Distribution of zeros of polynomials with positive coefficients

Walter Bergweiler, Alexandre Eremenko*

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Abstract

We describe the limit zero distributions of sequences of polynomials with positive coefficients.

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1 Introduction and results

In this paper we answer the following question of Ofer Zeitouni and Subhro Ghosh [8], which arises in the study of zeros of random polynomials [4].

Let P be a polynomial. Consider the discrete probability measure $\mu[P]$ in the plane which has an atom of mass $m/\deg P$ at every zero of P of multiplicity m. It is called the "empirical measure" in the theory of random polynomials.

Let μ_n be a sequence of empirical measures of some polynomials with positive coefficients, and suppose that $\mu_n \to \mu$ weakly. The question is how to characterize all possible limit measures μ. We give such a characterization in terms of logarithmic potentials.

Theorem 1. *For a measure μ to be a limit of empirical measures of polynomials with positive coefficients, it is necessary and sufficient that the following conditions are satisfied:*

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μ is symmetric with respect to the complex conjugation, $\mu(\mathbf{C}) \leq 1$, and the potential

$$u(z) = \int_{|\zeta|\leq 1} \log|z-\zeta|d\mu(\zeta) + \int_{|\zeta|>1} \log\left|1-\frac{z}{\zeta}\right| d\mu(\zeta) \tag{1.1}$$

has the property

$$u(z) \leq u(|z|). \tag{1.2}$$

The potential in Theorem 1 converges for every positive measure with the property $\mu(\mathbf{C}) < \infty$ to a subharmonic function $u \not\equiv -\infty$. If

$$\int_{|\zeta|>1} \log|\zeta|d\mu(\zeta) < \infty \quad \text{or} \quad \int_{|\zeta|<1} \log\frac{1}{|\zeta|}d\mu(\zeta) < \infty,$$

then the definition of u in Theorem 1 can be simplified to

$$\int_{\mathbf{C}} \log|z-\zeta|d\mu(\zeta) \quad \text{or} \quad \int_{\mathbf{C}} \log\left|1-\frac{z}{\zeta}\right| d\mu(\zeta),$$

respectively. When these integrals exist, they differ from the potential (1.1) only by additive constants.

Obrechkoff [7] proved that empirical measures of polynomials with non-negative coefficients satisfy

$$\mu(\{z \in \mathbf{C}\backslash\{0\} \colon |\arg z| \leq \alpha\}) \leq \frac{2\alpha}{\pi}\mu(\mathbf{C}\backslash\{0\}), \quad 0 \leq \alpha \leq \pi/2. \tag{1.3}$$

We call this the Obrechkoff inequality. The limits of these measures also satisfy (1.3).

Combining our result with Obrechkoff's theorem we conclude that (1.2) and symmetry of the measure imply (1.3). In particular we find that Obreschkoff's inequality is satisfied not only by polynomials with non-negative coefficients, but more generally by polynomials satisfying

$$|f(z)| \leq f(|z|), \quad z \in \mathbf{C}. \tag{1.4}$$

The converse does not hold; that is, the inequalities (1.4) and (1.2) do not follow from Obrechkoff's inequality. Indeed, let

$$P(z) = (z^2+1)^m(z^2-2z\cos\beta+1)$$

This polynomial has roots of multiplicity m at $\pm i$, and simple roots at $\exp(\pm i\beta)$. Obrechkoff's inequality is satisfied if $\beta \geq \pi/(2m+2)$. On the other hand, $P(1) < |P(-1)|$ for all m and $\beta \in (0, \pi/2)$.

We note that Obrechkoff's inequality is best possible [3]. For other results on the roots of polynomials with positive coefficients we refer to [1].

An important ingredient in our proof is the following theorem of De Angelis [2].

Theorem A. *Let*

$$f(z) = a_0 + \ldots + a_d z^d, \quad a_0 > 0,\ a_d > 0, \tag{1.5}$$

be a real polynomial. The following conditions are equivalent:

(*i*) *There exists a positive integer m such that all coefficients of f^m are strictly positive.*

(*ii*) *There exists a positive integer m_0 such that for all $m \geq m_0$, all coefficients of f^m are strictly positive.*

(*iii*) *The inequalities*

$$|f(z)| < f(|z|), \quad z \notin [0, \infty), \tag{1.6}$$

and

$$a_1 > 0, \quad a_{d-1} > 0 \tag{1.7}$$

hold.

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2 Proof of Theorem 1

We use some facts about subharmonic functions and potential theory which can be found in [6]. For the reader's convenience, they are stated in the Appendix.

We recall that the Riesz measure of a subharmonic function u is $(2\pi)^{-1}\Delta u$, where the Laplacian is understood as a Schwartz distribution. In particular the empirical measure of a polynomial P of degree d is the Riesz measure of the subharmonic function $(\log|P|)/d$. For the general properties of convergence of subharmonic functions we refer to [6, Theorem 3.2.13]. This result will be used repeatedly and is stated for the convenience of the reader as Theorem B in the Appendix.

The function u given by (1.1) satisfies

$$u(z) \leq O(\log|z|), \quad z \to \infty. \tag{2.1}$$

In turn, it is well known that every subharmonic function u in the plane which satisfies (2.1) can be represented in the form (1.1) plus a constant. We will call functions of this form simply "potentials"; see, for example [5, Theorem 4.2] (case $q = 0$).

Proof of Theorem 1. For a subharmonic function u we put

$$B(r, u) = \max_{|z|\leq r} u(z)$$

and notice that condition (1.2) can be rewritten as

$$B(r, u) = u(r), \quad r \geq 0, \tag{2.2}$$

in view of the Maximum Principle. This implies that $u(r)$ is strictly increasing for non-constant subharmonic functions u satisfying (1.2). Moreover, the Hadamard Three Circles Theorem implies that $u(r) = B(r, v)$ is convex with respect to $\log r$, so $u(r)$ is continuous for $r > 0$.

First we prove the necessity of our conditions. Let f_n be a sequence of polynomials with non-negative coefficients. Then $u_n = \log|f_n|/\deg f_n$ are subharmonic functions whose Riesz measures μ_n are the empirical measures of f_n. As the μ_n are probability measures, every sequence contains a subsequence for which the weak limit μ exists. This μ evidently satisfies $\mu(\mathbf{C}) \leq 1$, and μ is symmetric with respect to complex conjugation. Consider the potential u defined by (1.1). This is a subharmonic function, $u \not\equiv -\infty$, and we have $u_n + c_n \to u$ for suitable constants c_n.

For a complete discussion of the mode of convergence here we refer to the Appendix; what we need is that $u_n(r) + c_n \to u(r)$ at every point $r > 0$ and for all other points

$$\limsup_{n\to\infty} u_n(z) + c_n \leq u(|z|).$$

As the polynomials f_n have non-negative coefficients, they satisfy (1.4), and the u_n satisfy (1.2). Thus u satisfies (1.2).

In the rest of this section we prove sufficiency. We start with a measure μ such that the associated potential u in (1.1) satisfies (1.2) and

$$u(z) = u(\overline{z}). \tag{2.3}$$

The idea is to approximate u by potentials of the form $(\log|f_n|)/\deg f_n$, where the f_n are polynomials with real coefficients that satisfy the assumptions of Theorem A. Applying Theorem A we find that f_n^m has positive coefficients for some m. But f_n^m has the same empirical measure as f_n, which is close to μ.

If $u(z) = k\log|z|$, then we approximate u with

$$u_n(z) = k_n \log|z| + (1-k_n)\log|z+n|,$$

where k_n is a sequence of rational numbers such that $k_n \to k$, $0 \leq k_n \leq 1$. For the rest of the proof we assume that $u(z)$ is not of the form $k\log|z|$.

The approximation of u will be performed in several steps. In each step we modify the function obtained on the previous step, and starting with u obtain subharmonic functions $u_1, \ldots, u_5$. The corresponding Riesz measures will be denoted by $\nu_1, \ldots, \nu_5$. Each modification will preserve the asymptotic inequality (2.1).

1. Fix $\varepsilon > 0$ and define

$$u_1(z) = \max\{u(ze^{i\alpha})\colon |\alpha| \leq \varepsilon\}.$$

It is easy to see that u is the potential of some finite measure, and that $u_1 \to u$ when $\varepsilon \to 0$. This implies that the Riesz measure of u_1 is close (in the weak topology) to that of u.

Evidently, u_1 satisfies (1.2) and (2.3), and $u_1(re^{i\theta}) = u(r)$ for $|\theta| \leq \varepsilon$. Thus $u_1(re^{i\theta}) = u(r)$ does not depend on θ for $|\theta| \leq \varepsilon$.

2. Choose $\delta \in (0, \varepsilon)$ and consider the solution v of the Dirichlet problem in the sector

$$D = \{z\colon |\arg z| < \delta\}$$

with boundary conditions $u_1(z)$ and satisfying $v(z) = O(\log|z|)$ as $z \to \infty$. To prove the existence and uniqueness of v, we map D conformally onto the upper half-plane, and apply Poisson's formula to solve the Dirichlet problem.

The growth restriction near ∞ ensures that the solution of the Dirichlet problem is unique.

Let u_2 be the result of "sweeping out the Riesz measure" of u_1 out of the sector D. This means that

$$u_2(z) = \begin{cases} v(z) & \text{for } z \in D, \\ u_1(z) & \text{otherwise.} \end{cases}$$

Evidently, u_2 is subharmonic in the plane and satisfies (2.3). We shall prove that u_2 also satisfies the strict version of (1.2), namely

$$u_2(z) < u_2(|z|) \quad \text{for } z \notin [0, \infty). \tag{2.4}$$

In order to do so, we note first that u_1 is not harmonic in any neighborhood of the positive ray. This follows since $u_1(r)$ is not of the form $u_1(r) = c \log r$ and $u_1(re^{i\theta})$ does not depend on θ for $|\theta| \leq \varepsilon$. Because u_1 is subharmonic and v is harmonic this implies that $v(r) > u_1(r)$ for $r > 0$. As u_1 satisfies (1.2) we see that u_2 satisfies (2.4) for $\delta \leq |\arg z| \leq \pi$. In order to prove that u_2 satisfies (2.4) also for $|\arg z| \leq \delta$, let G be the plane cut along the negative ray and define

$$\psi_\alpha(z) = z^{\alpha/\pi} \quad \text{for } z \in G,$$

with the branch of the power chosen such that $\psi(z) > 0$ for $z > 0$. We claim that for $\alpha \in (\delta, \varepsilon)$, the function $v_\alpha = u_2 \circ \phi_\alpha$, extended by continuity to the negative ray, is subharmonic in the plane. Indeed, near the negative ray this function does not depend on $\arg z$ and it is subharmonic at all points except the negative ray, thus it is also subharmonic in a neighborhood of the negative ray.

The limit of these subharmonic functions v_α as $\alpha \to \delta + 0$ is the function v_δ which is thus subharmonic. But the Riesz measure of this function v_δ is supported on the negative ray, thus

$$v_\delta(z) = \int_0^1 \log|z + t| d\nu(t) + \int_{1+}^\infty \log\left|1 + \frac{z}{t}\right| d\nu(t),$$

with some non-negative measure ν. It is evident from this expression that for every $r > 0$ the function $t \mapsto v_\delta(re^{it})$ is strictly decreasing on $[0, \pi]$.

Thus for every $r > 0$, our function $t \mapsto u_2(re^{it})$ is strictly decreasing in the interval $[0, \delta]$. This, together with the fact that u_2 satisfies (2.3), completes the proof that u_2 satisfies (2.4).

3. Now we approximate our function u_2 by a function u_3 which is harmonic near 0. We set

$$u_3(z) = u_2(z + \varepsilon).$$

Then u_3 is harmonic near the origin, and using (2.4) and monotonicity of u_2 on the positive ray, we obtain

$$u_3(z) = u_2(z + \varepsilon) < u_2(|z + \varepsilon|) \leq u_2(|z| + \varepsilon) = u_3(|z|)$$

for $z \neq [0, \infty)$, so (2.4) is satisfied by u_3.

4. The subharmonic function u_3 we constructed has the following properties:

a) it satisfies (2.4),

b) it is harmonic near the origin,

c) it is harmonic in a neighborhood of the positive ray.

To construct a function which, in addition, is also harmonic near ∞ we consider the function

$$v(z) = u_3(1/z) + k \log |z|,$$

where $k = \nu_3(\mathbf{C})$. It is easy to see that this function is subharmonic, if we extend it to 0 appropriately. Notice that v satisfies (2.4), and it is harmonic in an angular sector containing the positive ray (in fact in the sector $|\arg z| < \delta$). The function $w(z) = v(z + \varepsilon)$ also satisfies (2.4) by the same argument that we used in Step 3 to show that u_3 satisfies (2.4). Moreover, it is harmonic near the origin and near infinity. Thus the function

$$u_4(z) = w(1/z) + k \log |z|$$

has all properties a), b), c) and in addition

d) it is harmonic in a punctured neighborhood of infinity.

5. As u_4 is harmonic in a neighborhood of the origin, it has a representation

$$u_4(z) = u_4(0) + \int \log \left| 1 - \frac{z}{\zeta} \right| d\nu_4(\zeta).$$

As u_4 satisfies (2.3), we can write

$$u_4(x+iy) = u_4(0) + cx + O(z^2), \quad z = x+iy \to 0,$$

where

$$c = \frac{d}{dx}\left(\int \log\left|1-\frac{x}{\zeta}\right| d\nu_4(\zeta)\right)\Big|_{x=0} = -\int \frac{\operatorname{Re}\zeta}{|\zeta|^2} d\nu_4(\zeta).$$

Property (2.4) of u_4 implies that $c \geq 0$. We may achieve $c > 0$ by adding to u_4 the potential $\varepsilon \log|1+z|$. This procedure changes c to $c+\varepsilon$. This also makes positive the linear term in the expansion at ∞. Thus we obtain a function u_5, close to our original potential u in the weak topology, which besides (2.3) and (2.4) also satisfies

$$u_5(x+iy) = \nu_5(\mathbf{C}) \log|z| + b/x + O(z^{-2}), \quad z \to \infty, \tag{2.5}$$

$$u_5(x+iy) = u_5(0) + ax + O(z^2), \quad z = x+iy \to 0, \tag{2.6}$$

with positive constants a and b.

6. In our final step we replace the Riesz measure of u_5 by a nearby discrete probability measure with finitely many atoms, each having rational mass.

Let μ be the Riesz measure of u_5. If $\mu(\mathbf{C}) < 1$ we change μ to a probability measure by adding an atom sufficiently far at the negative ray. Evidently, this procedure does not destroy our conditions (2.3) and (2.4), and we also still have (2.5) and (2.6) for certain positive constants a and b.

By our construction, the support of μ is disjoint from the open set

$$H = \{z\colon |\arg z| < \delta\} \cup \{z\colon |z| < \delta\} \cup \{z\colon |z| > 1/\delta\},$$

and replacing δ by a smaller number if necessary we may assume that this also holds after the atom on the negative ray was added.

Let μ_k be any sequence of symmetric discrete measures each having finitely many atoms of rational mass, supported outside H, and $\mu_k \to \mu$ weakly. Let w_k be the potential of μ_k. Clearly the w_k satisfy (2.3). We show that they also satisfy (2.4), provided k is large.

First we consider small $|z|$, noting that the w_k are harmonic for $|z| < \delta$. For $z = re^{i\theta}$ with $0 < r < \delta$ we thus have the expansion

$$w_k(z) = \sum_{n=0}^{\infty} a_{n,k} r^n \cos n\theta. \tag{2.7}$$

Hence

$$\frac{\partial^2}{\partial\theta^2} w_k(z) = -a_{1,k} r\cos\theta + \Phi_k(z) \tag{2.8}$$

with

$$\Phi_k(z) = -\sum_{n=2}^{\infty} a_{n,k} r^n n^2 \cos n\theta.$$

As the w_k are harmonic for $|z| < \delta$, the convergence to u_5 is locally uniformly there, and $\partial^2 w_k/\partial\theta^2$ also converges there locally uniformly to $\partial^2 u_5/\partial\theta^2$. For $0 < \eta < b$ and large k we thus have $a_{1,k} > \eta$ by (2.6). Moreover, for $0 < r_0 < \delta$ there exists $C > 0$ such that $|w_k(z)| \leq C$ for $|z| = r_0$ and all k. By Cauchy's inequalities we obtain $|a_{n,k} r_0^n| \leq C_1$ and hence

$$|\Phi_k(z)| \leq C_2 r^2 \quad \text{for } r \leq r_0/2.$$

This inequality, together with (2.8) shows that w_k satisfies (2.4) for $|z| < r_1$ with some r_1 independent of k.

The case of large $|z|$ is treated similarly, using (2.5) and the transformation

$$u(z) \mapsto \log|z| + u(1/z), \tag{2.9}$$

as we did before. Thus there exists $r_2 > 0$ such that w_k satisfies (2.4) for $|z| > r_2$.

We finally consider the case that $r_1 \leq |z| \leq r_2$. Recall that by the first statement of Lemma 1, $\partial^2 u/\partial\theta^2$ is negative on the positive ray, so we have a positive constant c such that $(\partial^2/\partial\theta^2) u(re^{i\theta}) < -c$ in some angular sector

$$S := \{z \colon |\arg z| < \beta,\ r_1 \leq |z| \leq r_2\}.$$

We conclude that

$$L(r) := u(r) - u(re^{i\beta}) \geq c_1 > 0 \quad \text{for } r_1 \leq r \leq r_2.$$

On the interval $[r_1, r_2]$ the convergence $w_k \to u$ is uniform, because u and w_k are harmonic in S. On the other hand, on the compact set

$$K := \{z \colon r_1 \leq |z| \leq r_2,\ |\arg z| \geq \beta\}$$

we have $w_k(z) \leq u(z) + c_1/2$ for all sufficiently large k. This follows from the general convergence properties of potentials of weakly convergent measures summarized in the Appendix. We conclude that w_k satisfies (2.4) also for $r_1 \leq |z| \leq r_2$, and hence for all $z \in \mathbf{C}$.

Now w_k is the empirical measure of some polynomial

$$f(z) = a_0 + a_1 z + \ldots + a_{d-1} z^{d-1} + a_d z^d,$$

and (2.4) implies that f satisfies (1.6). Clearly, $a_0 > 0$ and $a_d > 0$. Moreover, since $a_{1,k} > 0$ in (2.7), we see that $a_1 > 0$. The analogous expansion after the transformation (2.9) yields that $a_{d-1} > 0$. Thus the hypotheses of Theorem A are satisfied. Hence f^m has positive coefficients for some m. As the empirical measure of f and f^m coincide, we see that u_5 is a limit of empirical measures of polynomials with positive coefficients. As we may choose u_5 arbitrarily close to our original potential u by choosing ε sufficiently small, we see that u is also a limit of empirical measures of polynomials with positive coefficients. This completes the proof.

Appendix: Convergence of potentials

We frequently used various convergence properties of potentials of weakly convergent measures which we state here for the reader's convenience. An excellent reference for all this material is [6].

Let $\mu_n \to \mu$ be a sequence of weakly convergent positive measures. This means that for every continuous function ϕ with bounded support

$$\int \phi d\mu_n \to \int \phi d\mu, \quad n \to \infty.$$

If we restrict here to C^∞-functions ϕ with bounded support, we obtain convergence in the space D' of Schwartz distributions. Actually, for positive measures weak convergence is equivalent to D'-convergence.

Now the sequence of subharmonic functions

$$u_n(z) = \int_{|\zeta| \leq 1} \log|z - \zeta| d\mu_n(\zeta) + \int_{|\zeta|>1} \log\left|1 - \frac{z}{\zeta}\right| d\mu_n(\zeta)$$

converges in D' to the potential of the limit measure μ; that is, we have

$$\int \phi(z) u_j(z) dxdy \to \int \phi(z) u(z) dxdy \qquad (2.10)$$

for every test function ϕ. For the convenience of the reader we include a standard argument showing this.

First note that with

$$K(z,\zeta) = \begin{cases} \log|z-\zeta|, & |\zeta| \le 1, \\ \log|1-z/\zeta|, & |\zeta| > 1. \end{cases}$$

and

$$L(\zeta) = \int_{|z|\le R} \phi(z) K(z,\zeta) dxdy$$

we have

$$\int \phi(z) u_j(z) dxdy = \int \phi(z) \int K(z,\zeta) d\mu_j(\zeta) dxdy = \int L(\zeta) d\mu_j(\zeta).$$

Of course, this also holds with u_j and μ_j replaced by u and μ. Thus (2.10) is equivalent to

$$\int L(\zeta) d\mu_j(\zeta) \to \int L(\zeta)\, d\mu(\zeta). \tag{2.11}$$

Since $|\log|1-w|| \le 2|w|$ for $|w| \le 1/2$ we find that if $R > 1$, then

$$|K(z,\zeta)| \le \frac{2R}{|\zeta|} \quad \text{for } |z| \le R,\ |\zeta| > 2R.$$

Choosing R such that the support of ϕ is contained in $|z| \le R$ we conclude that

$$L(\zeta) \le \frac{C}{|\zeta|} \quad \text{for } |\zeta| > 2R$$

with some constant C.

To show that (2.11) holds we choose $\varepsilon > 0$ and fix $R_1 > 2R$ so large that $C/R_1 < \varepsilon/2$. Now L is continuous and we may write $L = L_1 + L_2$ with continuous functions L_1 and L_2, where L_1 has compact support and L_2 satisfies $L_2(\zeta) = 0$ for $|\zeta| \le R_1$ and $|L_2(\zeta)| \le |L(\zeta)| \le C/|\zeta|$ for $|\zeta| > R_1$. Then

$$\begin{aligned} \left| \int L_2(\zeta) d\mu_j(\zeta) - \int L_2(\zeta)\, d\mu(\zeta) \right| &\le \left| \int L_2(\zeta) d\mu_j(\zeta) \right| + \left| \int L_2(\zeta)\, d\mu(\zeta) \right| \\ &\le 2\frac{C}{R_1} \le \varepsilon \end{aligned}$$

since $\mu_j(\mathbf{C}) \le 1$ and $\mu(\mathbf{C}) \le 1$. We also have $\int L_1(\zeta) d\mu_j(\zeta) \to \int L_1(\zeta)\, d\mu(\zeta)$ by the definition of weak convergence, which is equivalent to convergence in D'. We obtain (2.11) and hence (2.10).

We cite Theorem 3.2.13 from [6] which says that this convergence of potentials also holds in several other senses.

Theorem B. *Let $u_j \not\equiv -\infty$ be a sequence of subharmonic functions converging in D' to the subharmonic function u. Then the sequence is uniformly bounded from above on any compact set. For every z we have*

$$\limsup_{n\to\infty} u_n(z) \le u(z). \tag{2.12}$$

More generally, if K is a compact set, and $f \in C(K)$, then

$$\limsup_{n\to\infty} \sup_K (u_n - f) \le \sup_K (u - f).$$

If $d\sigma$ is a positive measure with compact support such that the potential of $d\sigma$ is continuous, then there is equality in (2.12) *and $u(z) > -\infty$ for almost every z with respect to $d\sigma$. Moreover, $u_j d\sigma \to u\, d\sigma$ weakly.*

In this paper we deal with subharmonic functions satisfying (2.2), so $u(r)$ is increasing and convex with respect to $\log r$ on $(0, \infty)$. Choosing the length element on $[0, R]$ as $d\sigma$ in Theorem B, we conclude that $u_n \to u$ almost everywhere on the positive ray. For convex functions with respect to the logarithm this is equivalent to the uniform convergence on compact subsets of $(0, \infty)$. In particular, $u_n(r) \to u(r)$ at every point $r > 0$. As the u_n satisfy (1.2), we conclude that

$$\limsup_{n\to\infty} u_n(re^{i\theta}) \le u(r).$$

Choosing the uniform measure on the circle $|z| = r$ as $d\sigma$ in Theorem B, we conclude that $u(re^{i\theta}) \le u(r)$ almost everywhere with respect to $d\sigma$. As u is upper semi-continuous, we conclude that $u(re^{i\theta}) \le u(r)$. Thus (1.2) is preserved in the limit.

References

[1] R. Barnard, W. Dayawansa, K. Pearce and D. Weinberg, Polynomials with non-negative coefficients, Proc. Amer. Math. Soc., 113 (1991), 77–85.

[2] V. De Angelis, Asymptotic expansions and positivity of coefficients for large powers of analytic functions, Int. J. Math. Math. Sci., 16 (2003), 1003–1025.

[3] A. Eremenko and A. Fryntov, Remarks on the Obrechkoff inequality, preprint.

[4] S. Ghosh and O. Zeitouni, Large deviations for zeros of random polynomials with i.i.d. exponential coefficients, arXiv:1312.6195.

[5] W. K. Hayman and P. B. Kennedy, Subharmonic functions, vol. I, Academic Press, London, 1976.

[6] L. Hörmander, Notions of convexity, Birkhäuser, Boston, 1994.

[7] N. Obrechkoff, Sur un problème de Laguerre, Comptes Rendus, 177 (1923), 102–104.

[8] O. Zeitouni, Zeros of polynomials with real positive coefficients, http://mathoverflow.net/q/134998/25510

W. B.: Mathematisches Seminar, CAU Kiel,
Ludewig-Meyn-Str. 4, 24098 Kiel, Germany
bergweiler@math.uni-kiel.de

A. E.: Department of Mathematics
Purdue University
West Lafayette, IN 47907 USA
eremenko@math.purdue.edu