Entire functions with two radially distributed values

Walter Bergweiler, Alexandre Eremenko* and Aimo Hinkkanen

Abstract

We study entire functions whose zeros and one-points lie on distinct finite systems of rays. General restrictions on these rays are obtained. Non-trivial examples of entire functions with zeros and one-points on different rays are constructed, using the Stokes phenomenon for second order linear differential equations.

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1 Introduction

The zeros of an entire function can be arbitrarily assigned, but in general one cannot assign the preimages of two values [25]. Since this work of Nevanlinna, various necessary conditions which the sets of zeros and 1-points of an entire function must satisfy were found; see, e.g., [26, 29, 35]. Besides an intrinsic interest, these conditions are relevant to control theory [9, 5, 16].

In this paper we study the simplest setting when the zeros and 1-points lie on finitely many rays, or are close to finitely many rays. The word "ray" in this paper will always mean a ray from the origin. For an entire function f, we say that a value a is *radially distributed* if the set $f^{-1}(a)$ is contained in the union of finitely many rays.

We begin by recalling some classical results.

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Theorem A. (A. Edrei [15]) *Suppose that all zeros and* 1*-points of an entire function* f *are distributed on a finite set of rays, and let* ω *be the smallest angle between these rays. Then the order of* f *is at most* π/ω.

The most difficult part of this theorem is the conclusion that the order is finite. Some special cases of Theorem A under the a priori assumption that the order is finite were proved by Bieberbach [6].

In the 1920s, Biernacki [7] and Milloux [24] studied the case that all zeros of a transcendental entire function f lie on some line and all 1-points lie on a different line. Under certain additional hypotheses, including in particular the assumption that the order is not an integer, Biernacki showed that the lines must be parallel. Milloux obtained this result under less restrictive hypotheses, with additional conclusions on the order of the function. Both Biernacki and Milloux considered only functions of finite order, but by Edrei's Theorem A we know now that this is a consequence of the other hypotheses.

The following corollary from our main results completely describes the situation for the case that the lines intersect.

Theorem 1. *Suppose that all zeros of an entire function* f *lie on a line* L_1 *and all* 1*-points lie on a different line* L_2 *intersecting* L_1*. Then* f *is either of the form* $f(z) = e^{az+b}$ *or* $f(z) = 1 - e^{az+b}$*, or a polynomial of degree at most* 2.

This result is essentially contained in Milloux's paper. He did not mention the exceptional cases, perhaps assuming implicitly that there are indeed both zeros and 1-points. In his proof, Milloux omitted a detailed treatment of the case when L_2 is orthogonal to L_1. He also omitted the treatment of the case when f is of the form $f(z) = P(z)\prod_{n=1}^{\infty}(1 - z/a_n)$ where P is a polynomial and $\sum_n 1/|a_n| < \infty$. While these problems can be fixed by considering the function $f(z)\overline{f(-\overline{z})}$ in a suitable way, we include a proof of Theorem 1 below, based on different arguments and shorter than that of Milloux. In fact, we obtain Theorem 1 as a corollary of more general results.

The case of parallel lines was studied by Baker [3] and Kobayashi [21] who showed that if all zeros of a transcendental entire function f lie on a line L_1 and all 1-points lie on a different line L_2 parallel to L_1, then f has the form $f(z) = P(e^{az})$ with some $a \in \mathbb{C}\backslash\{0\}$ and a polynomial P.

Biernacki returned to the subject in 1929 and considered the more general situation where the zeros and 1-points do not lie exactly on certain lines or rays, but are only close to them. We say that points a_n (e.g., the zeros of

an entire function) accumulate in a direction φ if, for every $\varepsilon > 0$, the sector $\{z\colon |\arg z - \varphi| < \varepsilon\}$ contains infinitely many points a_n. If there is only one such direction φ (modulo 2π), we say that the points a_n accumulate in the direction φ only.

Theorem B. (M. Biernacki [8, p. 533]) *There does not exist a transcendental entire function f of finite order such that the zeros of f accumulate in a certain direction only and the* 1*-points of f accumulate in a different direction only.*

We complement the above theorems with several results. First we consider the situation when the zeros lie on a ray and the 1-points lie on two rays. Note that in view of Theorem A, these assumptions imply that the function is of finite order.

Theorem 2. *Let f be a transcendental entire function whose zeros lie on a ray L_0 and whose* 1*-points lie on two rays L_1 and L_{-1}, each of which is distinct from L_0. Suppose that the numbers of zeros and* 1*-points are infinite. Then $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$.*

It is remarkable that there are non-trivial examples of entire functions whose zeros lie on the positive ray while all 1-points lie on two rays that are not contained in the real line. As far as we know the functions given in the following theorem are the first non-trivial examples of entire functions with zeros and 1-points on finitely many rays.

Theorem 3. *For every integer $m \geq 3$, there exists an entire function f of order $1/2 + 1/m$ whose zeros are positive and whose* 1*-points lie on the two rays $\{z\colon \arg z = \pm 2\pi/(m+2)\}$.*

Theorem 1 implies that such functions do not exist for $m = 2$, except for the trivial case where f has no zeros at all and is of the form $f(z) = e^{az}$.

Taking $f(z^n)$ with f as in Theorem 3 we obtain an entire function whose zeros lie on n rays and whose 1-points lie on $2n$ rays distinct from those rays where the zeros lie.

Now we relax the condition that the zeros and 1-points are radially distributed. Let

$$A = \bigcup_{j=1}^{n} A_j, \quad A_j = \{te^{i\alpha_j} : t \geq 0\}, \quad 0 \leq \alpha_1 < \ldots < \alpha_n < 2\pi,$$

be a finite union of rays. We say that the a-points of an entire function f are *close* to the set A if they accumulate only in the directions given by A. Thus for every $\varepsilon > 0$ all but finitely many a-points are in the union of sectors

$$\bigcup_{j=1}^{n} \{z \colon |\arg z - \alpha_j| < \varepsilon\}.$$

Our next result describes the possible configurations of finite unions of rays A and B which, apart from the origin, are pairwise disjoint and have the property that the zeros are close to A and the 1-points are close to B. We will assume that the system of rays $A \cup B$ is *minimal* is the sense that for every ray $\{te^{i\alpha} \colon t \geq 0\}$ in A (respectively in B) there is a sequence (z_k) tending to ∞ such that $f(z_k) = 0$ (respectively $f(z_k) = 1$) for all k, and $\arg z_k \to \alpha$ as $k \to \infty$.

Theorem 4. *Let f be a transcendental entire function of order $\rho < \infty$ whose zeros are close to A and whose 1-points are close to B, with $A \cap B = \{0\}$. Suppose that the system $A \cup B$ is minimal.*

Then

$$\rho = \frac{\pi}{\omega} > \frac{1}{2}, \tag{1}$$

where ω is the largest *angle between adjacent rays in $A \cup B$, and there exists a system of rays $C = \bigcup_{j=1}^{2m} C_j \subset A \cup B$, with $m \geq 1$, partitioning the plane into $2m$ sectors S_j such that $\partial S_j = C_j \cup C_{j+1}$ for $1 \leq j \leq 2m-1$ and $\partial S_{2m} = C_{2m} \cup C_1$, with the following properties:*

(i) The angle of S_j at 0 is π/ρ when j is even, and at most π/ρ when j is odd.

(ii) Both boundary rays of an odd sector belong to the same set, A or B.

(iii) There are no rays of $A \cup B$ inside the even sectors.

(iv) If there are rays of A inside an odd sector, then the boundary rays of this sector belong to B. If there are rays of B inside an odd sector, then the boundary rays of this sector belong to A.

(v) If there are no rays of $A \cup B$ in an odd sector, then its opening angle is π/ρ.

Note that Theorem B is an immediate consequence of Theorem 4.

The next result – whose proof we will only sketch – shows that Theorem 4 is best possible.

Theorem 5. *Let A and B be systems of rays, satisfying conditions (i)–(v) of Theorem 4 for some $\rho \in (0, \infty)$. Then there exists an entire function of order ρ whose zeros are close to A and whose 1-points are close to B.*

Moreover, for all finite systems of rays A and B there exists an entire function of infinite order whose zeros are close to A and whose 1-points are close to B.

We note that Theorem 5 shows in particular that the hypothesis that f be of finite order is essential in Biernacki's Theorem B.

We illustrate our results by returning to the case of three distinct rays, which was already discussed in Theorems 2 and 3. Now we consider three rays

$$L_j = \{te^{ij\alpha} \colon t \geq 0\}, \quad j \in \{-1, 0, 1\},$$

with $\alpha \in (0, \pi)$. Theorems A, 1, 3 and 4 imply the following.

Theorem 3 shows that for certain $\alpha \in (0, \pi/2)$ there exists a transcendental entire function of order $\pi/(2\pi - 2\alpha)$ whose zeros lie on L_0 while its 1-points lie on $L_1 \cup L_{-1}$. This result has been extended in [17] to all $\alpha \in (0, \pi/3]$. It remains open whether such functions exist for $\alpha \in (\pi/3, \pi/2)$; see the discussion at the end of the paper on possible generalizations of this theorem.

If $\alpha = \pi/2$, then, according to Theorem 1, there is no transcendental entire function with all zeros on L_0 and all 1-points on $L_1 \cup L_{-1}$, unless it omits 0 or 1. However, the entire function $f(z) = 1/\Gamma(-z)$ has zeros on L_0 and 1-points close to the imaginary axis. This follows from Stirling's formula.

Finally, Theorem 4 implies that if $\alpha \in (\pi/2, \pi)$, then there is no transcendental entire function of finite order whose zeros are close to L_0 and whose 1-points are close to $L_1 \cup L_{-1}$.

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The plan of the paper is the following. In section 2 we first prove Theorem 4 and then deduce Theorem 1 and Theorem 2 from it. The proof of Theorem 5 showing the sharpness of Theorem 4 is then sketched in section 3. The proof of Theorem 3 is independent of the rest and will be given in section 4.

2 Proof of Theorems 4, 1 and 2

Proof of Theorem 4. If the a-points of f are close to a finite system of rays, then evidently $f(z+c)$ has the same property for every $c \in \mathbb{C}$, with the same rays. Therefore we may assume without loss of generality that

$$f(0) \notin \{0, 1\}. \tag{2}$$

We use the standard notation

$$M(r) = M(r, f) = \max_{|z| \le r} |f(z)|.$$

Let (r_k) be a sequence tending to ∞ with the property that

$$\log M(tr_k) = O(\log M(r_k)), \quad k \to \infty, \tag{3}$$

for every $t > 1$. Such sequences always exist for functions of finite order.

A sequence (r_k) is called a sequence of *Pólya peaks of order* $\lambda \in [0, \infty)$ for $\log M(r)$, if for every $\varepsilon > 0$ we have

$$\log M(tr_k) \le (1+\varepsilon) t^\lambda \log M(r_k), \quad \varepsilon \le t \le \varepsilon^{-1}, \tag{4}$$

when k is large enough. It is clear that every sequence of Pólya peaks satisfies (3). According to a result of Drasin and Shea [14], Pólya peaks of order λ exist for all finite $\lambda \in [\rho_*, \rho^*]$, where

$$\rho^* = \sup \left\{ p \in \mathbb{R} \colon \limsup_{r,t \to \infty} \frac{\log M(tr)}{t^p \log M(r)} = \infty \right\} \tag{5}$$

and

$$\rho_* = \inf \left\{ p \in \mathbb{R} \colon \liminf_{r,t \to \infty} \frac{\log M(tr)}{t^p \log M(r)} = 0 \right\}.$$

We always have

$$0 \le \rho_* \le \rho \le \rho^* \le \infty,$$

so when $\rho < \infty$, then there exist Pólya peaks of some (finite) order λ.

We refer to [19, Ch. III], [20, Ch. III] and [28] for the basic results on subharmonic functions used below. Fixing a sequence (r_k) with the property (3), we consider the two sequences (u_k) and (v_k) of subharmonic functions in $\mathbb{C}$ given by

$$u_k(z) = \frac{\log |f(r_k z)|}{\log M(r_k)} \quad \text{and} \quad v_k(z) = \frac{\log |f(r_k z) - 1|}{\log M(r_k)}. \tag{6}$$

In view of (3), these sequences are bounded from above on every compact subset of $\mathbb{C}$. It follows from (2) that the sequences $u_k(0)$ and $v_k(0)$ tend to 0. According to a well known compactness principle (see, for example, [19, Theorems 4.1.8, 4.1.9] or [20, Theorems 3.2.12, 3.2.13]), one can choose a subsequence of (r_k), which we do without changing notation, such that the limit functions

$$u(z) = \lim_{k\to\infty} \frac{\log|f(r_k z)|}{\log M(r_k)} \quad \text{and} \quad v(z) = \lim_{k\to\infty} \frac{\log|f(r_k z) - 1|}{\log M(r_k)} \tag{7}$$

exist and are subharmonic in $\mathbb{C}$. Here the convergence is in the Schwartz space $\mathscr{D}'$. It implies the convergence in L^1_{loc} and also the convergence of the Riesz measures, as the Laplacian is continuous in $\mathscr{D}'$.

The functions u and v are non-zero subharmonic functions in $\mathbb{C}$, and they have the following properties (we write $u^+ = \max\{u, 0\}$):

(a) $u^+ = v^+$.

(b) $\{z\colon u(z) < 0\} \cap \{z\colon v(z) < 0\} = \emptyset$.

(c) u is harmonic in $\mathbb{C}\backslash A$ and v is harmonic in $\mathbb{C}\backslash B$.

If (r_k) is a sequence of Pólya peaks of order $\lambda > 0$, then we have the additional property

(d) $u(0) = v(0) = 0$, and $\max\{u(z), v(z)\} \le |z|^\lambda$ for all $z \in \mathbb{C}$.

Properties (a) and (b) are evident. Property (c) holds because the Laplacian is continuous in $\mathscr{D}'$. Property (d) is a consequence of (2) and (4). Indeed, (2) and

$$u(0) \ge \limsup_{k\to\infty} u_k(0),$$

(see [19, (4.1.8)]) imply that $u(0) \ge 0$, while (4) yields $u(z) \le |z|^\lambda$ and thus, in particular, $u(0) = 0$. The same argument applies to v.

The components of the complement $\mathbb{C}\backslash(A \cup B)$ will be called *sectors of the system* $A \cup B$.

Lemma 1. *Let u and v be two non-zero subharmonic functions in the plane which satisfy (a), (b) and (c). Then either $u(z) \equiv v(z) \equiv c$ for some $c > 0$, or there exist an even number of rays $C_1, \dots, C_{2m}$, with $m \ge 1$, that belong to $A \cup B$ and partition the plane into sectors S_j, so that $\partial S_j = C_j \cup C_{j+1}$ for $1 \le j \le 2m-1$ and $\partial S_{2m} = C_{2m} \cup C_1$, such that $u(z) = v(z) > 0$ for z in the*

even sectors while $u(z) \le 0$ *and* $v(z) \le 0$ *for* z *in the odd sectors. Moreover, in each odd sector one of the functions* u, v *is strictly negative and the other is zero. If* $u(z) = 0$ *in* S_{2k+1} *then* $\partial S_{2k+1} \subset A$*, and if* $v(z) = 0$ *in* S_{2k+1} *then* $\partial S_{2k+1} \subset B$*.*

Proof of Lemma 1. If D is a sector of the system $A \cup B$, and if at some point z_0 in D, we have $\max\{u(z_0), v(z_0)\} > 0$, then $u(z) = v(z) > 0$ for all points $z \in D$. Indeed, both u and v are harmonic in D by (c), and (a) gives

$$u(z_0) = v(z_0) > 0. \tag{8}$$

If $\min\{u(z_1), v(z_1)\} < 0$ for some $z_1 \in D$, then this also holds in a neighborhood of z_1, and one of the functions u and v must be zero in this neighborhood by (b). Then it is identically equal to zero in D which contradicts (8). Thus u and v are non-negative in D, and the minimum principle implies that they are positive. Then they are equal in D by (a). Such sectors D will be called *positive* sectors.

If one of the functions u and v is constant, then both functions are equal to the same positive constant. This follows from (a) and (b) and the assumption that neither u nor v is identically zero. For the rest of the proof we assume that both u and v are non-constant.

Suppose that some ray $L \subset A \cup B$ has the property that positive sectors D_1 and D_2 are adjacent to L on both sides, that is, $L = \partial D_1 \cap \partial D_2$. (We will see in a moment that $D_1 \ne D_2$). Then we have $u(z) = v(z)$ for $z \in D = D_1 \cup D_2 \cup L$, in view of (a), and u and v must be positive and harmonic in D in view of (c).

If there are no non-positive sectors, then u and v are equal, positive and harmonic in $\mathbb{C}\backslash\{0\}$, which is impossible under the current assumption that they are non-constant. So there is at least one non-positive sector. In particular, $D_1 \ne D_2$ in the previous paragraph.

Let D be a positive sector. Let z_0 be a point on $\partial D \cap \partial D'$, where D' is a non-positive sector. This means that $u(z) \le 0$ and $v(z) \le 0$ in D'. Then $u(z_0) = v(z_0) = 0$. Indeed, $u(z_0) \ge 0$ and $v(z_0) \ge 0$ by the upper semi-continuity of subharmonic functions. As D' is not thin at z_0 (in the sense of potential theory, see [28]) we obtain that $u(z_0) = v(z_0) = 0$.

Let C be the union of those rays in $A \cup B$ which separate a positive and a non-positive sector. It follows from the above considerations that C can be written in the form $C = \bigcup_{j=1}^{2m} C_j$, with $m \ge 1$, with rays $C_j \subset A \cup B$ so that in the sector S_j between C_j and C_{j+1} the functions u and v are positive for

even j and non-positive for odd j. Moreover, we have $u(z) = v(z) = 0$ for $z \in C$. Note that the sectors with respect to the system C may be unions of several sectors and rays of the system $A \cup B$.

Let S_{2j-1} be an odd sector. Then $u(z) \le 0$ in S_{2j-1}. If $u(z) = 0$ in S_{2j-1}, then u is not harmonic on either of the two rays in ∂S_{2j-1}, so v is harmonic on these two rays. As $v(z) > 0$ in the adjacent sectors to S_{2j-1}, we obtain that $v(z) \not\equiv 0$ in S_{2j-1}, so $v(z) < 0$ in S_{2j-1} by maximum principle. In this case we evidently have $\partial S_{2j-1} \subset A$. Similarly, if $v(z) = 0$ in S_{2j-1} then $u(z) < 0$ in S_{2j-1} and $\partial S_{2j-1} \subset B$.

This proves Lemma 1.

We return to the proof of Theorem 4. Lemma 1 does not exclude the possibility that the set of rays C_j is empty, and thus the whole plane coincides with one positive sector. In this case u and v are identically equal to the same positive constant. The following argument shows that this is impossible.

Suppose first that

$$\rho^* = 0 \tag{9}$$

in (5). Then

$$\log M(tr) \le c_0 t^{1/4} \log M(r), \quad t > t_0,\ r > r_0, \tag{10}$$

with some $c_0 > 0$. This implies (3), thus from every sequence $r_k \to \infty$ one can choose a subsequence such that the functions (6) have limits u, v satisfying Lemma 1.

We claim that these functions must be constant when (10) holds. Indeed, (10) implies that

$$u(z) \le c_0 |z|^{1/4}, \quad |z| > t_0. \tag{11}$$

If u is not constant, then there exists at least one positive sector D of the system C, so that $u(z) > 0,\ z \in D$ and $u(z) = 0,\ z \in \partial D$, and u is harmonic in D. But every positive harmonic function in a sector, zero on the boundary must have the form

$$c|z|^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}(\arg z - \theta_0)\right), \quad z \in D, \tag{12}$$

where $\alpha \le 2\pi$ is the opening angle of the sector. This can be seen by transforming the sector D conformally to a half-plane, for which the result is standard, see, for example, [10, Theorem I]. So $\pi/\alpha \ge 1/2$ in contradiction to (11). This proves the claim that u and v coincide with the same positive constant function when (10) holds.

Let $z_k = r_k e^{i\beta_k} \to \infty$ be a sequence of zeros of f, such that $\beta_k \to \beta$ where β is an argument of some ray of the system A. Using the sequence r_k we define functions u_k, v_k by (6), and consider some limit functions u, v. These functions are both equal to a positive constant c under our assumption (9).

We have $u_k \to u$ in $\mathscr{D}'$. According to Azarin [1], this convergence also holds in the following sense: for every $\varepsilon > 0$ the set

$$\{z\colon |z| < 2,\ |u(z) - u_k(z)| > \varepsilon\}$$

can be covered by disks the sum of whose radii is at most ε, when k is large enough. Let D be the closed disk with the center at $e^{i\beta}$ of radius $\delta < 1/2$, where δ is also so small that D can meet only one ray in $A \cup B$. Choosing $\varepsilon < \min\{\delta/2, c/2\}$, we see that for large k, there is a disk B_k around $e^{i\beta}$ such that $u_k(z) \geq u(z)/2 = c/2 > 0$ for $z \in \partial B_k$. This means that $\log|f(z)| \geq (c/3)\log M(r_k)$ for z on the circles

$$\{z\colon z/r_k \in \partial B_k\}.$$

Each of these circles encloses the zero z_k of f. Thus by Rouché's theorem, each of them also contains a 1-point of f. As δ can be arbitrarily small, the argument of this 1-point is close to β. This is a contradiction with our assumption that the zeros and 1-points are close to disjoint systems of rays. This contradiction shows that

$$\rho^* > 0. \tag{13}$$

So for some $\lambda > 0$ there exists a sequence of Pólya peaks of order λ. Using this sequence to define u_k, v_k as in (6), we obtain limit functions u, v satisfying conditions $(a), (b), (c)$ of Lemma 1, and in addition, condition (d). This condition (d) excludes the possibility that u and v are equal to the same constant in Lemma 1. So there exist at least one positive and at least one non-positive sector of the system C. If D is a positive sector of opening α, then (12) holds in D. Comparing this with the condition (d) before Lemma 1, we conclude that $\pi/\alpha = \lambda$. As α is an angle between certain two rays of a finite system $A \cup B$, there are only finitely many possibilities for λ. On the other hand, the possible orders λ of Pólya peaks of a given function fill an interval $[\rho_*, \rho^*]$ which contains ρ. We conclude that

$$\rho = \rho_* = \rho^*,$$

and in particular, ρ^* is finite and $\rho > 0$.

Using (5), we obtain

$$\log M(tr) \le t^{\rho+1} \log M(r), \quad r > r_0,\ t > t_0.$$

This implies that (3) holds for every sequence $r_k \to \infty$.

Also this shows that the angle of every even sector at the origin is equal to π/ρ, proving the first statement of (i) of Theorem 4.

For $r_0 > 0$ such that $M(r_0) > 1$ we consider the curve mapping $[r_0, \infty)$ to $\mathscr{D}' \times \mathscr{D}'$ given by

$$r \mapsto \left(\frac{\log |f(rz)|}{\log M(r)}, \frac{\log |f(rz) - 1|}{\log M(r)} \right).$$

Let F be the limit set of this curve when $r \to \infty$. It consists of pairs (u, v) satisfying (a), (b) and (c) and thus satisfying the conclusions of Lemma 1. As a limit set of a curve, F is closed and connected. In each sector of the system $A \cup B$ either both of the functions u and v are positive, or one is negative. We conclude that the sectors S_j and their classification into three classes (positive sectors, those where $u(z) < 0$ and those where $v(z) < 0$) is independent on the choice of the sequence r_k. In particular, u, v are never constant even if the r_k are not Pólya peaks. Further, by Lemma 1 we deduce that (ii) of Theorem 4 holds.

One consequence of this is that $f(z) \to 0$ in every closed subsector T of an odd sector where $u(z) < 0$. Indeed, if T contains points z_k where $|f(z_k)| \ge \delta > 0$ with $z_k \to \infty$ and $\arg z_k \to \beta$, we choose $r_k = |z_k|$ and, after choosing a subsequence, consider the limits (7). As $u(e^{i\beta}) < 0$, we have $u(z) < -\varepsilon$ in a neighborhood of $e^{i\beta}$ for some $\varepsilon > 0$. Convergence $u_k \to u$ in $\mathscr{D}'$ implies convergence in L^1_{loc}, and we obtain

$$u_k(w) \le \frac{1}{\pi r^2} \int_{|z-w|<r} u_k(z) dx dy < -\varepsilon/2, \quad z = x + iy,$$

where k is large enough, w is in a neighborhood of $e^{i\beta}$, and $r > 0$ is small enough. This contradicts our assumption that $|f(z_k)| \ge \delta$ and proves that $f(z) \to 0$ in T. So such a sector cannot contain rays of B.

Similarly, an odd sector in which $v(z) < 0$ cannot contain rays of A. This proves statement (iv) of Theorem 4.

Next we prove statement (iii). Suppose that a ray $L = \{te^{i\beta} \colon t \ge 0\}$ of the set A lies inside an even sector S_{2j}. By assumption, there is an infinite

sequence of zeros (z_k) of the form $z_k = r_k e^{i\beta_k}$ with $r_k \to \infty$ and $\beta_k \to \beta$. Passing to a subsequence we may assume that the limits in (7) exist. We obtain a positive harmonic function u in a neighborhood of $e^{i\beta}$. Using the same argument as before, with reference to Azarin's and Rouché's theorems we conclude that there must be a sequence of 1-points whose arguments tend to β, and this contradicts our assumption that zeros and 1-points are close to disjoint systems of rays.

A similar argument shows that there are no rays from B inside any even sector S_{2j}.

This proves (iii) and the fact that the even sectors of the system C coincide with the positive sectors of the system $A \cup B$.

We next prove the second statement of (i). Let S_{2j-1} be an odd sector with angle π/γ at the origin. Consider again the limit functions u and v obtained from the Pólya peaks r_k. Then we have (d) with $\lambda = \rho$. One of the two subharmonic functions, say u, is negative in S_{2j-1}, and zero on the boundary of S_{2j-1}. Let h be the least harmonic majorant of u in $G = S_{2j-1} \cap \{z\colon |z| < 1\}$. Then h is a negative harmonic function in G, equal to zero on the straight segments of ∂G. It follows that

$$\int_{re^{it} \in G} h(re^{it})\, dt \le -cr^\gamma, \quad r < 1,$$

where $c > 0$ and $\gamma = \pi/\alpha$; here α is the angle of G at the origin. To prove this, notice that the normal derivative of h cannot be zero on the rectilinear part of the boundary of G by the reflection principle, thus h has a harmonic majorant of the form $-cr^\gamma \cos(\gamma(t - t_0))$, where t_0 is the argument of the bisector of G. Then u satisfies the same inequality

$$u(re^{it}) \le -cr^\gamma \cos(\gamma(t - t_0))$$

in G. Combining this with property (d) and using, for $0 < r < 1$, the fact that

$$0 = 2\pi u(0) \le \int_0^{2\pi} u(re^{it})\, dt \le -c \int_{re^{it} \in G} r^\gamma \cos(\gamma(t - t_0))\, dt + \int_{re^{it} \notin G} r^\rho\, dt,$$

we obtain that $\gamma \ge \rho$ so that $\alpha \le \pi/\rho$. This proves the second part of (i). By this and (iii), we also obtain $\rho = \pi/\omega$ in (1). That $\rho > 1/2$, that is, $\omega < 2\pi$, follows since there are at least two sectors.

Finally, if there are no rays of A inside S_{2j-1}, then u is harmonic in S_{2j-1}. Hence, if $u < 0$ in S_{2j-1}, then u is of the form (12), and it is a harmonic continuation from an adjacent even sector, so we must have $\alpha = \pi/\rho$. A similar argument applies if v is negative and there are no rays of B inside S_{2j-1}. This proves (v) and completes the proof of Theorem 4.

Proof of Theorem 1. According to Theorem A, the order ρ of f is finite.

First we deal with the case when f is a polynomial, following Baker [2]. Without loss of generality, we may assume that L_1 is the real line. Then $f = cg$, where g is a real polynomial with all zeros real. Thus all zeros of f' are real. Similarly we conclude that all zeros of f' lie on L_2, and hence the point z_0 of intersection of L_1 and L_2 is the only possible zero of f'. Thus $f(z) = c_1(z - z_0)^n + c_2$ for some $n \geq 1$ and some $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq 0$. Such a function f can satisfy the assumptions of Theorem 1 only if f is a polynomial of degree at most 2. Notice that this argument can be extended to functions of order less than 2, but we do not use this.

Suppose now that f is transcendental. Then we use Theorem 4. This theorem implies that there exists at least one even sector. If there is only one even sector, and its angle is greater than π, then the odd sector does not contain rays of $A \cup B$, so by (v) its opening must be the same as the opening of the even sector, which is a contradiction.

If there are two even sectors, then the odd sectors contain no rays of $A \cup B$. It follows from (i) and (v) that all sectors must have opening $\pi/2$. Then by (ii) the zeros are close to the boundary of a quadrant, and the 1-points are close to the the boundary of the opposite quadrant. But by hypothesis the zeros lie on a line and the 1-points lie on a line. We conclude that the zeros are actually close to one ray and the 1-points are close to another ray. But then there is only one even sector, contradicting our assumption at the beginning of this paragraph.

The only remaining possibility is that there is one even sector with opening π. Then $\rho = 1$, and we assume without loss of generality that this sector is the upper half plane and the 1-points are real. This means that B consists of two rays whose union is the real line. Using the notation of the proof of Theorem 4, and choosing a sequence (r_k) of Pólya peaks of order 1, we obtain $u(z) = \operatorname{Im} z$ and $v(z) = \operatorname{Im}^+ z$. This implies that

$$N(r_k, 1, f) \sim \frac{1}{\pi} \log M(r_k), \quad k \to \infty. \tag{14}$$

Now let $g(z) = f(z)\overline{f(\overline{z})}$. As all 1-points of f are real, $f(z) = 1$ implies that

$g(z) = 1$, so if $g \not\equiv 1$, we will have from (14) that

$$N(r_k, 1, g) \geq (1 - o(1)) \frac{1}{\pi} \log M(r_k), \quad k \to \infty. \tag{15}$$

Now define the subharmonic function w in the plane by

$$w(z) = \lim_{k \to \infty} \frac{\log |g(r_k z)|}{\log M(r_k)}.$$

Above, $M(r_k)$ always refers to $M(r_k, f)$. It is evident that $w(z) = u(z) + u(\overline{z}) = 0$. Together with (15) this implies that $g(z) \equiv 1$. We conclude that with this normalization, f has the form $f(z) = \exp(icz + id)$, where c and d are real. This completes the proof of Theorem 1.

Proof of Theorem 2. We assume without loss of generality that L_0 is the positive ray. The order of f must be finite by Theorem A, so Theorem 4 is applicable. As there are only three rays, the number m in Theorem 4 must be 1. So we have one even sector of opening π/ρ and one odd sector of opening at most π/ρ. Thus $\rho \leq 1$. In view of (ii), the common boundary of the odd and even sector is $L_1 \cup L_{-1}$. So L_0 lies inside the odd sector. The possibility that $\rho = 1$ is excluded by (1) and Theorem 1. Hence $\rho < 1$, and the function f is of genus zero and thus of the form

$$f(z) = cg(z), \quad g(z) = z^n \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_k}\right),$$

where $c \in \mathbb{C} \setminus \{0\}$ and n is a non-negative integer, and (z_k) is a sequence of positive numbers tending to ∞. If c is real, we conclude that the rays L_1 and L_{-1} are symmetric with respect to L_0 which proves Theorem 2 in this case.

Suppose now that L_1 and L_{-1} are not symmetric with respect to L_0 so that c is not real. Let us set $a = 1/c$. Then $f(z) = 1$ is equivalent to $g(z) = a$. We consider the function $h(z) = (g(z) - a)/(\overline{a} - a)$. In view of the symmetry of g, the zeros of h lie on the rays L_1 and L_{-1}, while the 1-points lie on the reflected rays $\overline{L_1}$ and $\overline{L_{-1}}$. Since L_0 lies in the odd sector, which has angle $< \pi$ at the origin, it follows that the two rays L_1, L_{-1} are interlaced with the two rays $\overline{L_1}$, $\overline{L_{-1}}$. This contradicts (ii) and (iii) of Theorem 4 applied to h, and completes the proof of Theorem 2.

3 Sketch of the proof of Theorem 5

We only indicate the construction of examples showing that Theorem 4 is best possible, as this construction is well-known, see for example [13], where a similar construction was used for the first time.

We fix $\rho \in (1/2, \infty)$ and construct a ρ-trigonometrically convex function h such that the union of the even sectors coincides with the set

$$\{re^{it} \colon r > 0,\ h(t) > 0\},$$

and such that h is trigonometric except at the arguments of some rays inside the odd sectors. If there are no rays of $A \cup B$ in the odd sectors at all, then ρ must be an integer, and we just take h to be of the form $h(t) = \cos(\rho(t - t_0))$, where t_0 is the argument of the bisector of one of the sectors.

Then we discretize the Riesz mass of the subharmonic function

$$w(re^{it}) = r^\rho h(t),$$

as it is done in [1], and obtain an entire function g with zeros on some rays $A \cup B$ which lie in the odd sectors, and such that

$$\lim_{r\to\infty} r^{-\rho} \log |g(rz)| = w(z).$$

If there are odd sectors with opening π/ρ, then h must be trigonometric on the intervals corresponding to these sectors, so we multiply g by a canonical product of order smaller than ρ to achieve that g has infinitely many zeros on all those rays of the system $A \cup B$ which lie inside the odd sectors. Then we label the odd sectors with labels 0 and 1: if the boundary of an odd sector belongs to A, we label it with 1, and if the boundary belongs to B we label it with 0.

Let S_j be an odd sector labeled with 1. Consider the component D_j of the set $\{z \colon |g(z)| < 2\}$ which is asymptotic to S_j. Let p be a quasiconformal map of the disk $\{z \colon |z| < 2\}$ onto itself, equal to the identity mapping on the boundary, and such that $p(0) = 1$, whose complex dilatation is supported on the set $\{z \colon 3/2 \le |z| \le 2\}$. We define

$$G(z) = \begin{cases} p(g(z)), & z \in \bigcup_j D_j, \\ g(z), & \text{otherwise.} \end{cases}$$

Here the union is over all odd sectors labeled with 1. This G is a quasiregular map of the plane, whose complex dilatation is supported on a small set E in the sense that

$$\int_E \frac{dxdy}{x^2+y^2} < \infty.$$

Then the theorem of Teichmüller–Wittich–Belinski [22, §V.6] guarantees the existence of a quasiconformal map ϕ such that $f = G \circ \phi$ is an entire function, and $\phi(z)/z \to 1$ as $z \to \infty$. It is easy to verify that f has all the required properties.

For the construction of infinite order functions, let $A = \bigcup_{j=1}^m \{te^{i\alpha_j} : t \geq 0\}$ and $B = \bigcup_{k=1}^n \{te^{i\beta_k} : t \geq 0\}$ be two finite systems of rays with $A \cap B = \{0\}$. Again we only sketch the argument.

First we note that by [27, Part III, Problems 158–160] there exists an entire function E such that $z^2(E(z) + 1/z)$ is bounded outside the half-strip $S = \{z\colon \operatorname{Re} z > 0, |\operatorname{Im} z| < \pi\}$. In particular, E is bounded outside S. Considering $F(z) = \delta(E(z) - c)/((z-a)(z-b))$, where $\delta > 0$ is small, $c \in \mathbb{C}$, and a and b are c-points of E, we obtain an entire function F such that

$$|F(z)| \leq \frac{1}{|z|^2+1} \leq \frac{1}{\operatorname{dist}(z,S)^2}, \quad z \notin S,$$

where $\operatorname{dist}(z,S)$ denotes the distance from z to S. For some large $R > 0$ we now consider the functions

$$a_j(z) = 1 + z \exp F(e^{-i\alpha_j} z - R) \quad \text{and} \quad b_k(z) = z \exp F(e^{-i\beta_k} z - R).$$

With $S_j = \{e^{i\alpha_j}(z+R)\colon z \in S\}$ we find, noting that $|e^w - 1| \leq 2|w|$ for $|w| \leq 1$, that

$$\begin{aligned} |a_j(z) - z - 1| &\leq \left| z \left(\exp F(e^{-i\alpha_j} z - R) - 1 \right) \right| \leq 2|zF(e^{-i\alpha_j} z - R)| \\ &\leq \frac{2|z|}{\operatorname{dist}(e^{-i\alpha_j} z - R, S)^2} = \frac{2|z|}{\operatorname{dist}(z, S_j)^2}, \quad z \notin S_j. \end{aligned}$$

Similarly, with $T_k = \{e^{i\beta_k}(z+R)\colon z \in S\}$ we have

$$|b_k(z) - z| \leq \frac{2|z|}{\operatorname{dist}(z, T_k)^2}, \quad z \notin T_k.$$

We choose $\varepsilon > 0$ so small that the sectors $U_j = \{z\colon |\arg(z - e^{i\alpha_j}R/2) - \alpha_j| \le \varepsilon\}$ and $V_k = \{z\colon |\arg(z - e^{i\beta_k}R/2) - \beta_k| \le \varepsilon\}$ are disjoint and put

$$G(z) = \begin{cases} a_j(z), & z \in U_j, \\ b_k(z), & z \in V_k. \end{cases}$$

Then

$$G(z) = z + O(1), \quad z \in \bigcup_{j=1}^{m} \partial U_j \cup \bigcup_{k=1}^{n} \partial V_k.$$

This allows us to extend G to a quasiregular map of the plane which satisfies

$$G(z) = z + O(1), \quad z \in \mathbb{C} \setminus \left(\bigcup_{j=1}^{m} U_j \cup \bigcup_{k=1}^{n} V_k \right)$$

and whose dilatation K_G satisfies $K_G(z) = 1 + O(1/|z|)$ as $z \to \infty$. Again the theorem of Teichmüller–Wittich–Belinski yields the existence of a quasiconformal map ϕ such that $f = G \circ \phi$ is entire and $\phi(z)/z \to 1$ as $z \to \infty$. It is not difficult to show that the zeros of f are close to A and the 1-points of f are close to B.

We note that the method does not actually require that the rays that form A are distinct from those that form B. Indeed, if we want that both zeros and 1-points accumulate at $\{te^{i\alpha_j} : t \ge 0\}$, we only have to choose $a_j(z) = c + z \exp F(e^{-i\alpha_j} z - R)$ with a constant c different from 0 and 1.

4 Proof of Theorem 3

We consider differential equations

$$-y'' + \left((-1)^\ell z^m + E\right) y = 0, \quad \ell \in \{0, 1\}, \quad m \ge 3, \quad E \in \mathbb{C}. \tag{16}$$

Here m is an integer, so all solutions are entire functions. The equation has the following symmetry property. Set

$$\varepsilon = e^{\pi i/(m+2)}, \quad \omega = \varepsilon^2.$$

If $y_0(z, E)$ is a solution of (16) then

$$y_k(z, E) = y_0(\omega^{-k} z, \omega^{2k} E) \tag{17}$$

satisfies the same equation, while

$$y_0(\varepsilon^{-k}z, \varepsilon^{2k}E)$$

with an odd k satisfies (16) with $(-1)^\ell z^m$ replaced by $(-1)^{\ell+1}z^m$.

The Stokes sectors are defined as follows. When $\ell = 0$, they are $S_0 = \{z\colon |\arg z| < \pi/(m+2)\}$, and $S_k = \omega^k S_0$ for $k \in \mathbb{Z}$. When $\ell = 1$, the Stokes sectors are $S_0 = \{z\colon 0 < \arg z < 2\pi/(m+2)\}$ and $S_k = \omega^k S_0$ for $k \in \mathbb{Z}$.

To obtain a discrete sequence of eigenvalues, one imposes boundary conditions of the form

$$y(z) \to 0, \quad z \to \infty, \quad z \in S_n \cup S_k, \tag{18}$$

for some n and k. The exact meaning of (18) is that $y(z) \to 0$ when $z \to \infty$ along any interior ray from the origin contained in the union of the two sectors.

We will denote such a boundary condition by (n, k). It is known [31] that when $n \neq k \pm 1$ (modulo $m+2$), then the boundary value problem (n, k) has a discrete spectrum with a sequence of eigenvalues tending to infinity. (For completeness, we include the argument below.) Moreover, K. Shin [30] proved that these eigenvalues always lie on a ray from the origin. In particular, when S_n and S_k are symmetric with respect to the positive ray, these eigenvalues are positive. All other cases can be reduced to this case using the symmetry of the differential equation stated above: if ω_1 and ω_2 are bisectors of S_n and S_k, then the eigenvalues lie on the ray $\{t/(\omega_1\omega_2)\colon t \geq 0\}$.

From now on we assume that $\ell = 0$ in (16). For each E the equation (16) has a solution tending to zero as $z \to \infty$ in S_0. More precisely, there is a unique solution $y_0(z, E)$ satisfying

$$y_0(z, E) = (1 + o(1))z^{-m/4} \exp\left(-\frac{2}{m+2}z^{(m+2)/2}\right) \tag{19}$$

as $z \to \infty$ in any closed subsector of $S_0 \cup S_1 \cup S_{-1}$; see [31, Thm 6.1]. Notice the simple but important fact that this principal part of the asymptotics does not depend on E. The function $y_0(z, E)$ is actually an entire function of the two variables z and E, and its asymptotics when $E \to \infty$ while z is fixed are also known [31, Thm 19.1]; this implies that the entire function $E \mapsto y_0(z_0, E)$ has order

$$\rho = \frac{1}{2} + \frac{1}{m}.$$

Now we define y_k by (17). Then $y_k \to 0$ as $z \to \infty$ in S_k. The boundary problem (n, k) thus has a solution when y_n and y_k are linearly dependent as functions of z. This means that their Wronskian vanishes. But the Wronskian, evaluated at $z = 0$, is an entire function of E, and its order is less than 1. Thus its zeros, which are the eigenvalues of the problem, form a sequence tending to infinity, as mentioned above.

As y_0, y_1, y_{-1} satisfy the same differential equation, we have

$$y_{-1} = C(E)y_0 + \tilde{C}(E)y_1.$$

The asymptotics of y_1 and y_{-1} in S_0 (which follow from (19)) show that $\tilde{C} = -\omega$, so

$$y_{-1} = C(E)y_0 - \omega y_1. \tag{20}$$

By differentiating this with respect to z we obtain

$$y'_{-1} = C(E)y'_0 - \omega y'_1. \tag{21}$$

Solving (20) and (21) by Cramer's rule, we obtain

$$C(E) = W_{-1,1}/W_{0,1},$$

where $W_{i,j}$ is the Wronskian of y_i and y_j. This shows that C is an entire function (because $W_{0,1}$ is never 0). It has the same order ρ that y_0 has as a function of E.

In view of (20), the zeros of C are exactly the eigenvalues λ_j of the problem (18) with $(n, k) = (-1, 1)$. So all zeros of C are positive by Shin's result. Substituting $(z, E) \mapsto (\omega^{-1}z, \omega^2 E)$ to (20), we obtain

$$y_0 = C(\omega^2 E)y_1 - \omega y_2.$$

Using this to eliminate y_0 from (20) we obtain

$$y_{-1} = \big(C(E)C(\omega^2 E) - \omega\big)\, y_1 - C(E)\omega y_2.$$

We conclude that the zeros of the entire function

$$g(E) := C(E)C(\omega^2 E) - \omega \tag{22}$$

are the eigenvalues of the problem $(-1, 2)$. Therefore, these zeros lie on the ray $\{z = t\omega^{-1} \colon t \geq 0\}$. So if we define $f(E) = -\omega^{-1}g(\omega^{-1}E)$ and $h(E) = C(E)/\sqrt{\omega}$, then

$$f(E) = 1 - h(\omega^{-1}E)h(\omega E),$$

the zeros of f are on the positive ray and the 1-points on two other rays. This completes the proof of Theorem 3.

Remarks. Once it is known that two entire functions C and g satisfy (22) and the zeros of each function lie on a ray, the order of both functions and the angles between the rays can be determined from Theorem 4 and Theorem 2.

Equations of the type (22) occur for the first time in the work of Sibuya and his students [31, 33, 32] for the simplest case when $m = 3$.

It was later discovered that these equations also arise in the context of exactly solvable models of statistical mechanics on two-dimensional lattices and in quantum field theory [11, 12].

The interesting question is to which angles Theorem 3 generalizes. Concerning the approach that makes use of differential equations, the following comments are in order. If $m > 2$ is not an integer, equation (16) and its solutions are defined on the Riemann surface of the logarithm, but Sibuya's solution y_0 is still entire as a function of E. We found no source where this fact is proved, but it is stated and used in [11, p. 576], [12, p. R231] and [34]. Shin's result, which we used above seems to generalize to non-integer $m \geq 4$, see [30, Theorem 11] which we use with $\ell = 1$ and $\ell = 2$. On the other hand, numerical evidence in [4] (see Figs. 14, 15, 20) shows that for $m < 4$ our function $g(E)$ in (22) does not have radially distributed zeros on one ray, even if finitely many of the zeros are discarded.

By extending our method Theorem 3 is generalized in [17] to all angles in $(0, \pi/3]$. The question remains open for angles in $(\pi/3, \pi/2)$ other than $2\pi/5$.

References

[1] V. S. Azarin, Asymptotic behavior of subharmonic functions of finite order (Russian), Mat. Sb. (N.S.) 108(150) (1979), no. 2, 147–167, 303. English transl.: Math. USSR, Sb. 36, 135–154 (1980).

[2] I. N. Baker, Entire functions with linearly distributed values, Math. Z. 86 (1964) 263–267.

[3] I. N. Baker, Entire functions with two linearly distributed values, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), no. 2, 381–386.

[4] C. M. Bender, S. Boettcher and P. N. Meisinger, PT-symmetric quantum mechanics, J. Math. Phys. 40 (1999), no. 5, 2201–2229.

[5] W. Bergweiler and A. Eremenko, Goldberg's constants, J. Anal. Math. 119, no. 1, (2013) 365–402.

[6] L. Bieberbach, Über eine Vertiefung des Picardschen Satzes bei ganzen Funktionen endlicher Ordnung, Math. Z. 3 (1919), 175–190.

[7] M. Biernacki, Sur le déplacement des zéros des fonctions entières par leur dérivation, Comptes Rendus 175 (1922), 18–20.

[8] M. Biernacki, Sur la théorie des fonctions entières, Bulletin de l'Académie polonaise des sciences et des lettres, Classe des sciences mathématiques et naturelles, Série A (1929), 529–590.

[9] V. Blondel, Simultaneous stabilization of linear systems, Springer, Berlin, 1994.

[10] H. P. Boas and R. P. Boas, Short proofs of three theorems on harmonic functions, Proc. Amer. Math. Soc. 102 (1988), no. 4, 906–908.

[11] P. Dorey, C. Dunning and R. Tateo, On the relation between Stokes multipliers and the T-Q systems of conformal field theory, Nuclear Physics B 563 (1999) 573–602.

[12] P. Dorey, C. Dunning and R. Tateo, The ODE/IM correspondence, J. Phys. A 40 (2007), no. 32, R205–R283.

[13] D. Drasin, Value distributions of entire functions in regions of small growth, Ark. Mat. 12 (1974), 281–296.

[14] D. Drasin and D. F. Shea, Pólya peaks and the oscillation of positive functions, Proc. Amer. Math. Soc. 34 (1972), 403–411.

[15] A. Edrei, Meromorphic functions with three radially distributed values, Trans. Amer. Math. Soc. 78 (1955), 276–293.

[16] A. Eremenko, Simultaneous stabilization, avoidance and Goldberg's constants, arXiv:1208.0778.

[17] A. Eremenko, Entire functions, PT-symmetry and Voros's quantization scheme, arXiv:1510.02504.

[18] A. A. Goldberg and I. V. Ostrovskii, Distribution of values of meromorphic functions, Amer. Math. Soc., Providence, RI, 2008.

[19] L. Hörmander, The analysis of linear partial differential operators I, 2nd ed., Springer, Berlin, 1990.

[20] L. Hörmander, Notions of convexity, Birkhäuser, Boston, 1994.

[21] T. Kobayashi, An entire function with linearly distributed values, Kodai Math. J. 2 (1979), no. 1, 54–81.

[22] O. Lehto and K. I. Virtanen, Quasiconformal mappings in the plane, Springer, New York – Heidelberg, 1973.

[23] B. Ya. Levin, Distribution of zeros of entire functions, Amer. Math. Soc., Providence, RI, 1970.

[24] H. Milloux, Sur la distribution des valeurs des fonctions entières d'ordre fini, à zéros reels, Bull. Sci. Math. (2) 51(1927), 303–319.

[25] R. Nevanlinna, Über die Konstruktion von meromorphen Funktionen mit gegebenen Wertzuordnungen, Festschrift zur Gedächtnisfeier für Karl Weierstraß, Westdeutscher Verlag, Köln – Opladen, 1966, pp. 579–582.

[26] M. Ozawa, On the zero-one set of an entire function, Kodai Math. Sem. Rep. 28 (1977), no. 4, 311–316.

[27] G. Pólya and G. Szegő, Problems and theorems in analysis. Vol. I: Series, integral calculus, theory of functions, Springer, New York, 1972.

[28] T. Ransford, Potential theory in the complex plane, Cambridge Univ. Press, 1995.

[29] L. A. Rubel and C.-C. Yang, Interpolation and unavoidable families of meromorphic functions, Michigan Math. J. 20 (1974), no. 4, 289–296.

[30] K. Shin, The potential $(iz)^m$ generates real eigenvalues only, under symmetric rapid decay boundary conditions, J. Math. Phys. 46 (2005), no. 8, 082110, 17pp.

[31] Y. Sibuya, Global theory of a second order linear ordinary differential equation with a polynomial coefficient, North-Holland, Amsterdam, 1975.

[32] Y. Sibuya, Non-trivial entire solutions of the functional equation $f(\lambda)+f(\omega\lambda)f(\omega^{-1}\lambda)=1$, Analysis 8 (1998), 271–295.

[33] Y. Sibuya and R. Cameron, An entire solution of the functional equation $f(\lambda)+f(\omega\lambda)f(\omega^{-1}\lambda)=1$, Lecture Notes Math. 312, Springer, Berlin, 1973, pp. 194–202.

[34] T. Tabara, Asymptotic behavior of Stokes multipliers for $y''-(x^\sigma+\lambda)y=0$, $(\sigma\geq 2)$ as $\lambda\to\infty$, Dynamics of Continuous, Discrete and Impulsive Systems 5 (1999), 93–105.

[35] J. Winkler, Zur Existenz ganzer Funktionen bei vorgegebener Menge der Nullstellen und Einsstellen, Math. Z. 168 (1979), 77–86.

W. B.: Mathematisches Seminar
Christian-Albrechts-Universität zu Kiel
Ludewig-Meyn-Str. 4
24098 Kiel
Germany

A. E.: Department of Mathematics
Purdue University
West Lafayette, IN 47907
USA

A. H.: Department of Mathematics
University of Illinois at Urbana–Champaign
1409 W. Green St.
Urbana, IL 61801
USA