LYAPUNOV EXPONENTS AND RELATED CONCEPTS FOR ENTIRE FUNCTIONS

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Abstract. Let f be an entire function and denote by $f^{\#}$ the spherical derivative of f and by f^n the n-th iterate of f. For an open set U intersecting the Julia set $J(f)$, we consider how fast $\sup_{z\in U}(f^n)^{\#}(z)$ and $\int_U (f^n)^{\#}(z)^2 dx\, dy$ tend to ∞. We also study the growth rate of the sequence $(f^n)^{\#}(z)$ for $z \in J(f)$.

1. Introduction and results

The Julia set $J(f)$ of a rational or entire function f, which we always assume to be neither constant nor rational of degree 1, is the set of all points where the iterates f^n of f do not form a normal family. Let

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$$

be the spherical derivative of f. Marty's theorem yields that a point $\xi \in \mathbb{C}$ is contained in $J(f)$ if and and only if

$$\sup_{n\in\mathbb{N}} \sup_{z\in U} (f^n)^{\#}(z) = \infty$$

for every neighborhood U of ξ. Putting

$$\mu(U, f) = \sup_{z\in U} f^{\#}(z),$$

we thus see that the sequence $(\mu(U, f^n))_{n\in\mathbb{N}}$ is unbounded. It is not difficult to see that it actually tends to ∞ and we are interested in the question of how fast it tends to ∞.

We shall also consider how fast the sequence $((f^n)^{\#}(z))_{n\in\mathbb{N}}$ can grow for a point $z \in J(f)$. A result of Przytycki [25] says that for a rational function f there exist points $z \in J(f)$ such that $((f^n)^{\#}(z))_{n\in\mathbb{N}}$ has essentially the same growth rate as $(\mu(U, f^n))_{n\in\mathbb{N}}$; see (1.3) and the subsequent remarks for the precise statement. We shall see that, in contrast, for certain transcendental entire functions f the sequence $(\mu(U, f^n))_{n\in\mathbb{N}}$ may grow much faster than any of the sequences $((f^n)^{\#}(z))_{n\in\mathbb{N}}$.

Let

$$M(r, f) = \max_{|z|=r} |f(z)|$$

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be the maximum modulus of f and denote by $M^n(r,f)$ the iterate of $M(r,f)$ with respect to the first variable; that is,

$$M^1(r,f) = M(r,f) \quad \text{and} \quad M^{n+1}(r,f) = M(M^n(r,f),f).$$

It is easy to see that $M^n(R,f)\to\infty$ if R is sufficiently large.

Theorem 1.1. *Let f be an entire function and let U be an open set intersecting the Julia set of f. Then, for any $R>0$, there exists $m\in\mathbb{N}$ such that*

$$\mu(U,f^n) \geq \log M^{n-m}(R,f)$$

for large n.

For $f(z)=z^d$ we have $\log M^n(R,f)=d^n\log R$ and $\mu(U,f^n)\sim d^n/2$ as $n\to\infty$ if U intersects the unit circle. So Theorem 1.1 gives the correct order of magnitude for polynomials.

Next we show that analogous results hold if the supremum of the spherical derivative is replaced by the normalized spherical area

$$S(U,f) = \frac{1}{\pi}\int_U f^{\#}(z)^2 dx\, dy.$$

For a rational function f of degree d we have

$$c\, d^n \leq S(U,f^n) \leq d^n.$$

for some positive constant c and thus [30, Theorem 1]

$$\lim_{n\to\infty}\frac{1}{n}\log S(U,f^n) = \log d.$$

Since $\mu(U,f)\geq\sqrt{S(U,f)}$ this implies that

$$\lim_{n\to\infty}\frac{1}{n}\log \mu(U,f^n) \geq \frac{1}{2}\log d. \tag{1.1}$$

Barrett and Eremenko [3, inequality (13) and the remarks following it] showed that we always have strict inequality in (1.1), but that the constant $1/2$ on the right hand side cannot be replaced by a larger constant.

When dealing with rational functions, it is more systematic to consider

$$\|f'(z)\| = f^{\#}(z)(1+|z|^2) = |f'(z)|\frac{1+|z|^2}{1+|f(z)|^2}$$

instead of $f^{\#}(z)$, and this is the quantity considered in [3]. We note that Theorem 1.1 holds if $f^{\#}$ is replaced by $\|f'\|$ in the definition of $\mu(U,f)$. An analogous remark applies to the results below.

Theorem 1.2. *Let f be a transcendental entire function and let U be an open set intersecting the Julia set of f. Then, for any $R>0$, there exists $m\in\mathbb{N}$ such that*

$$S(U,f^n) \geq \log M^{n-m}(R,f)$$

for large n.

This result gives the right order of magnitude for the growth of $S(U, f^n)$.

Theorem 1.3. *Let f be a transcendental entire function and let U be a bounded open subset of $\mathbb{C}$. Then there exists $R > 0$ such that*

$$S(U, f^n) \le \log M^n(R, f)$$

for all $n \in \mathbb{N}$.

We note that it is easy to deduce from Theorems 1.1 and 1.2 that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(U, f^n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log S(U, f^n) = \infty \tag{1.2}$$

for a transcendental entire function f. The second equation answers a question from [30], where it was shown that this holds under various additional hypotheses.

We now consider how fast $(f^n)^{\#}(z)$ can tend to ∞ for a point $z \in J(f)$. A result of Przytycki says that for rational functions the maximal growth rate of the sequence $((f^n)^{\#}(z))$ over all $z \in J(f)$ is essentially the same as the one obtained when restricting to periodic points z only. More precisely, Przytycki showed ([25], the proof is reproduced in [19]) that if f is a rational function, then

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{z\in\mathbb{C}} \log \|(f^n)'(z)\| = \sup_{z\in \operatorname{Per}(f)} \lim_{n\to\infty} \frac{1}{n} \log \|(f^n)'(z)\|, \tag{1.3}$$

where $\operatorname{Per}(f)$ denotes the set of periodic points of f. Note that if z is a periodic point of f, say $f^p(z) = z$ and $\lambda = (f^p)'(z)$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \|(f^n)'(z)\| = \lim_{n\to\infty} \frac{1}{n} \log (f^n)^{\#}(z) = \frac{\log |\lambda|}{p}. \tag{1.4}$$

The limit on the right hand side is called the *Lyapunov exponent* of f at z and denoted by $\chi(f, z)$. More generally,

$$\overline{\chi}(f, z) = \limsup_{n\to\infty} \frac{1}{n} \log (f^n)^{\#}(z) \quad \text{and} \quad \underline{\chi}(f, z) = \liminf_{n\to\infty} \frac{1}{n} \log (f^n)^{\#}(z)$$

are called the *upper* and *lower Lyapunov exponent* of f at z. For some recent results on Lyapunov exponents we refer to [18, 19, 23] for rational maps and to [14, 23] for entire functions in the exponential family. Lyapunov exponents of general entire functions do not seem to have been studied yet.

On the left hand side of (1.3) one may replace the supremum over all $z \in \mathbb{C}$ by the supremum over all $z \in U$, if U is an open set intersecting $J(f)$. Thus (1.3) takes the form

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(U, f^n) = \sup_{z\in \operatorname{Per}(f)} \chi(f, z), \tag{1.5}$$

Eremenko and Levin [16, Theorem 3] showed that if f is a polynomial of degree $d \ge 2$, then there exists a periodic point z such that $\chi(f, z) \ge \log d$, with strict inequality unless f is conjugate to the monomial $z \mapsto z^d$. It follows from (1.1) and (1.3) that if f is a rational function of degree $d \ge 2$, then there exists a periodic point z such that $\chi(f,z) > (\log d)/2$; see also [12, 19, 31] for related results. Finally, (1.2) and (1.5) suggest that if f is a transcendental entire function, then

$$\sup_{z\in \operatorname{Per}(f)} \chi(f,z) = \infty.$$

It follows from the results in [6] that this is indeed the case.

Theorem 1.4. *Let f be a transcendental entire function. Then the set of all z such that $\chi(f,z) = \infty$ is dense in $J(f)$.*

The essential statement here is that there exists $z \in J(f)$ with $\chi(f,z) = \infty$. Once this is known, it is easy to see that the set of all such points is dense in $J(f)$. Note that such points cannot be periodic since $\chi(f,z) < \infty$ for a periodic point z by (1.4).

It seems plausible that Theorem 1.4 can be improved by giving a lower bound for $(f^n)^{\#}(z)$ which depends on the maximum modulus of f. However, Theorem 1.6 below will show that such a lower bound will have to be much smaller than that given in Theorems 1.1 and 1.2.

We can give such a lower bound for functions in the Eremenko-Lyubich class B consisting of all transcendental entire functions for which the set of critical and (finite) asymptotic values is bounded. In fact, we only need to assume that f has a logarithmic singularity over ∞. This includes functions in B since for such functions all singularities over ∞ are logarithmic.

The lower order $\lambda(f)$ of an entire function f is defined by

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}. \tag{1.6}$$

Taking the limes superior in (1.6) yields the order $\rho(f)$.

Theorem 1.5. *Let f be a transcendental entire function with a logarithmic singularity over ∞. Then the set of all z such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log\log (f^n)^{\#}(z) \geq \log(1+\lambda(f)) \tag{1.7}$$

is dense in $J(f)$.

If f has a logarithmic singularity over ∞, then $\lambda(f) \geq 1/2$; see, e.g., [7, Proof of Corollary 2] or [22, p. 1788] for this observation. Hence Theorem 1.5 yields the following result.

Corollary 1.1. *Let f be a transcendental entire function with a logarithmic singularity over ∞. Then the set of all z such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log\log (f^n)^{\#}(z) \geq \log\frac{3}{2}$$

is dense in $J(f)$.

Theorems 1.5 and Corollary 1.1 are sharp. More precisely, we have the following result.

Theorem 1.6. *For each $\rho \in [1/2, \infty)$ there exists $f \in B$ with $\lambda(f) = \rho(f) = \rho$ such that if $z \in \mathbb{C}$ satisfies $\chi(f, z) = \infty$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log\log (f^n)^{\#}(z) \leq \log(1+\rho). \tag{1.8}$$

Combining Theorem 1.6 with Theorems 1.1 and 1.2 we see that there exist entire functions f for which $\mu(U, f^n)$ and $S(U, f^n)$ grow substantially faster than $(f^n)^{\#}(z)$ for any $z \in J(f)$.

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2. Background from complex dynamics and function theory

For an introduction to the iteration theory of entire functions we refer to [4, 29]. A basic result of the theory is the following lemma.

Lemma 2.1. *The Julia set of a transcendental entire function is the closure of the set of repelling periodic points.*

For rational functions this result was obtained by both Fatou and Julia, for transcendental entire functions it is due to Baker [2].

The *exceptional set* $E(f)$ of an entire function f is the set of all $z \in \mathbb{C}$ for which the backward orbit

$$O^-(z) = \bigcup_{n=0}^{\infty} f^{-n}(z)$$

is finite. It is a simple consequence of Picard's theorem that $E(f)$ contains at most one point. The following result is sometimes called the "blowing-up property" of the Julia set.

Lemma 2.2. *Let f be entire, $U \subset \mathbb{C}$ open with $U \cap J(f) \neq \emptyset$ and $K \subset \mathbb{C} \backslash E(f)$ compact. Then $f^n(U) \supset K$ for all large $n \in \mathbb{N}$.*

The *escaping set*

$$I(f) = \{x \in \mathbb{R}^m \colon f^n(x) \to \infty\},$$

introduced in [15], plays an important role in transcendental dynamics. Its subset

$$A(f) = \left\{ z \in \mathbb{C} \colon \text{ there exists } l \in \mathbb{N} \text{ with } |f^n(z)| > M(R, f^{n-l}) \text{ for } n \geq l \right\}, \tag{2.1}$$

where $R > \min_{z \in J(f)} |z|$ and $J(f)$ is the Julia set, is called the *fast escaping set*. It was introduced in [9] and has also turned out to be very useful in transcendental dynamics. A thorough study of this set is given in [28] where it is also shown that

$$A(f) = \left\{ z \in \mathbb{C} \colon \text{ there exists } l \in \mathbb{N} \text{ with } |f^n(z)| > M^{n-l}(R, f) \text{ for } n \geq l \right\}, \tag{2.2}$$

with R so large that

$$M^n(R, f) \to \infty \quad \text{as } n \to \infty. \tag{2.3}$$

The equivalence of (2.1) and (2.2) is also apparent from the following lemma proved in [11, Lemma 2.1].

Lemma 2.3. *Let f be a transcendental entire function and $\varepsilon > 0$. Then there exists $R > 0$ such that if $r > R$ and $n \in \mathbb{N}$, then*

$$M((1+\varepsilon)r, f^n) \geq M^n(r, f).$$

The following lemma [26, Lemma 2.2] is a a consequence of Hadamard's three circles theorem; that is, the convexity of $\log M(r, f)$ in $\log r$.

Lemma 2.4. *Let f be a transcendental entire function. Then there exists $R > 0$ such that, for all $r \geq R$ and all $c > 1$,*

$$\log M(r^c, f) \geq c \log M(r, f).$$

The next lemma can be found in [15] for the escaping set and in [9, 28] for the fast escaping set.

Lemma 2.5. *Let f be entire. Then $J(f) = \partial I(f) = \partial A(f)$.*

The next lemma consists of Koebe's distortion theorem and Koebe's one quarter theorem. Here and in the following we denote by $D(a, r)$ the open disk of radius r around a.

Lemma 2.6. *Let $g\colon D(a, r) \to \mathbb{C}$ be univalent, $0 < \rho < 1$ and $z \in D(a, \rho r)\backslash\{a\}$. Then*

$$\frac{1}{(1+\rho)^2} \leq \frac{|g(z) - g(a)|}{|g'(a)| \cdot |z - a|} \leq \frac{1}{(1-\rho)^2}$$

and

$$\frac{1-\rho}{(1+\rho)^3} \leq \frac{|g'(z)|}{|g'(a)|} \leq \frac{1+\rho}{(1-\rho)^3}.$$

Moreover,

$$g(D(a, r)) \supset D\left(g(a), \frac{1}{4}|g'(a)|r\right).$$

Koebe's theorems are usually only stated for the special case that $a = 0$, $r = 1$, $g(0) = 0$ and $g'(0) = 1$, but the above version follows immediately from this special case.

The following lemma is Harnack's inequality.

Lemma 2.7. *Let $u\colon D(a, r) \to \mathbb{R}$ be a positive harmonic function, $0 < \rho < 1$ and $z \in D(a, \rho r)$. Then*

$$\frac{1-\rho}{1+\rho} \leq \frac{u(z)}{u(a)} \leq \frac{1+\rho}{1-\rho}.$$

3. Proof of Theorem 1.1

The following lemma is similar to results given in [8, 13, 24]. Here and in the following we denote by $D(a, r)$ the open disk around a of radius r.

Lemma 3.1. *Let $f\colon D(a,r)\to\mathbb{C}$ be holomorphic and $K,L>0$. Suppose that $|f(a)|\le K$ and that $|f'(z)|\le L$ whenever $|f(z)|=K$. Then*

$$|f(z)|<K\exp\left(\frac{2L}{K}|z-a|\right)\quad \text{for } z\in D\left(a,\frac{r}{2}\right).$$

Proof. We follow the arguments in [8, p. 303] and put $u(z)=\log(|f(z)|/K)$ so that $|\nabla u|=|f'/f|$. With $G=\{z\in D(a,r)\colon |f(z)|>K\}$ we then have $a\notin G$ and

$$|\nabla u(z)|\le\frac{L}{K}\quad\text{for } z\in D(a,r)\cap\partial G.$$

Let $z\in G\cap D(a,r/2)$ and put $d(z)=\operatorname{dist}(z,\partial G)$. Since $a\notin G$ we have $d(z)\le|z-a|$ and thus there exists $z_1\in\partial G\cap\partial D(z,d(z))\cap D(a,r)$. For $0<s<1$ we put $z_s=z+s(z_1-z)=sz_1+(1-s)z$ and deduce from Harnack's inequality that

$$u(z_s)\ge\frac{1-s}{1+s}u(z).$$

It follows that

$$\frac{u(z)}{(1+s)d(z)}\le\frac{u(z_s)}{(1-s)d(z)}=\frac{u(z_s)-u(z_1)}{(1-s)|z-z_1|}=\frac{u(z_s)-u(z_1)}{|z_s-z_1|}.$$

Passing to the limit as $s\to1$ we obtain

$$u(z)\le2|\nabla u(z_1)|d(z)\le\frac{2L}{K}d(z)\le\frac{2L}{K}|z-a|\quad\text{for } z\in G\cap D\left(a,\frac{r}{2}\right),$$

from which the conclusion follows. ∎

Proof of Theorem 1.1. Since, by Lemma 2.1, repelling periodic points are dense in $J(f)$, we may assume without loss of generality that $U=D(a,r)$ for some repelling periodic point a and some $r>0$. Since a is periodic we have $|f^n(a)|\le K$ for some K and all n. Lemma 2.5 implies that there exists $b\in A(f)\cap D(a,r/2)$. Thus $|f^n(b)|\ge M^{n-m}(R,f)$ for some $m\in\mathbb{N}$ and all $n\ge m$. Here $R>0$ is chosen such that (2.3) is satisfied.

With

$$L=\frac{K}{r}\log\frac{M^{n-m}(R,f)}{K}$$

we thus have

$$|f^n(b)|\ge K\exp\left(\frac{Lr}{K}\right)>K\exp\left(\frac{2L}{K}|b-a|\right).$$

Applying Lemma 3.1 we see that there exists $\xi\in D(0,r)$ with $|(f^n)'(\xi)|>L$ and $|f^n(\xi)|=K$. It follows that

$$(f^n)^{\#}(\xi)>\frac{L}{1+K^2}=\frac{K}{r(1+K^2)}\log\frac{M^{n-m}(R,f)}{K}.$$

We may assume here that $r < K/(1+K^2)$ so that the first term on the right side is greater than 1. From this we can deduce that

$$(f^n)^{\#}(\xi) \geq \log M^{n-m}(R,f)$$

for large n. □

4. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We may assume that 0 is a periodic point. Since [21, p. 13] the Nevanlinna characteristic $T(r,f)$ and the Ahlfors-Shimizu characteristic $T_0(r,f)$ satisfy

$$\left|T(r,f) - T_0(r,f) - \log^+|f(0)|\right| \leq \frac{1}{2}\log 2,$$

this yields that

$$|T(r,f^n) - T_0(r,f^n)| \leq C \tag{4.1}$$

for some constant C independent of n.

Choosing $r_1 > 1$ such that $E(f) \subset D(0,r_1)$ and $r_2 > r_1$ we then have

$$f^k(U) \supset D(0,r_2)\backslash D(0,r_1)$$

for some $k \in \mathbb{N}$ by Lemma 2.2. Hence

$$S(U,f^{n+k}) \geq S(D(0,r_2)\backslash D(0,r_1),f^n) = S(r_2,f^n) - S(r_1,f^n), \tag{4.2}$$

with $S(r,f) = S(D(0,r),f)$.

We use the standard estimates

$$\begin{aligned}\frac{1}{2}S(\sqrt{r},f)\log r &= \int_{\sqrt{r}}^{r}\frac{S(\sqrt{r},f)}{t}dt \leq \int_0^r \frac{S(t,f)}{t}dt \\ &= T_0(r,f) \leq S(r,f)\log r + T_0(1,f)\end{aligned}$$

which may also be written as

$$\frac{T_0(r,f) - T_0(1,f)}{\log r} \leq S(r,f) \leq \frac{T_0(r^2,f)}{\log r}. \tag{4.3}$$

Now (4.2) and (4.3) give

$$S(U,f^{n+k}) \geq \frac{T_0(r_2,f^n) - T_0(1,f^n)}{\log r_2} - \frac{T_0(r_1^2,f^n)}{\log r_1} \geq \frac{T_0(r_2,f^n)}{\log r_2} - 2\frac{T_0(r_1^2,f^n)}{\log r_1}$$

and thus

$$S(U,f^{n+k}) \geq \frac{T(r_2,f^n)}{\log r_2} - 2\frac{T(r_1^2,f^n)}{\log r_1} - \frac{2C}{\log r_1}$$

by (4.1). With the standard estimate

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r}T(R,f)$$

this yields

$$S(U,f^{n+k}) \geq \frac{1}{3}\frac{\log M(\frac{1}{2}r_2,f^n)}{\log r_2} - 2\frac{\log^+ M(r_1^2,f^n)}{\log r_1} - \frac{2C}{\log r_1}. \tag{4.4}$$

Lemma 2.3 implies that $M(r_2/2, f^n) \geq M^n(r_2/4, f)$ if r_2 was chosen large enough. Thus (4.4) yields

$$\begin{aligned} S(U, f^{n+k}) &\geq \frac{1}{3} \frac{\log M^n(\frac{1}{4}r_2, f)}{\log r_2} - 2\frac{\log^+ M^n(r_1^2, f)}{\log r_1} - \frac{2C}{\log r_1} \\ &\geq \frac{1}{3} \frac{\log M^n(\frac{1}{4}r_2, f)}{\log r_2} - 3\frac{\log^+ M^n(r_1^2, f)}{\log r_1}, \end{aligned}$$

provided r_1 is chosen large enough.

With $R_1 = M(r_1^2, f)$ and $R_2 = M(r_2/4, f)$ this takes the form

$$S(U, f^{n+k}) \geq \frac{1}{3} \frac{\log M^{n-1}(R_2, f)}{\log R_2} \frac{\log R_2}{\log r_2} - 3\frac{\log^+ M^{n-1}(R_1, f)}{\log R_1} \frac{\log R_1}{\log r_1}. \tag{4.5}$$

Since f is transcendental,

$$\frac{\log M(r, f)}{\log r} \to \infty$$

as $r \to \infty$. This implies that we may choose r_1 and r_2 such that

$$\frac{\log R_1}{\log r_1} \geq 1 \quad \text{and} \quad \frac{\log R_2}{\log r_2} \geq 12\frac{\log R_1}{\log r_1}.$$

We may also assume that $M^{n-1}(R_2, f) \geq 1$ for all n. Thus (4.5) yields

$$S(U, f^{n+k}) \geq \left(4\frac{\log^+ M^{n-1}(R_2, f)}{\log R_2} - 3\frac{\log^+ M^{n-1}(R_1, f)}{\log R_1}\right) \frac{\log R_1}{\log r_1} \tag{4.6}$$

if r_1 was chosen large enough.

Since $M(r, f) \geq r$ for large r, Lemma 2.4 yields that there exists R_0 such that for $r \geq R_0$ and $c \geq 1$ we have

$$\log M^n(r^c, f) \geq c \log M^n(r, f). \tag{4.7}$$

This is equivalent to saying that

$$\frac{\log M^n(r, f)}{\log r}$$

is a non-decreasing function of r for $r \geq R_0$. Since we may assume that $R_1 \geq R_0$ this yields

$$\frac{\log M^{n-1}(R_2, f)}{\log R_2} \geq \frac{\log M^{n-1}(R_1, f)}{\log R_1}.$$

It now follows from (4.6) that

$$S(U, f^{n+k}) \geq \frac{\log M^{n-1}(R_2, f)}{\log R_2}$$

if r_2 (and hence R_2) is large enough.

We now choose l such that $M^l(R, f) \geq R^{\log R_2}$ and deduce that

$$S(U, f^{n+k}) \geq \frac{\log M^{n-1-l}(M^l(R_2, f), f)}{\log R_2} \geq \frac{\log M^{n-1-l}(R^{\log R_2}, f)}{\log R_2}.$$

Using (4.7) again this finally yields

$$S(U, f^{n+k}) \geq \log M^{n-1-l}(R, f).$$

The conclusion now follows with $m = l + k + 1$. □

Proof of Theorem 1.3. Choose $R \geq e^4$ with $M^n(R, f) \to \infty$ as $n \to \infty$ such that $U \subset D\left(0, \sqrt{R}\right)$. Then

$$S(U, f^n) \leq S\left(\sqrt{R}, f^n\right) \leq \frac{2T_0(R, f^n)}{\log R} \leq \frac{1}{2}\left(T(R, f^n) + C\right)$$
$$\leq \frac{1}{2}\left(\log M(R, f^n) + C\right) \leq \log M^n(R, f)$$

for large n by (4.1) and (4.3). Increasing R if necessary we may achieve that this holds for all $n \in \mathbb{N}$. □

5. Proof of Theorem 1.4

We will use the following result [6, Theorem 1.2] already quoted in the introduction.

Lemma 5.1. *Let f be a transcendental entire function and let $p \in \mathbb{N}$, $p \geq 2$. Then there exists a sequence (a_k) of fixed points of f^p such that $(f^p)'(a_k) \to \infty$ as $k \to \infty$.*

Proof of Theorem 1.4. We apply this lemma for $p = 2$. We may assume that all a_k are repelling fixed points of f^2. Then there exist $r_k > 0$ such that f^2 is univalent in the disk $D_k = D(a_k, r_k)$. Moreover, we may assume that there exists an increasing sequence (λ_k) tending to ∞ such that

$$\left|(f^2)'(z)\right| \geq \lambda_k > 1 \quad \text{for } z \in D_k \tag{5.1}$$

and that there exists a domain W_k satisfying $\overline{W_k} \subset D_k$ such that $f^2\colon W_k \to D_k$ is univalent. This implies that for every $n_k \in \mathbb{N}$ there exists a domain V_k satisfying $\overline{V_k} \subset D_k$ such that $f^{2n_k}\colon V_k \to D_k$ is univalent.

We put $D_0 = U$. By the Ahlfors islands theorem (see [21, Section 5] or [5]), for each $k \in \mathbb{N}$ there exist $m_k \in \mathbb{N}$ and a subdomain U_{k-1} of D_{k-1} such that $f^{m_k}\colon U_{k-1} \to D_j$ is univalent for some $j \in \{k, k+1, k+2\}$. We may assume that this holds for $j = k$ since otherwise we may restrict to a subsequence of (a_k). Thus $f^{m_k}\colon U_{k-1} \to D_k$ is univalent for $k \in \mathbb{N}$.

We conclude that for each $l \in \mathbb{N}$ there exists subdomain X_l of U_0 such that

$$f^{m_l} \circ f^{2n_{l-1}} \circ f^{m_{l-1}} \circ \cdots \circ f^{m_2} \circ f^{2n_1} \circ f^{m_1}\colon X_l \to D_l$$

is univalent, with $\overline{X_{l+1}} \subset X_l$. It follows that there exists

$$z \in \bigcap_{l=1}^{\infty} X_l = \bigcap_{l=1}^{\infty} \overline{X_l}.$$

We show that we can achieve $\chi(f, z) = \infty$ by choosing the sequence (n_k) rapidly increasing.

In order to do so we note that once the sequences (m_l) and (U_l) are fixed, there are also sequences (α_l) and (β_l) of positive numbers such that

$$\left|(f^k)'(\zeta)\right| \geq \alpha_l \quad \text{and} \quad \left|(f^k)(\zeta)\right| \leq \beta_l \quad \text{for } 0 \leq k \leq m_l \text{ and } \zeta \in U_{l-1}. \tag{5.2}$$

Here, as usual, $f^0(\zeta) = \zeta$, so for $k = 0$ the first inequality just means that $\alpha_l \leq 1$.

With the sequence (n_k) still to be determined, we define sequences (N_l) and (M_l) by

$$N_l = \sum_{k=1}^{l}(2n_k + m_k) \quad \text{and} \quad M_l = N_l - 2n_l = N_{l-1} + m_l$$

so that $M_l < N_l < M_{l+1}$. We may choose (n_k) such that

$$n_l = \frac{1}{2}(N_l - M_l) \geq \frac{1}{4}(N_l + m_{l+1}) + 1 = \frac{1}{4}M_{l+1} + 1 \tag{5.3}$$

and

$$\lambda_l^{n_l/2} \geq \frac{1+\beta_{l+1}^2}{\prod_{k=1}^{l+2}\alpha_k} \geq \frac{1+\beta_{l+1}^2}{\prod_{k=1}^{l+1}\alpha_k} \tag{5.4}$$

for all l.

Suppose first that $n \in \mathbb{N}$ is such that $N_l < n \leq M_{l+1} = N_l + m_{l+1}$ for some $l \in \mathbb{N}$. We deduce from (5.1), (5.2), (5.3) and (5.4) that

$$\begin{aligned} |(f^n)'(z)| &= \left|(f^{n-N_l} \circ f^{2n_l} \circ f^{m_l} \circ \cdots \circ f^{m_2} \circ f^{2n_1} \circ f^{m_1})(z)\right| \\ &\geq \prod_{k=1}^{l} \lambda_k^{n_k} \cdot \prod_{k=1}^{l+1} \alpha_k \geq \lambda_l^{n_l} \cdot \prod_{k=1}^{l+1}\alpha_k \geq (1+\beta_{l+1}^2)\lambda_l^{n_l/2} \\ &\geq (1+\beta_{l+1}^2)\lambda_l^{M_{l+1}/8} \geq (1+\beta_{l+1}^2)\lambda_l^{n/8} \end{aligned} \tag{5.5}$$

and hence, using (5.2) again, that

$$\left|(f^n)^{\#}(z)\right| \geq \frac{|(f^n)'(z)|}{1+\beta_{l+1}^2} \geq \lambda_l^{n/8}. \tag{5.6}$$

Suppose next that $M_{l+1} < n \leq N_{l+1} = M_{l+1} + 2n_{l+1}$ for some $l \in \mathbb{N}$. Using the same arguments as before we find that

$$\begin{aligned} |(f^n)'(z)| &= \left|(f^{n-M_{l+1}} \circ f^{m_{l+1}} \circ f^{2n_l} \circ \cdots \circ f^{m_2} \circ f^{2n_1} \circ f^{m_1})(z)\right| \\ &\geq \lambda_{l+1}^{(n-M_{l+1})/2} \cdot \prod_{k=1}^{l} \lambda_k^{n_k} \cdot \prod_{k=1}^{l+1}\alpha_k \geq \lambda_{l+1}^{(n-M_{l+1})/2}(1+\beta_{l+1}^2)\lambda_l^{n_l/2} \end{aligned}$$

if $n - M_{l+1}$ is even while

$$\begin{aligned} |(f^n)'(z)| &= \left|(f^{n-M_{l+1}} \circ f^{m_{l+1}} \circ f^{2n_l} \circ \cdots \circ f^{m_2} \circ f^{2n_1} \circ f^{m_1})(z)\right| \\ &\geq \alpha_{l+2} \cdot \lambda_{l+1}^{(n-M_{l+1}-1)/2} \cdot \prod_{k=1}^{l} \lambda_k^{n_k} \cdot \prod_{k=1}^{l+1}\alpha_k \geq \lambda_{l+1}^{(n-M_{l+1}-1)/2}(1+\beta_{l+1}^2)\lambda_l^{n_l/2} \end{aligned}$$

if $n - M_{l+1}$ is odd. Thus

$$|(f^n)'(z)| \geq \lambda_{l+1}^{(n-M_{l+1}-1)/2}(1+\beta_{l+1}^2)\lambda_l^{n_l/2}$$

in both cases. Since (λ_k) is increasing and

$$n - M_{l+1} - 1 + n_l \geq n - \frac{3}{4}M_{l+1} \geq \frac{n}{4}$$

by (5.3), we find that

$$|(f^n)'(z)| \geq (1+\beta_{l+1}^2)\lambda_l^{(n-M_{l+1}-1+n_l)/2} \geq (1+\beta_{l+1}^2)\lambda_l^{n/8},$$

which is the same inequality as (5.5). We conclude that (5.6) and hence

$$\frac{1}{n}\log\left|(f^n)^{\#}(z)\right| \geq \frac{1}{8}\log\lambda_l$$

holds for all $n \geq N_1$. Since l and hence λ_l tend to ∞ with n, this yields $\chi(f,z)=\infty$.

To prove that the set of all ζ with $\chi(f,\zeta)=\infty$ is dense in $J(f)$ we note that if this holds for $\zeta = z$, then it also holds for $\zeta = f^n(z)$ if $n \in \mathbb{N}$. More generally, it holds for all ζ for which there exist $m, n \in \mathbb{N}$ such that $f^m(\zeta) = f^n(z)$ and $(f^m)'(\zeta) \neq 0$. The set of all such ζ is easily seen to be dense in $J(f)$, using the Ahlfors island theorem – or the simpler result that if $a_1, a_2, a_3 \in \mathbb{C}$ are distinct, then the family of all functions holomorphic in a domain which have no simple a_j-points for all j is normal. □

Remark 5.1. Given a sequence (V_k) of open sets intersecting $J(f)$, one may choose the sequences (m_k) and (U_k) in the above proof such that $f^{l_k}(U_{k-1}) \subset V_k$ for some $l_k \leq m_k$. Using this it is not difficult to see that one may choose z with the additional property that the orbit of z is dense in $J(f)$.

Similarly, given any sequence (c_k) of positive real numbers tending to ∞, one may choose z such that $|f^k(z)| \leq c_k$ for all large k. This can be achieved by choosing (n_k) large; a similar idea appears in [27].

6. Proof of Theorem 1.5

We recall the logarithmic change of variable for a function f in the Eremenko-Lyubich class; see [17, §2]. For simplicity we will assume that all singularities of the inverse are in the unit disk and that $|f(0)| < 1$. The general case can be reduced to this. Let U be a logarithmic tract of f, that is, a component of $\{z \in \mathbb{C}\colon |f(z)| > 1\}$. Let $H = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$ be the right half-plane and $W = \exp^{-1}(U)$. Then there exists a $2\pi i$-periodic holomorphic function $F\colon W \to H$ satisfying $\exp F(z) = f(e^z)$, and the restriction of F to a component of W maps this component biholomorphically onto H.

We call F the function obtained from f by a logarithmic change of variable. The main tool when working with the Eremenko-Lyubich class is the inequality

$$|F'(z)| \geq \frac{1}{4\pi}\operatorname{Re} F(z) \quad \text{for } z \in W \tag{6.1}$$

obtained by them. We will also need the following lower bound for F'.

Lemma 6.1. *Let $F\colon W \to H$ be the function obtained from a logarithmic change of variable as above. For $z \in W$ let $z_1 \in \partial W$ with $|z_1 - z| = \operatorname{dist}(z, \partial W)$. Then*

$$|F'(\zeta)| \geq \frac{1}{8\pi}\operatorname{Re} F(z) \tag{6.2}$$

for all ζ in the straight line segment from z to z_1.

Proof. Let $G\colon H\to W$ be the branch of the inverse of F with $G(F(\zeta))=\zeta$. Since G is univalent in $D(F(\zeta),\operatorname{Re}F(\zeta))$, Koebe's one quarter theorem yields that

$$W\supset G(H)\supset G(D(F(\zeta),\operatorname{Re}F(\zeta)))\supset D\left(\zeta,\frac{1}{4}|G'(F(\zeta))|\operatorname{Re}F(\zeta)\right)$$

and hence

$$\frac{\operatorname{Re}F(\zeta)}{4|F'(\zeta)|}=\frac{1}{4}|G'(F(\zeta))|\operatorname{Re}F(\zeta)\le\operatorname{dist}(\zeta,\partial W)=|\zeta-z_1|. \tag{6.3}$$

We note that (6.1) follows from this by noting that $\operatorname{dist}(\zeta,\partial W)\le\pi$. In fact, this is the proof of (6.1) given in [17].

To prove (6.2), we write $\zeta=z+s(z_1-z)=sz_1+(1-s)z$ with $0<s<1$ and put $u(z)=\operatorname{Re}F(z)$. Harnack's inequality yields that

$$u(\zeta)\ge\frac{1-s}{1+s}u(z).$$

Hence

$$|\zeta-z_1|=(1-s)|z-z_1|\le 2\frac{u(\zeta)}{u(z)}|z-z_1|=2|z-z_1|\frac{\operatorname{Re}F(\zeta)}{\operatorname{Re}F(z)}.$$

Together with (6.3) this yields

$$\frac{1}{4|F'(\zeta)|}\le\frac{2|z-z_1|}{\operatorname{Re}F(z)},$$

from which the conclusion follows since $|z-z_1|\le\pi$. ☐

For $f\in B$ and the function $F\colon W\to H$ obtained from f by the logarithmic change of variable we put $\alpha=\inf\{\operatorname{Re}z\colon z\in W\}$. As in [10, §3] we consider the function $h\colon(\alpha,\infty)\to(0,\infty)$ defined by

$$h(x)=\max_{\operatorname{Re}z=x}\operatorname{Re}F(z)=\max_{y\in\mathbb{R}}\operatorname{Re}F(x+iy).$$

Note that h is increasing by the maximum principle. Moreover, h is convex by analogy to Hadamard's three circles theorem.

Lemma 6.2. *For $x>\alpha$ let $z_x\in W$ with*

$$\operatorname{Re}z_x=x\quad\textit{and}\quad\operatorname{Re}F(z_x)=h(x).$$

Then for each $t\in(0,h(x)]$ there exists $\zeta_t\in D(z_x,\operatorname{dist}(z_x,\partial W))$ such that

$$\operatorname{Re}F(\zeta_t)=t\quad\textit{and}\quad|F'(\zeta_t)|\ge\frac{h(x)}{8\pi}. \tag{6.4}$$

Moreover, if $t\ge 8\pi$ and if U_t is the component of $F^{-1}(D(F(\zeta_t),4\pi))$ that contains ζ_t, then

$$|F'(z)|\ge\frac{h(x)}{96\pi}\quad\textit{for } z\in U_t \tag{6.5}$$

and

$$U_t\subset D(\zeta_t,2). \tag{6.6}$$

Proof. It follows from Lemma 6.1 that there exists $\zeta_t \in D(z_x, \operatorname{dist}(z_x, \partial W))$ satisfying (6.4).

Let now $t \geq 8\pi$ and, as in the proof of Lemma 6.1, let $G\colon H \to W$ be the branch of the inverse of F with $G(F(\zeta_t)) = \zeta_t$. Since G is univalent in $D(F(\zeta_t), \operatorname{Re} F(\zeta_t))$, Koebe's distortion theorem implies that

$$|G'(w)| \leq |G'(F(\zeta_t))| \frac{1+4\pi/t}{(1-4\pi/t)^3} \leq 12|G'(F(\zeta_t))| \quad \text{for } w \in D(F(\zeta_t), 4\pi)$$

and hence that

$$|F'(z)| \geq \frac{1}{12}|F'(\zeta_t)| \quad \text{for } z \in U_t,$$

which together with (6.4) yields (6.5). Koebe's distortion theorem and (6.4) also yield that if $z \in U_t$, then

$$|z - \zeta_t| \leq |G'(F(\zeta_t))| \frac{4\pi/t}{(1-4\pi/t)^2} \leq |G'(F(\zeta_t))| \frac{16\pi}{t} = \frac{16\pi}{|F'(\zeta_t)|t} \leq \frac{128\pi^2}{h(x)t}.$$

Since $8\pi \leq t \leq h(x)$ this yields (6.6). ☐

Lemma 6.3. *Let $(x_n)_{n\geq 0}$ be a sequence of positive numbers satisfying*

$$x_n > \max\{\alpha, 8\pi\} \quad \textit{and} \quad x_{n+1} \leq h(x_n)$$

for all $n \geq 0$. Then there exists $u \in W$ such that

$$|\operatorname{Re} F^n(u) - x_n| \leq 4\pi \quad \textit{and} \quad |(F^n)'(u)| \geq \frac{1}{(96\pi)^n} \prod_{j=0}^{n-1} h(x_j)$$

for all $n \geq 1$.

Proof. First we choose z_0 with $\operatorname{Re} z_0 = x_0$ and $\operatorname{Re} F(z_0) = h(x_0)$. By Lemma 6.2 there exist $\zeta_0 \in D(z_0, 2)$ such that with $\xi_1 = F(\zeta_0)$ we have $\operatorname{Re} \xi_1 = x_1$ and such that the component V_1 of $F^{-1}(D(\xi_1, 4\pi)$ that contains ζ_0 satisfies $V_1 \subset D(\zeta_0, 2)$ and

$$|F'(z)| \geq \frac{1}{96\pi} h(x_0) \quad \text{for } z \in V_1. \tag{6.7}$$

We now choose a point z_1 with $\operatorname{Re} z_1 = \operatorname{Re} \xi_1 = x_1$ and $|\operatorname{Im} z_1 - \operatorname{Im} \xi_1| \leq \pi$ such that $\operatorname{Re} F(z_1) = h(x_1)$. Using Lemma 6.2 again, we see that there exists $\zeta_1 \in D(z_1, \pi) \subset D(\xi_1, 2\pi)$ such that with $\xi_2 = F(\zeta_1)$ we have $\operatorname{Re} \xi_2 = x_2$, and the component U_2 of $F^{-1}(D(\xi_2, 4\pi)$ that contains ζ_1 satisfies $U_2 \subset D(\zeta_1, 2)$ and

$$|F'(z)| \geq \frac{1}{96\pi} h(x_1) \quad \text{for } z \in U_2. \tag{6.8}$$

Note that that $U_2 \subset D(\xi_1, 2 + 2\pi)$. Since $F\colon V_1 \to D(\xi_1, 4\pi)$ is biholomorphic we deduce that there exists a domain V_2 satisfying $\overline{V_2} \subset V_1$ such that $F\colon V_2 \to U_2$ is biholomorphic. Hence $F^2\colon V_2 \to D(\xi_2, 4\pi)$ is biholomorphic. Moreover, it follows from (6.7) and (6.8) that

$$|(F^2)'(z)| \geq \frac{1}{(96\pi)^2} h(x_0) h(x_1) \quad \text{for } z \in V_2.$$

Inductively we thus find a sequence (ξ_n) of points satisfying $\operatorname{Re}\xi_n = x_n$ and a sequence (V_n) of domains satisfying $\overline{V_n} \subset V_{n-1}$ such that $F^n \colon V_n \to D(\xi_n, 4\pi)$ is biholomorphic

$$|(F^n)'(z)| \geq \frac{1}{(96\pi)^n} \prod_{j=0}^{n-1} h(x_j) \quad \text{for } z \in V_n.$$

The conclusion now follows by choosing $u \in \bigcap_{n=1}^{\infty} V_n$. □

Proof of Theorem 1.5. By hypothesis we have

$$\log M(r, f) \geq r^{\lambda(f) - o(1)}$$

as $r \to \infty$. In terms of F this takes the form

$$h(x) \geq e^{(\lambda - \varepsilon(x))x}$$

where $\lambda = \lambda(f)$ and $\varepsilon(x) \to 0$. We may assume here that ε is non-increasing, since otherwise we may replace it by

$$\varepsilon^*(x) = \sup_{t \geq x} \varepsilon(t).$$

We now consider, for $x > 1$,

$$\delta(x) = \max\left\{\varepsilon(x), \frac{1}{\log x}\right\}. \tag{6.9}$$

Then δ is non-increasing. We now choose x_0 large,

$$x_1 = \frac{\lambda - \delta(x_0)}{1 + \delta(x_0)} x_0 \quad \text{and} \quad x_{n+1} = \frac{1 + \lambda}{1 + \delta(x_n)} x_n \tag{6.10}$$

for $n \geq 1$. It follows from (6.10) that there exists a sequence (η_n) tending to 0 such that

$$x_n = (1 + \lambda + \eta_n)^n, \tag{6.11}$$

provided x_0 was chosen large enough.

Induction shows that

$$\sum_{j=0}^{n-1} (\lambda - \delta(x_j)) x_j \geq (1 + \delta(x_{n-1})) x_n \tag{6.12}$$

for all $n \geq 1$. Indeed, this holds for $n = 1$ by the choice of x_1 and assuming that (6.12) holds we obtain, using that $\delta(x)$ is non-increasing,

$$\begin{aligned}
\sum_{j=0}^{n} (\lambda - \delta(x_j)) x_j &= \sum_{j=0}^{n-1} (\lambda - \delta(x_j)) x_j + (\lambda - \delta(x_n)) x_n \\
&\geq (1 + \delta(x_{n-1})) x_n + (\lambda - \delta(x_n)) x_n \\
&= (1 + \lambda + \delta(x_{n-1}) - \delta(x_n)) x_n \\
&\geq (1 + \lambda) x_n = (1 + \delta(x_n)) x_{n+1}.
\end{aligned}$$

It follows from (6.11) that Lemma 6.3 is applicable if x_0 was chosen large enough. With u as in this lemma we thus have

$$\begin{aligned}|(F^n)'(u)| &\geq \frac{1}{(96\pi)^n}\prod_{j=0}^{n-1} h(x_j)\\ &\geq \exp\left(\sum_{j=0}^{n-1}(\lambda-\delta(x_j))x_j - n\log(96\pi)\right)\\ &\geq \exp((1+\delta(x_{n-1}))x_n - n\log(96\pi)).\end{aligned} \tag{6.13}$$

Since $\exp F(u) = f(e^u)$ we find, with $z = e^u$, that

$$(F^n)'(u) = z\frac{(f^n)'(z)}{f^n(z)}.$$

Since $\operatorname{Re} F^n(u) \to \infty$ and thus $|f^n(z)| \to \infty$ by (6.11) and Lemma 6.3, we have

$$(f^n)^{\#}(z) \geq \frac{|(f^n)'(z)|}{2|f^n(z)^2|} = \frac{|(F^n)'(u)|}{2|z|\exp(\operatorname{Re} F^n(u))}$$

for large n. Combined with (6.9) and (6.13) this yields

$$\begin{aligned}(f^n)^{\#}(z) &\geq \exp((1+\delta(x_{n-1}))x_n - n\log(96\pi) - x_n - 4\pi\log(2|z|))\\ &= \exp(\delta(x_{n-1})x_n - n\log(96\pi) - 4\pi\log(2|z|))\\ &\geq \exp\left(\frac{x_n}{\log x_n} - n\log(96\pi) - 4\pi\log(2|z|)\right).\end{aligned}$$

For large n we thus have

$$(f^n)^{\#}(z) \geq \exp\left(\frac{x_n}{2\log x_n}\right)$$

and hence

$$\begin{aligned}\log\log(f^n)^{\#}(z) &\geq \log x_n - \log\log x_n - \log 2\\ &= n\log(1+\lambda+\eta_n) - \log n - \log\log(1+\lambda+\eta_n) - \log 2,\end{aligned}$$

from which (1.7) follows.

Once it is known that there exists one point z satisfying (1.7), it follows as in the proof of Theorem 1.4 that the set of all such z is dense in $J(f)$. □

7. Proof of Theorem 1.6

Mittag-Leffler's function

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}$$

satisfies $\rho(E_\alpha) = \lambda(E_\alpha) = 1/\alpha$. It was shown in [1, Section 4] that f is in the Eremenko-Lyubich class if $0 < \alpha < 2$. Since $E_2(z) = \cosh\sqrt{z}$ this also holds for $\alpha = 2$.

For $0 < \alpha < 2$ and $\rho = 1/\alpha$ we have (see [20, p. 85] and [1, Section 4]), for sufficiently small $\delta > 0$,

$$E_\alpha(z) = \varrho \exp(z^\varrho) + O\left(\frac{1}{|z|}\right) \quad \text{for } |\arg(z)| \leq \frac{\alpha\pi}{2} + \delta,$$
$$E'_\alpha(z) = \varrho^2 z^{\varrho-1} \exp(z^\varrho) + O\left(\frac{1}{|z|^2}\right) \quad \text{for } |\arg(z)| \leq \frac{\alpha\pi}{2} + \delta,$$
$$E_\alpha(z) = O\left(\frac{1}{|z|}\right) \quad \text{for } \frac{\alpha\pi}{2} + \delta < |\arg(z)| \leq \pi,$$
$$E'_\alpha(z) = O\left(\frac{1}{|z|^2}\right) \quad \text{for } \frac{\alpha\pi}{2} + \delta < |\arg(z)| \leq \pi.$$

This implies that there exists constants A and B such that

$$|E'_\alpha(z)| \leq A|z|^{\rho-1}|E_\alpha(z)| + B$$

for all $z \in \mathbb{C}$. With $C = A + B$ we thus have

$$|E'_\alpha(z)| \leq C|z|^{\rho-1}|E_\alpha(z)| \quad \text{if } |z| \geq 1 \text{ and } |E_\alpha(z)| \geq 1. \tag{7.1}$$

Since $E_2(z) = \cosh\sqrt{z}$ the last estimate also holds for $\alpha = 2$.

We consider the function

$$f(z) = \eta E_\alpha(z)$$

where $0 < \eta < 1$. Since $E_\alpha \in B$ we have $f \in B$. By choosing η small we can achieve that

$$|f(z)| < 1 \quad \text{and} \quad |f'(z)| < 1 \quad \text{for } |z| \leq 1. \tag{7.2}$$

Moreover, (7.1) implies that

$$|f'(z)| \leq C|z|^{\rho-1}|f(z)| \quad \text{if } |z| \geq 1 \text{ and } |f(z)| \geq 1. \tag{7.3}$$

Suppose now that z satisfies $\chi(f, z) = \infty$ and put $z_n = f^n(z)$ for $n \geq 0$ so that $z_0 = z$. It follows from (7.2) that if $|z_N| \leq 1$ for some $N \geq 0$, then $(f^n)'(z) \to 0$ and hence $(f^n)^\#(z) \to 0$ as $n \to \infty$. Thus we may assume that $|z_n| \geq 1$ for all $n \geq 0$. Using (7.3) we see that

$$|(f^n)'(z)| = \prod_{j=0}^{n-1} |f'(z_j)| \leq C^n \prod_{j=0}^{n-1} |z_j|^{\rho-1}|z_j| = \frac{C^n|z_n|}{|z_0|} \prod_{j=0}^{n-1} |z_j|^\rho.$$

Hence

$$(f^n)^\#(z) \leq \frac{|(f^n)'(z)|}{|(f^n)(z)|^2} \leq \frac{C^n}{|z_n|} \prod_{j=0}^{n-1} |z_j|^\rho. \tag{7.4}$$

If

$$|z_n| > \prod_{j=0}^{n-1} |z_j|^\rho, \tag{7.5}$$

then (7.4) yields that $(f^n)^\#(z) \leq C^n$. Since $\chi(f,z) = \infty$, we deduce that (7.5) cannot hold for infinitely many n. Thus there exists $n_0 \in \mathbb{N}$ such that

$$|z_n| \leq \prod_{j=0}^{n-1} |z_j|^\rho \quad \text{for } n \geq n_0.$$

We put $t_n = \log |z_n|$. Then the last inequality takes the form

$$t_n \leq \rho \sum_{j=0}^{n-1} t_j \quad \text{for } n \geq n_0.$$

This implies that

$$\sum_{j=0}^{n} t_j = t_n + \sum_{j=0}^{n-1} t_j \leq (1+\rho) \sum_{j=0}^{n-1} t_j.$$

Induction yields that

$$\sum_{j=0}^{n} t_j \leq c_0 (1+\rho)^n \quad \text{for } n \geq n_0 - 1, \tag{7.6}$$

with

$$c_0 = (1+\rho)^{-n_0+1} \sum_{j=0}^{n_0-1} t_j.$$

Using (7.4) and (7.6) we find for $n \geq n_0$ that

$$\log (f^n)^\#(z) \leq \log \left(C^n \prod_{j=0}^{n-1} |z_j|^\rho \right) = n \log C + \rho \sum_{j=0}^{n-1} t_j \leq n \log C + \rho c_0 (1+\rho)^{n-1}.$$

Hence

$$\log (f^n)^\#(z) = O((1+\rho)^n)$$

as $n \to \infty$, which yields (1.8).

Remark 7.1. The proof of Theorem 1.5 shows that if $f \in B$ or, more generally, if f has a logarithmic singularity over ∞, then there exists $z \in I(f)$ satisfying (1.7). In particular, there exists $z \in I(f)$ with $\chi(f,z) = \infty$. We do not know whether this holds for all transcendental entire functions f.

On the other hand, it follows from the proof of Theorem 1.6 that in general there does not exist $z \in A(f)$ with $\chi(f,z) = \infty$. Indeed, it is easily seen that if f is as there, $z \in A(f)$ and $z_n = f^n(z_n)$, then (7.5) holds for large n. As shown in the proof of Theorem 1.6 this is incompatible with $\chi(f,z) = \infty$.

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