t))\gamma^{-1}(t)$. *(Here* γ^{-1} *denotes the inverse function of* γ*.) If* f *is an entire function of infinite lower order which satisfies* $T(r) \leq \gamma(r)$ *for arbitrarily large* r*, then*

$$\liminf_{r\to\infty} \frac{\log M(r)}{\Psi(T(r))} \leq \pi. \tag{11}$$

We give two applications of Theorem 2.

COROLLARY 2 *If*

$$\lambda_l = \liminf_{r\to\infty} \frac{\log_{l+1} T(r)}{\log r} < \infty,$$

for some positive integer l*, then*

$$\liminf_{r\to\infty} \frac{\log M(r)}{T(r)\log T(r)\cdots\log_l T(r)} \leq \pi\lambda_l. \tag{12}$$

Here $\log_0 x = x$ and $\log_{l+1} x = \log(\log_l x)$ for $l \geq 0$ and sufficiently large x. The quantity λ_l is called the lower l-order of f (cf. [9]). Corollary 2 follows if we choose $\gamma(r) = \exp_l(r^{\lambda_l+\varepsilon})$ where $\varepsilon > 0$. For the next result we choose $\gamma(r) = \exp((\mu+\varepsilon)r^K)$.

Corollary 3 *If*

$$\mu = \liminf_{r\to\infty} \frac{\log T(r)}{(\log r)^K} < \infty$$

for some $K > 1$, then

$$\liminf_{r\to\infty} \frac{\log M(r)}{T(r)[\log T(r)]^{(K-1)/K}} \leq \pi K \mu^{1/K}. \tag{13}$$

To prove that our results are best possible in a certain sense, we assume again that $\log\gamma(r)$ is convex in $\log r$. Since we are interested in functions of infinite lower order, we shall also assume that $(\log\gamma(r))/(\log r) \to \infty$ as $r \to \infty$. We define $h(x) = \gamma(e^x)$. It follows that $h'(x)/h(x)$ is increasing and unbounded. It is not difficult to show (cf. [14, p. 253], see also Lemma 3 below) that this implies that

$$\left(\frac{h'(x)}{h(x)}\right)' \leq \left(\frac{h'(x)}{h(x)}\right)^{3/2} \tag{14}$$

for x outside an exceptional set of finite measure. We assume that this exceptional set is bounded, that is, we assume that (14) holds for all large x.

Theorem 3 *Suppose that $\gamma(r)$ and $h(x)$ are as above and define $\Psi(t)$ as in Theorem 2. Then there exists an entire function f satisfying*

$$T(r) \sim \gamma(r)$$

and

$$\log M(r) \sim \pi r \gamma'(r) \sim \pi A(r) \sim \pi \Psi(T(r)).$$

We remark that the hypotheses imposed on h and γ may also be expressed in terms of Ψ or the function ϕ defined by (8). An easy computation shows that the hypotheses of Theorem 3 are satisfied if and only if $\phi(x)$ is non-decreasing, tends to ∞, and satisfies $\phi'(x) \leq \sqrt{\phi(x)}$ for all large x.

In particular, it follows from Theorem 3 that the constant π in (2), (3), (4), (11), (12), and (13) cannot be replaced by a smaller one.

We have stated our results in terms of the Ahlfors-Shimizu characteristic $T(r)$. We remark that all our results hold for the Nevanlinna characteristic $T_N(r)$ as well. Similarly, we may replace $A(r)$ by $s(r)$ where

$$s(r) = \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\theta})d\theta,$$

since

$$T_N(r) = \int_0^r \frac{s(t)}{t}dt + O(1).$$

Our proofs are based on Baernstein's $*$-function [2]. This function yields (1), if $r \to \infty$ through a sequence of Pólya peaks of order λ. The main idea in the proof of Theorem

1 is to replace the Pólya peaks by a suitable other sequence. It is conceivable that this method can be used to deal with other problems concerning functions of infinite lower order. Theorem 2 will be deduced from Theorem 1. For the proof of Theorem 3, we shall use a result of Warschawski [16] concerning conformal mappings of strips and a result of Al-Katifi [1] concerning approximation of subharmonic functions by $\log|f|$ where f is entire.

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2 Lemmas for Real Functions

LEMMA 1 *Let $\Phi(x)$ be increasing and differentiable for $x \geq x_0$ and assume that $\Phi(x) \geq cx$ for some positive constant c and arbitrarily large x. Then there exist sequences (x_j), (M_j), and (ε_j) such that $x_j \to \infty$, $M_j \to \infty$, and $\varepsilon_j \to 0$ as $j \to \infty$, $\Phi'(x_j) \geq c/8$, and*

$$\Phi(x_j + h) \leq \Phi(x_j) + \Phi'(x_j)h + \varepsilon_j$$

for $|h| \leq M_j/\Phi'(x_j)$.

Proof. First we prove that there exist arbitrarily large u such that

(15) $$\Phi(5u) \geq 2\Phi(2u) \geq 2cu.$$

Assume that $\Phi(x) \geq cx/2$ holds for all large x. Then the second inequality in (15) holds for all large u. To prove that the first one holds for arbitrarily large u, we assume that this is false and define n by $(5/2)^{n-1} < x \leq (5/2)^n$. Then

$$\Phi(x) \leq \Phi((\frac{5}{2})^n) \leq 2\Phi((\frac{5}{2})^{n-1}) \leq \cdots \leq K2^{n-1} \leq K(\frac{4}{5})^{n-1}x < cx$$

for a suitable constant K and all large x. This contradicts our hypothesis. Now assume that $\Phi(x) < cx/2$ for arbitrarily large x. Then there exist arbitrarily large s and t such that $\Phi(s) = cs/2$, $\Phi(t) = ct$, and $cx/2 \leq \Phi(x) \leq cx$ for $s < x < t$. We define $u = s/2$. If $5u \geq t$, then

$$\Phi(5u) \geq \Phi(t) = ct \geq cs = 2\Phi(2u) = 2cu.$$

And if $5u \leq t$, then

$$\Phi(5u) \geq 5cu/2 = 5\Phi(2u)/2 > 2\Phi(2u) = 2cu.$$

Hence there exist arbitrarily large u satisfying (15).

It suffices to prove that if $M > 0$ and $\varepsilon > 0$, then there exists v satisfying $v > u$, $\Phi'(v) \geq c/8$, and

(16) $$\Phi(v + h) \leq \Phi(v) + \Phi'(v)h + \varepsilon$$

for $|h| \le M/\Phi'(v)$. For $a > 0$ and $u < x < u(1+1/a)$ we define

$$F_a(x) = \frac{u\Phi(2u)}{(a+1)u - ax}.$$

If $a > 1$, then $u(1+1/a) < 2u$ so that $F_a(x) \ge \Phi(2u) \ge \Phi(x)$ for $u < x < u(1+1/a)$. If $a < 1/8$, then $u(1+1/a) > 9u > 5u$ and

$$F_a(5u) = \frac{\Phi(2u)}{1-4a} < 2\Phi(2u) \le \Phi(5u).$$

We define

$$b = \inf\{a; F_a(x) \ge \Phi(x) \text{ for } u < x < u(1+1/a)\}.$$

The above observations show that $1/8 \le b \le 1$. We define $F = F_b$ and $w = u(1+1/b)$. Then we have $F(x) \ge \Phi(x)$ for $u < x < w$ and there exists v satisfying $2u < v < w$ and $F(v) = \Phi(v)$. We deduce that $F'(v) = \Phi'(v)$.

We have

$$F(v) = \frac{u\Phi(2u)}{(b+1)u - bv} = \frac{u\Phi(2u)}{b(w-v)}$$

and

$$\frac{F'(v)}{F(v)} = \frac{1}{w-v}.$$

We deduce that

$$F'(v) = \frac{F(v)}{w-v} \ge \frac{\Phi(2u)}{w-u} \ge cb \ge \frac{c}{8} \tag{17}$$

and

$$w - v = \frac{F(v)}{F'(v)} \ge \frac{\Phi(2u)}{F'(v)} > \frac{M}{F'(v)} \tag{18}$$

if u is large enough. Moreover,

$$v - u \ge u > \frac{M}{F'(v)} \tag{19}$$

for large u. Next we have

$$\frac{F'(v+t)}{F'(v)} = (\frac{w-v}{w-v-t})^2 = (\frac{F'(v)(w-v)}{F'(v)(w-v) - tF'(v)})^2 = (\frac{F(v)}{F(v) - tF'(v)})^2.$$

if $u < v + t < w$. It follows that

$$(1 - \frac{\varepsilon}{M})F'(v) \le F'(v+t) \le (1 + \frac{\varepsilon}{M})F'(v)$$

if $|t| \le M/F'(v)$ and if u is large enough. We deduce that

$$\begin{aligned} F(v+h) &= F(v) + \int_0^h F'(v+t)dt \\ &\le F(v) + (1 + \frac{\varepsilon}{M})F'(v)h \\ &\le F(v) + F'(v)h + \varepsilon, \end{aligned}$$

if $0 < h \le M/F'(v)$. Similarly, we find that

$$F(v+h) \le F(v) + F'(v)h + \varepsilon \tag{20}$$

for $-M/F'(v) \le h < 0$. Altogether, we see that (20) holds for all h satisfying $|h| \le M/F'(v)$. Moreover, if $|h| \le M/F'(v)$, then $u < v+h < w$ by (18) and (19). Using $F(v) = \Phi(v)$ and $F'(v) = \Phi'(v)$ we deduce that

$$\Phi(v+h) \le F(v+h) \le F(v) + F'(v)h + \varepsilon = \Phi(v) + \Phi'(v)h + \varepsilon$$

for $|h| \le M/\Phi'(v)$. This, together with (17), completes the proof of the lemma.

LEMMA 2 *If $M > 0$ and if $\Phi(x)$ is a convex, increasing, and differentiable function, then there exists a set E of finite measure such that (16) holds for all $v \in E$, provided $|h| \le M/\Phi'(v)$.*

Proof. Since Φ is convex, Φ' is nondecreasing and a classical lemma of Borel (cf. [13]) implies that

$$\Phi'(v + \frac{M}{\Phi'(v)}) \le (1 + \frac{\varepsilon}{M})\Phi'(v) \tag{21}$$

except possibly for a set of v-values having finite measure. Similarly,

$$(1 - \frac{\varepsilon}{M})\Phi'(v) \le \Phi'(v - \frac{M}{\Phi'(v)}) \tag{22}$$

except possibly for a set of finite measure. As in the proof of Lemma 1 we deduce that (16) holds outside a possible exceptional set of finite measure.

LEMMA 3 *Let $a(x)$ and $b(x)$ be increasing and differentiable for $x \ge x_0$ and suppose that $a(x) \le b(x)$ for arbitrarily large x. Define*

$$\varphi(t) = \frac{1}{\frac{d}{dt}b^{-1}(t)}.$$

If $K > 1$, then

$$a'(x) \le K\varphi(a(x))$$

on a set of x-values of upper density at least $(K-1)/K$.

Proof. Lemmas of this type are standard in Nevanlinna theory (cf. e. g. [14, p. 253]), but we include the short proof for completeness. We define

$$E_y = \{x | a'(x) > K\varphi(a(x)), x_0 \le x \le y\}.$$

Then the conclusion follows from the inequality

$$\begin{aligned} K\int_{E_y} dx &\le \int_{E_y} \frac{a'(x)}{\varphi(a(x))}dx \\ &\le \int_{x_0}^{y} \frac{a'(x)}{\varphi(a(x))}dx \\ &= \int_{a(x_0)}^{a(y)} \frac{du}{\varphi(u)} \\ &= b^{-1}(a(y)) - b^{-1}(a(x_0)) \\ &\le y - b^{-1}(a(x_0)), \end{aligned}$$

which holds for arbitrarily large y.

3 Proofs of Theorems 1 and 2

Proof of Theorem 1. We put $r = e^x$ so that $\Phi(x) = \log \gamma(r)$, choose (x_j), (M_j), and (ε_j) according to Lemma 1, and define $r_j = e^{x_j}$ and $\mu_j = r\gamma'(r_j)/\gamma(r_j)$, that is, $\mu_j = \Phi'(x_j)$. Then we have

$$\gamma(r) \leq e^{\varepsilon_j} (\frac{r}{r_j})^{\mu_j} \gamma(r_j) = (1 + o(1))(\frac{r}{r_j})^{\mu_j} \gamma(r_j). \tag{23}$$

for $|\log(r/r_j)| \leq M_j/\mu_j$. Since f has infinite order, we have $\gamma(r) \geq r^4$ for arbitrarily large r, that is, $\Phi(x) \geq 4x$ for arbitrarily large x. Hence $\mu_j \geq 1/2$. Let

$$m^*(re^{i\theta}) = \sup \frac{1}{2\pi} \int_E \log |f(re^{i\varphi})| d\varphi$$

be the function introduced by Baernstein [2]. Here the supremum is taken over all measurable subsets E of $(-\pi, \pi)$ whose measure equals 2θ. For z in the upper half-plane, we define

$$v_j(z) = m^*(z^{1/2\mu_j})$$

which is possible since $\mu_j \geq 1/2$. Then we define $R_j = (r_j)^{2\mu_j}$ and

$$V_j(R) = \max_{0 \leq \theta \leq \pi} v_j(Re^{i\theta}).$$

We have

$$V_j(R) \leq T_N(R^{1/2\mu_j}) = T(R^{1/2\mu_j}) + O(1) \leq \gamma(R^{1/2\mu_j}) + O(1) \leq (1 + o(1))(\frac{R}{R_j})^{1/2} \gamma(r_j)$$

if $|\log(R/R_j)^{1/2\mu_j}| \leq M_j/\mu_j$, that is, if $e^{-2M_j} \leq R/R_j \leq e^{2M_j}$. Following Baernstein we estimate $v_j(R_j e^{i\theta})$ by the Poisson integral for the half-disc and we find that (cf. [2])

$$v_j(R_j e^{i\theta}) \leq (1 + o(1))\gamma(r_j) \sin \frac{\theta}{2}.$$

We deduce that

$$m^*(r_j e^{i\varphi}) = v(R_j e^{i2\mu_j \varphi}) \leq (1 + o(1))\gamma(r_j) \sin \mu_j \varphi$$

and hence that

$$\begin{aligned} \log M(r_j) &= \lim_{\varphi \to 0} \frac{\pi}{\varphi} m^*(r_j e^{i\varphi}) \\ &\leq (1 + o(1))\pi\gamma(r_j) \lim_{\varphi \to 0} \frac{\sin \mu_j \varphi}{\varphi} \\ &= (1 + o(1))\pi\gamma(r_j)\mu_j \qquad (24) \\ &= (1 + o(1))\pi r_j \gamma'(r_j). \end{aligned}$$

This proves (a). To prove (b), we note that Lemma 2 implies that (23) and hence (24) hold for any sequence (r_j) which tends to ∞ and is outside some exceptional set of logarithmic density zero, with suitable sequences (M_j) and (ε_j).

Proof of Theorem 2. We apply Lemma 3 for $a(x) = T(e^x)$ and $b(x) = \gamma(e^x)$. It follows that $A(r) \leq K\Psi(T(r))$ on a set of positive upper logarithmic density, provided $K > 1$. Theorem 1 (b) implies that $\log M(r) \leq (1 + o(1))\pi K\Psi(T(r))$ on a set of positive upper logarithmic density. Letting K tend to 1, we deduce Theorem 2.

4 Proof of Theorem 3

For the proof of Theorem 3, we need a result of Warschawski [16] concerning conformal mappings of strips.

Lemma 4 *Let $\theta(x)$ be positive and continuously differentiable for all x and suppose that $\theta(x)$ satisfies the conditions $\theta'(x) = o(1)$ as $x \to \infty$ and*

$$\int_0^\infty \frac{(\theta'(x))^2}{\theta(x)} dx < \infty.$$

Let F be a conformal mapping which maps the strip $\{z : |\mathrm{Im}\ z| < \theta(x)\}$ onto the strip $\{z : |\mathrm{Im}\ z| < \frac{\pi}{2}\}$ in such a way that $F(z) \to \pm\infty$ as $z \to \pm\infty$. Then there exists a real constant λ such that

$$F(x+iy) = \frac{\pi}{2}\int_0^x \frac{dt}{\theta(t)} + \lambda + i\frac{\pi y}{2\theta(x)} + o(1)$$

as $x \to \infty$.

Proof of Theorem 3. From our assumption that $(\log\gamma(r))/(\log r) \to \infty$ we deduce that $\log h(x)/x \to \infty$ as $x \to \infty$. Hence the function

$$\theta(x) = \frac{\pi}{2}\Big(\int_x^\infty \frac{dt}{h(t)}\Big) h(x)$$

is well-defined. First we prove that $\theta(x)$ satisfies the hypotheses of Lemma 4. To do this, we note that

$$1 \le \frac{h''(x)h(x)}{(h'(x))^2} \le 1 + \sqrt{\frac{h(x)}{h'(x)}} = 1 + o(1). \tag{25}$$

by the convexity of $\log h$ and (14). We deduce that

$$\frac{1}{h'(x)} = \int_x^\infty \frac{h''(t)}{(h'(t))^2}dt \ge \int_x^\infty \frac{dt}{h(t)} \ge (1-o(1))\int_x^\infty \frac{h''(t)}{(h'(t))^2}dt = (1-o(1))\frac{1}{h'(x)}$$

so that

$$\theta(x) = (1+o(1))\frac{\pi h(x)}{2h'(x)} = o(1).$$

Moreover, we have

$$\begin{aligned}
0 &\ge \theta'(x) \\
&= \frac{\pi}{2}\Big(-1 + \Big(\int_x^\infty \frac{dt}{h(t)}\Big)h'(x)\Big) \\
&\ge \frac{\pi}{2}\Big(-1 + \Big(\int_x^\infty \frac{h''(t)}{(h'(t))^2}\Big(1+\sqrt{\frac{h(t)}{h'(t)}}\Big)^{-1} dt\Big)h'(x)\Big) \\
&\ge \frac{\pi}{2}\Big(-1 + \Big(1+\sqrt{\frac{h(x)}{h'(x)}}\Big)^{-1}\Big(\int_x^\infty \frac{h''(t)}{(h'(t))^2}dt\Big)h'(x)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\pi}{2}\sqrt{\frac{h(x)}{h'(x)}}\Big(1+\sqrt{\frac{h(x)}{h'(x)}}\Big)^{-1} \\
&\geq -\frac{\pi}{2}\sqrt{\frac{h(x)}{h'(x)}} \\
&\geq -(1+o(1))\sqrt{\frac{\pi}{2}}\sqrt{\theta(x)}.
\end{aligned}
$$

Therefore $\theta'(x) \to 0$ as $x \to \infty$ and

$$\int_x^\infty \frac{(\theta'(t))^2}{\theta(t)}dt \leq (1+o(1))\sqrt{\frac{\pi}{2}}\int_x^\infty \frac{-\theta'(t)}{\sqrt{\theta(t)}} = (2+o(1))\sqrt{\frac{\pi}{2}}\sqrt{\theta(x)} < \infty.$$

Hence Lemma 4 is applicable.

We shall also need further relations between $h(x)$ and $\theta(x)$. It follows from the definition of $\theta(x)$ that

$$\frac{h'(x)}{h(x)} = \frac{\theta'(x)}{\theta(x)} + \frac{\pi}{2\theta(x)}.$$

Hence

$$h(x) = c\exp\Big(\frac{\pi}{2}\int_0^x \frac{dt}{\theta(t)}\Big)\theta(x)$$

for some constant c. This implies that

$$h'(x) = c\exp\Big(\frac{\pi}{2}\int_0^x \frac{dt}{\theta(t)}\Big)\big(\frac{\pi}{2}+\theta'(x)\big) \sim c\frac{\pi}{2}\exp\Big(\frac{\pi}{2}\int_0^x \frac{dt}{\theta(t)}\Big).$$

Let now $F(z)$ be a conformal map as in Lemma 4. We may choose F such that $\lambda = \log(c\pi^2/2)$. By Lemma 4, we have

$$\exp F(\log(re^{i\varphi})) \sim e^\lambda \exp\Big(\frac{\pi}{2}\int_0^{\log r}\frac{dt}{\theta(t)} + i\frac{\pi\varphi}{2\theta(\log r)}\Big).$$

Now we define $u(z) = \text{Re } \exp F(\log z)$ if $|\arg\, z| \leq \theta(\log|z|)$ and $u(z) = 0$ else. Then $u(z)$ is harmonic for $|\arg\, z| \neq \theta(\log|z|)$ and subharmonic elsewhere. Moreover, we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\varphi})d\varphi &\sim \frac{1}{2\pi}e^\lambda \exp\Big(\frac{\pi}{2}\int_0^{\log r}\frac{dt}{\theta(t)}\Big)\int_{-\pi}^{\pi}\cos\Big(\frac{\pi\varphi}{2\theta(\log r)}\Big)d\varphi \\
&= \frac{c\pi}{4}\exp\Big(\frac{\pi}{2}\int_0^{\log r}\frac{dt}{\theta(t)}\Big)\frac{2\theta(\log r)}{\pi}\int_{-\pi}^{\pi}\cos u\; du \\
&= h(\log r) \\
&= \gamma(r).
\end{aligned}
$$

To obtain an entire function which has the desired properties, we use a result of Al-Katifi [1, Theorem 2.1] which says that there exists an entire function f satisfying

$$|\log|f(z)| - u(z)| \leq K\Big(\log|z| + \log^+\frac{1}{\delta(z)}\Big),$$

where K is a constant and where $\delta(z)$ denotes the distance of z from the curves given by $|\arg z| = \theta(\log|z|)$. As shown by Al-Katifi [1, Theorem 2.3], the last two equations imply that $T(r) \sim \gamma(r)$. Formal differentiation with respect to $\log r$ yields $A(r) \sim r\gamma'(r)$, and a result of London [10, p. 502] shows that this formal differentiation is justified by the right inequality of (25). Moreover, we can show that $\log M(r) \sim u(r) \sim \pi r\gamma'(r)$. (We omit the technical details.)

Finally, if $u < v$, then

$$\begin{aligned}
0 &\leq \Psi(v) - \Psi(u) \\
&= h'(h^{-1}(v)) - h'(h^{-1}(u)) \\
&= \int_u^v \frac{d}{dt} h'(h^{-1}(t))dt \\
&= \int_u^v \frac{h''(h^{-1}(t))}{h'(h^{-1}(t))}dt \\
&\leq (1+o(1)) \int_u^v \frac{h'(h^{-1}(t))}{h(h^{-1}(t))}dt \\
&\leq (1+o(1)) \frac{h'(h^{-1}(v))}{h(h^{-1}(v))}(v-u) \\
&= (1+o(1)) \frac{\Psi(v)}{v}(v-u)
\end{aligned}$$

as $u \to \infty$. It follows that $\Psi(v) \sim \Psi(u)$ if $v \sim u$. Hence $\Psi(T(r)) \sim \Psi(\gamma(r)) = r\gamma'(r)$. This completes the proof of Theorem 3.

5 Remarks

1. In the introduction, we remarked that (2) follows from (1) if there exist Pólya peaks for $T(r)$ of order μ for some $\mu \geq \frac{1}{2}$. As explained there, this follows from the fact that if (r_j) is a sequence of Pólya peaks of order μ for $T(r)$, then $A(r_j) \sim \mu T(r_j)$ as $j \to \infty$. To prove the latter assertion, choose a positive sequence (δ_j) which tends to zero and define $r_j' = (1+\delta_j)r_j$. Then

$$\begin{aligned}
(1-o(1))A(r_j)\delta_j &= A(r_j)\log(1+\delta_j) \\
&= A(r_j)\int_{r_j}^{r_j'} \frac{dt}{t} \\
&\leq T(r_j') - T(r_j) \\
&\leq \Big((1+o(1))\Big(\frac{r_j'}{r_j}\Big)^{\mu} - 1\Big)T(r_j) \\
&= \Big((1+o(1))(1+\delta_j)^{\mu} - 1\Big)T(r_j) \\
&= (1+o(1))\mu\delta_j T(r_j)
\end{aligned}$$

so that $A(r_j) \leq (1+o(1))\mu T(r_j)$. Similarly, by choosing δ_j negative, we obtain the opposite inequality $A(r_j) \geq (1+o(1))\mu T(r_j)$.

2. It is clear that the conclusion of Theorem 1 (b) does not hold, if we assume only that $\gamma(r)$ is increasing and differentiable. This is also true in the special case $\gamma(r) = T(r)$. In

fact, there exist entire functions (of infinite lower order) satisfying

$$\lim_{\substack{r\to\infty\\ r\in G}} \frac{\log M(r)}{A(r)} = \infty$$

for some set G of upper logarithmic density one. This can be seen by considering a gap series. We omit the technical details, but we sketch the idea. If the gaps are large, then there exist large intervals $[r, R]$ such that $T(t) \sim \log M(t)$ and $A(r) \sim A(t)$ for $t \in [r, R]$. It follows that if $r \le t \le R$, then

$$(1 - o(1))A(t) \le A(r) = \log\left(\frac{r}{t}\right) \int_r^t \frac{A(r)}{u} du \le \log\left(\frac{r}{t}\right) T(t) \le \log\left(\frac{r}{t}\right) \log M(t),$$

and our assertion follows, if the intervals $[r, R]$ are chosen such that $r/R \to 0$. This can be achieved by a suitable choice of the gaps and the coefficients of the function.

3. Although the hypothesis that $\log \gamma(r)$ be convex in $\log r$ cannot be omitted in Theorem 1 (b), it is probably not necessary in Theorem 2. On the other hand, it is satisfied in most applications like Corollaries 2 and 3. Also, in terms of Ψ and ϕ, where Ψ and ϕ are defined by (10) and (8), this hypothesis means that $\Psi(t)/t$ and $\phi(x)$ are non-decreasing. These are exactly the hypotheses that were made in the proofs of (6) and (9).

4. It is easy to see that the exponent $\frac{3}{2}$ in (14) may be replaced by any constant less than 2. I do not know, however, whether a condition of this type is necessary for the validity of Theorem 3.

5. It is not difficult to see that for any entire function f, there exists a function $\gamma(r)$ which satisfies the hypotheses of Theorems 1 (b) and 2. We also note that for the function $\Psi(t)$ defined in Theorem 2 we always have

$$\int_{t_0}^{\infty} \frac{dt}{\Psi(t)} = \infty,$$

so that in some sense Theorem 2 gives a better estimate for $\log M(r)$ in terms of $T(r)$ than the one obtained from (6) and (7).

6. It does not seem possible to prove Theorems 1 and 2 by using only (5) and growth lemmas for real functions like the classical one of Borel. However, using these methods we can obtain weaker estimates of this type where the constant π appearing in Theorems 1 and 2 is replaced by $2e$ (cf. [8, Chapter 1]).

7. In the proof of Theorem 3, we have used a result of Al-Katifi [1] concerning subharmonic functions. An alternative method to obtain an entire function f with the desired properties is to define f by a suitable contour integral (cf. [11] or [8, p. 81]).

8. It was proved by Petrenko [15] that (1) also holds for meromorphic functions. For functions of infinite lower order, however, there is a difference between entire and meromorphic functions. As proved by Dai, Drasin, and Li [4], there is an extra factor $\log \phi(\log T(r))$ in the denominator in (9) if f is meromorphic, and this result is sharp. It is quite clear from [4] that the results of this paper do not hold for meromorphic functions. An interesting problem is to determine the sharp bounds for $\log M(r)$ in terms of $A(r)$ and $T(r)$ under the hypotheses of Theorems 1 and 2 for meromorphic f.

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