On the Bank–Laine conjecture

Walter Bergweiler and Alexandre Eremenko*

Abstract

We resolve a question of Bank and Laine on the zeros of solutions of $w'' + Aw = 0$ where A is an entire function of finite order.

AMS 2010 Subj. Class: 34A20, 30D15.

Keywords: entire function, linear differential equation, complex oscillation, quasiconformal surgery, Bank–Laine function.

1 Introduction and result

The asymptotic distribution of zeros of solutions of linear differential equations with polynomial coefficients is described quite precisely by asymptotic integration methods; cf. [10] and [11, Chapter 8]. While certain differential equations with transcendental coefficients such as the Mathieu equation were considered early on, the first general results concerning the frequency of the zeros of the solutions of

$$w'' + Aw = 0 \tag{1.1}$$

with a transcendental entire function A appear to be due to Bank and Laine [2, 3].

For an entire function f, denote by $\rho(f)$ the order and by $\lambda(f)$ the exponent of convergence of the zeros of f. If A is a polynomial of degree n, then $\rho(w) = 1 + n/2$ for every solution w of (1.1), while $\rho(w) = \infty$ for every solution w if A is transcendental.

Let w_1 and w_2 be linearly independent solutions of (1.1). Bank and Laine proved that if A is transcendental and $\rho(A) < \frac{1}{2}$, then

$$\max\{\lambda(w_1), \lambda(w_2)\} = \infty.$$

*Supported by NSF grant DMS-1361836.

It was shown independently by Rossi [19] and Shen [20] that this actually holds for $\rho(A) \leq \frac{1}{2}$. Bank and Laine also showed that in the case of non-integer $\rho(A)$ we always have

$$\max\{\lambda(w_1), \lambda(w_2)\} \geq \rho(A), \tag{1.2}$$

and they gave examples of functions A of integer order for which there are solutions w_1 and w_2 both without zeros.

A problem left open by their work – which later became known as the Bank–Laine conjecture – is whether $\max\{\lambda(w_1), \lambda(w_2)\} = \infty$ whenever $\rho(A)$ is not an integer. This question has attracted considerable interest; see [13] for a survey, as well as, e.g., [8], [9] and [12, Chapter 5].

We answer this question by showing that the estimate (1.2) is best possible for a dense set of orders in the interval $(1, \infty)$.

Theorem. *Let p and q be odd integers. Then there exists an entire function A of order*

$$\rho(A) = 1 + \frac{\log^2(p/q)}{4\pi^2}$$

for which the equation (1.1) *has two linearly independent solutions w_1 and w_2 such that $\lambda(w_1) = \rho(A)$ while w_2 has no zeros.*

By an extension of the method it should be possible to achieve any prescribed order $\rho(A) > 1$; see Remark 2 at the end.

If w_1 and w_2 are linearly independent solutions of (1.1), then the Wronskian $W(w_1, w_2) = w_1 w_2' - w_1' w_2$ is a non-zero constant. The solutions are called normalized if $W(w_1, w_2) = 1$.

It is well-known that the ratio $F = w_2/w_1$ satisfies the Schwarz differential equation (see, for example [11]):

$$S[F] := \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = 2A.$$

These meromorphic functions F are completely characterized by a topological property: they are locally univalent. More precisely, consider the equivalence relation on meromorphic functions $F_1 \sim F_2$ if $F_1 = L \circ F_2$, where L is a fractional linear transformation. Then the map $F \mapsto S[F]$ is a bijection between the equivalence classes of locally univalent meromorphic functions and all entire functions.

Normalized solutions w_1, w_2 are recovered from F by the formulas

$$w_1^2 = \frac{1}{F'}, \quad w_2^2 = \frac{F^2}{F'}.$$

So zeros of F are zeros of w_2 and poles of F are zeros of w_1.

A meromorphic function F is locally univalent if and only if $E = F/F'$ is an entire function with the property that $E(z) = 0$ implies $E'(z) \in \{-1, 1\}$. Such entire functions E are called *Bank–Laine functions*. If w_1 and w_2 is a normalized system of solutions of (1.1) and $F = w_2/w_1$, then

$$E = \frac{F}{F'} = w_1 w_2.$$

The converse is also true: every Bank–Laine function is the product of two linearly independent solutions of (1.1).

It turns out that the Schwarzian derivative has the following factorization:

$$2S[F] = B[F/F'],$$

where

$$B[E] := -2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2}. \tag{1.3}$$

Thus every Bank–Laine function E is a product of two linearly independent solutions of (1.1) with $4A = B[E]$, a fact discovered by Bank and Laine [2, 3].

A considerable part of the previous research related to the Bank–Laine conjecture has concentrated on the study of Bank–Laine functions. There are a number of papers where Bank–Laine functions of finite order with various other properties are constructed [1, 4, 6, 14, 15, 16, 18]. In all examples constructed so far, for which the order could be determined, it was an integer. In our construction we have $\rho(E) = \rho(A)$; see Remark 1. Thus our theorem also yields the first examples of Bank–Laine functions of finite non-integral order.

In the proof of our theorem we use the fact that the functions F have a topological characterization. Starting with two elementary locally univalent functions, we paste them together by a quasiconformal surgery. The resulting function is locally univalent, and the asymptotics of $\log |F/F'|$ can be explicitly computed. A different kind of quasiconformal surgery was used in [4, 13].

Acknowledgments. We are grateful to Jim Langley for a very detailed reading of the manuscript and many helpful suggestions. The first author also thanks him for an illuminating discussion on the problem in 2012. We are also thankful to David Drasin, Gary Gundersen and Ilpo Laine for their comments.

2 Proof of the theorem

For every integer $m \geq 0$ we consider the polynomial

$$P_m(z) = \sum_{k=0}^{2m} (-1)^k \frac{z^k}{k!}.$$

Then the entire function

$$g_m(z) = P_m(e^z) \exp e^z$$

satisfies

$$g_m'(z) = \left(P_m'(e^z) + P_m(e^z)\right) e^z \exp e^z = \frac{1}{(2m)!} \exp\left(e^z + (2m+1)z\right)$$

and thus it has the following properties:

a) $g_m'(z) \neq 0$ for all $z \in \mathbf{C}$,

b) g_m is increasing on $\mathbf{R}$, and satisfies $g_m(x) \to 1$ as $x \to -\infty$ as well as $g_m(x) \to +\infty$ as $x \to +\infty$.

From now on, we fix two distinct non-negative integers m and n, and will sometimes omit them from notation. Notice that g_m and g_n are locally univalent entire functions. We are going to restrict g_m to the upper half-plane H^+ and g_n to the lower half-plane H^-, and then paste them together, using a quasiconformal surgery, producing an entire function F. Then our Bank–Laine function will be $E = F/F'$ and thus $A = B[E]/4$ as in (1.3).

It follows from b) that there exists an increasing diffeomorphism $\phi\colon \mathbf{R} \to \mathbf{R}$ such that $g_m(x) = (g_n \circ \phi)(x)$ for $x \in \mathbf{R}$. Let

$$k = \frac{2m+1}{2n+1}.$$

We show that the asymptotic behavior of the diffeomorphism ϕ is the following:

$$\phi(x) = x + O(e^{-x/2}), \quad \phi'(x) \to 1, \quad x \to +\infty, \tag{2.1}$$

and

$$\phi(x) = kx + c + O(e^{-\delta|x|}), \quad \phi'(x) \to k, \quad x \to -\infty, \tag{2.2}$$

with

$$c = \frac{1}{2n+1} \log \frac{(2n+1)!}{(2m+1)!} \quad \text{and} \quad \delta = \frac{1}{2}\min\{1, k\}.$$

In order to prove (2.1), we note that

$$\log g_m(x) = e^x + O(x) = e^x \left(1 + O(xe^{-x})\right), \quad x \to +\infty.$$

The equation $g_m(x) = g_n(\phi(x))$ easily implies that $\frac{2}{3}x \le \phi(x) \le 2x$ for large x. Thus we also have

$$\begin{aligned} \log g_n(\phi(x)) &= e^{\phi(x)} \left(1 + O\big(\phi(x) e^{-\phi(x)}\big)\right) \\ &= e^{\phi(x)} \left(1 + O\big(xe^{-2x/3}\big)\right), \quad x \to +\infty. \end{aligned}$$

Combining the last two equations we obtain

$$e^{\phi(x)-x} = 1 + O(xe^{-2x/3}), \quad x \to +\infty,$$

from which the first statement in (2.1) easily follows. For the second statement in (2.1) we use

$$\phi' = \frac{g_m'}{g_m} \frac{g_n \circ \phi}{g_n' \circ \phi}, \tag{2.3}$$

so that

$$\begin{aligned} \phi'(x) &= \frac{(2n)!}{(2m)!} e^{(2m+1)x - (2n+1)\phi(x)} \frac{P_n(e^{\phi(x)})}{P_m(e^x)} \\ &\sim e^{(2m+1)x - (2n+1)\phi(x) + 2n\phi(x) - 2mx} \\ &= e^{x - \phi(x)} = 1 + o(1), \quad x \to +\infty. \end{aligned}$$

In order to prove (2.2) we note that

$$P_m(w) = e^{-w} + \frac{w^{2m+1}}{(2m+1)!} + O(w^{2m+2}), \quad w \to 0,$$

and thus

$$P_m(w)e^w = 1 + \frac{w^{2m+1}}{(2m+1)!} + O(w^{2m+2}), \quad w \to 0.$$

Hence

$$\begin{aligned} g_m(x) &= 1 + \frac{e^{(2m+1)x}}{(2m+1)!} + O(e^{(2m+2)x}) \\ &= 1 + \frac{e^{(2m+1)x}}{(2m+1)!}(1 + O(e^x)), \quad x \to -\infty. \end{aligned}$$

The equation $g_m(x) = g_n(\phi(x))$ now yields

$$\frac{(2m+1)!}{(2n+1)!} e^{(2n+1)\phi(x) - (2m+1)x} = 1 + O(e^x) + O(e^{\phi(x)}), \quad x \to -\infty$$

and hence

$$\phi(x) = \frac{2m+1}{2n+1}x + \frac{1}{2n+1}\log\frac{(2n+1)!}{(2m+1)!} + O(e^x) + O(e^{\phi(x)}), \quad x \to -\infty,$$

which gives the first statement in (2.2). For the second statement in (2.2) we use (2.3) and obtain

$$\begin{aligned} \phi'(x) &\sim \frac{(2n)!}{(2m)!} e^{(2m+1)x - (2n+1)\phi(x)} = \frac{(2n)!}{(2m)!} e^{(2m+1)x - (2n+1)(kx + c + o(1))} \\ &= \frac{(2n)!}{(2m)!} e^{-(2n+1)c + o(1)} = k + o(1). \end{aligned}$$

Let $D = \mathbf{C} \backslash \mathbf{R}_{\leq 0}$, and $p \colon D \to \mathbf{C}$, $p(z) = z^\mu$, the principal branch of the power. Here μ is a complex number to be determined so that p maps D onto the complement G of a logarithmic spiral Γ, with

$$p(x + i0) = p(kx - i0), \quad x < 0. \tag{2.4}$$

It will be convenient to consider also the map $z \to \mu z$ obtained from p by a logarithmic change of the variable: if $w = p(z)$ then $\log w = \mu \log z$, cf. Figure 1.

This shows (taking $x = 0$ in Figure 1) that with $a_- = \log k - i\pi$ and $a_+ = i\pi$ we have $\mathrm{Re}(\mu a_-) = \mathrm{Re}(\mu a_+)$; that is, $\mathrm{Re}(\mu(\log k - i\pi)) = \mathrm{Re}(\mu i \pi)$. Moreover, $\mathrm{Im}(i\pi/\mu) = \pi$. A simple computation now yields that

$$\mu = \frac{2\pi}{4\pi^2 + \log^2 k}(2\pi - i \log k).$$

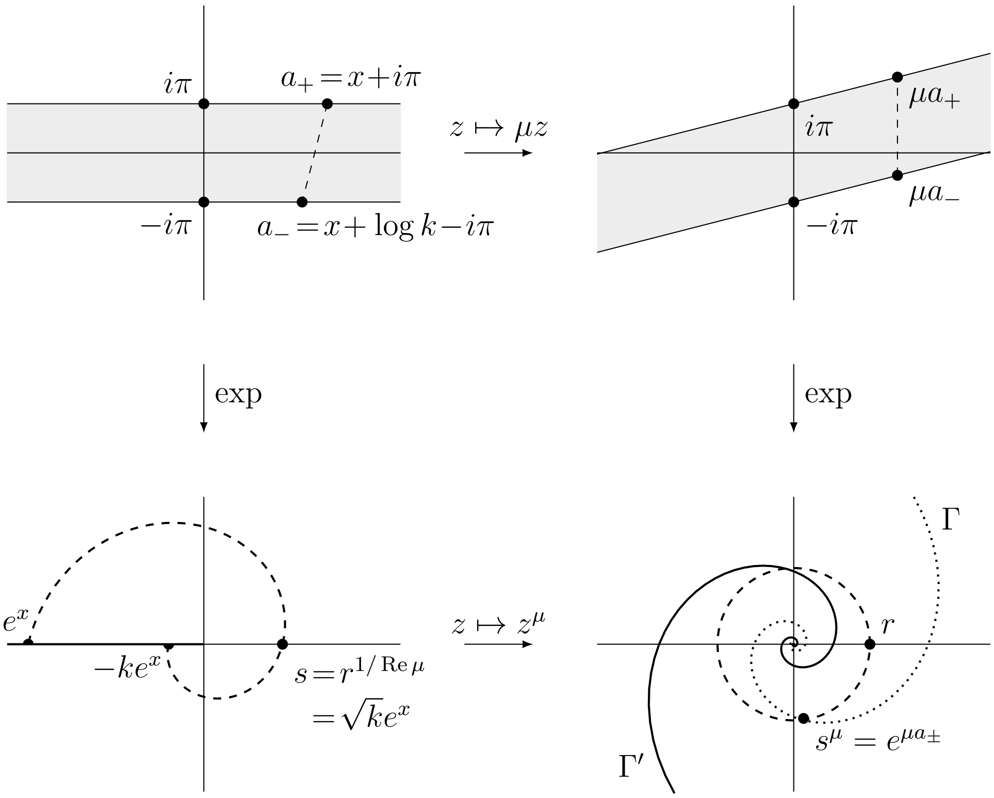

Figure 1: Sketch of the map p and the logarithmic change of variable, for $k = \frac{1}{5}$ and $\mu \approx 0.9384 + 0.2403i$. (The actual spirals Γ and Γ' wind much slower than drawn.)

The inverse map $h = p^{-1}$ is a conformal homeomorphism $h\colon G \to D$. Let $\Gamma' = p(\mathbf{R}_{\geq 0})$. The two logarithmic spirals Γ and Γ' divide the plane into two parts, G^+ and G^- which are images under p of the upper and lower half-planes, respectively.

The function V defined by

$$V(z) = \begin{cases} (g_m \circ h)(z), & z \in G^+, \\ (g_n \circ h)(z), & z \in G^-, \end{cases}$$

is analytic in $G^+ \cup G^-$ and has a jump discontinuity on Γ and Γ'. In view of (2.1), (2.2) and (2.4), this discontinuity can be removed by a small change in the independent variable. In order to do so, we consider the strip $\Pi =$

$\{z\colon |\operatorname{Im} z| < 1\}$ and define a quasiconformal homeomorphism $\tau\colon \mathbf{C} \to \mathbf{C}$, commuting with the complex conjugation, which is the identity outside of Π and satisfies

$$\tau(x) = \phi(x), \quad x > 0, \quad \text{and} \quad \tau(kx) = \phi(x), \quad x < 0. \tag{2.5}$$

Our homeomorphism can be given by an explicit formula: for $y = \operatorname{Im} z \in (-1, 1)$ we put

$$\tau(x+iy) = \begin{cases} \phi(x) + |y|(x - \phi(x)) + iy, & x \geq 0 \\ \phi(x/k) + |y|(x - \phi(x/k)) + iy, & x < 0. \end{cases}$$

The Jacobian matrix D_τ of τ is given for $x > 0$ and $0 < |y| < 1$ by

$$D_\tau(x+iy) = \begin{pmatrix} \phi'(x) + |y|(1 - \phi'(x)) & \pm(x - \phi(x)) \\ 0 & 1 \end{pmatrix},$$

and we see using (2.1) that

$$D_\tau(x+iy) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 < |y| < 1,\ x \to \infty,$$

Similarly, using (2.2) we find that

$$D_\tau(x+iy) \to \begin{pmatrix} 1 & \mp c \\ 0 & 1 \end{pmatrix}, \quad 0 < |y| < 1,\ x \to -\infty.$$

We conclude that τ is quasiconformal in the plane.

Now we modify V to obtain a continuous function and define $U\colon \mathbf{C} \to \mathbf{C}$,

$$U(z) = \begin{cases} (g_m \circ h)(z), & z \in G^+ \cup \Gamma \cup \Gamma' \cup \{0\}, \\ (g_n \circ \tau \circ h)(z), & z \in G^-. \end{cases} \tag{2.6}$$

It follows from (2.4) and (2.5) that U is continuous and quasiregular in the plane. The existence theorem for solutions of the Beltrami equation [17, §V.1] yields that there exists a a quasiconformal homeomorphism $\psi\colon \mathbf{C} \to \mathbf{C}$ with the same Beltrami coefficient as U. The function $F = U \circ \psi^{-1}$ is then entire.

We note that U is regular in $\mathbf{C}\backslash X$, where $X = p(\Pi^-)$, and Π^- is the lower half of Π. Let $\Delta = \{z\colon |z| > 1\}$. It is easy to see that $X \cap \Delta$ has finite logarithmic area; that is,

$$\int_{X\cap\Delta} \frac{dx\,dy}{x^2+y^2} = \int_{\Pi^-\cap\Delta} \frac{|p'(z)|^2}{|p(z)|^2} dx\,dy = |\mu|^2 \int_{\Pi^-\cap\Delta} \frac{dx\,dy}{x^2+y^2} < \infty.$$

Thus the Beltrami coefficient of U (and hence of ψ) satisfies the hypotheses of the Teichmüller–Wittich–Belinskii theorem [17, §V.6]. This theorem shows that ψ is conformal at ∞ and may thus be normalized to satisfy

$$\psi(z) \sim z, \quad z \to \infty. \tag{2.7}$$

Now we want to differentiate the asymptotic relation (2.7). We write $\psi(z) = z + \psi_0(z)$ so that $\psi'(z) = 1 + \psi_0'(z)$. Then $|\psi_0(z)| \leq \alpha(z)$ for some function α satisfying $\alpha(z) = o(z)$ as $z \to \infty$. We may assume that $\alpha(z) \to \infty$ as $z \to \infty$. We use the Cauchy formula

$$\psi_0'(z) = \frac{1}{2\pi i} \int_{C_z} \frac{\psi_0(\zeta)}{(\zeta - z)^2} d\zeta$$

with a circle C_z centered at z. Choosing the radius $\beta(z)$ of this circle to satisfy

$$\alpha(z) = o(\beta(z)), \quad \beta(z) = o(z), \quad z \to \infty$$

and putting $Y = \{z\colon \operatorname{dist}(z, X) \leq \beta(z)\}$ we obtain

$$\psi_0'(z) \to 0, \quad z \to \infty,\ z \in \mathbf{C}\backslash Y. \tag{2.8}$$

We also have

$$\operatorname{meas}\{\theta \in [0, 2\pi]\colon re^{i\theta} \in Y\} \to 0, \quad r \to \infty.$$

Let $Y' = \psi(Y)$. Using (2.7) we see that also

$$\operatorname{meas}\{\theta \in [0, 2\pi]\colon re^{i\theta} \in Y'\} \to 0, \quad r \to \infty. \tag{2.9}$$

We put $E = F/F'$. As $F'(z) \neq 0$ for all $z \in \mathbf{C}$ by construction, E is entire. As all zeros of F are simple, all residues of F'/F are equal to 1, so $E'(z) = 1$ at every zero z of E, which implies the Bank–Laine property.

First we prove that E is of finite order. In order to do this, we use the standard terminology of Nevanlinna theory; see [7] or [12]. The counting function of the sequence of zeros of g_m and g_n is of order 1, so the counting function of the zeros of U in (2.6) is of finite order. Then (2.7) shows that the counting function of zeros of F, and hence the counting function of the zeros of E, is also of finite order; that is, $\log N(r, 1/E) = O(\log r)$. Similarly, $\log\log m(r, F) = O(\log r)$, so by the Lemma on the logarithmic derivative [7, Chapter 3, Theorem 1.3] we have $\log m(r, 1/E) = \log m(r, F'/F) = O(\log r)$. Thus $\log T(r, E) = O(\log r)$ so that E is of finite order.

Now we estimate more precisely the growth of the Nevanlinna proximity function $m(r, 1/E) = m(r, F'/F)$. The "small arcs lemma" of Edrei and Fuchs [7, Chapter 1, Theorem 7.3] permits us to discard the exceptional set $Y' = \psi(Y)$. Outside of this set we have $\psi'(z) \to 1$ in view of (2.8), therefore

$$\int_{\{\theta\in[0,2\pi]\colon\, re^{i\theta}\in\mathbf{C}\setminus Y'\}} \left|\log|\psi'(re^{i\theta})|\right| d\theta = o(1), \quad r \to \infty. \tag{2.10}$$

Furthermore, as $h(z) = z^{1/\mu}$, we have

$$\int_0^{2\pi} \left|\log|h'(re^{i\theta})|\right| d\theta = O(\log r), \quad r \to \infty. \tag{2.11}$$

Now we have in $\psi^{-1}(D^+\setminus Y)$

$$\frac{F'}{F} = \left(\frac{g_m'}{g_m} \circ h \circ \psi^{-1}\right)(h' \circ \psi^{-1})(\psi^{-1})'. \tag{2.12}$$

According to (2.10) and (2.11), the contribution of h' and $(\psi^{-1})'$ to $m(r, F'/F)$ is $O(\log r)$. Using the explicit form of g_m'/g_m we obtain, outside small neighborhoods of the zeros of g_m whose contribution can be neglected again by the small arcs lemma of Edrei and Fuchs,

$$\log^+\left|\frac{g_m'(z)}{g_m(z)}\right| \sim \operatorname{Re}^+ z, \quad z \to \infty. \tag{2.13}$$

Now the image of the circle $\{z\colon |z| = r\}$ under $h(z) = z^{1/\mu}$ is the part of the logarithmic spiral which connects two points on the negative real axis and intersects the positive real axis at $r^{1/\operatorname{Re}\mu}$; cf. Figure 1. By (2.7), the image of this circle under $h \circ \psi^{-1}$ is an arc close to this part of the logarithmic spiral. It now follows from (2.10), (2.11), (2.12) and (2.13) that the part of $m(r, F'/F)$ which comes from $\psi^{-1}(G^+\setminus Y)$ has order

$$\rho = \frac{1}{\operatorname{Re}\mu} = 1 + \frac{\log^2 k}{4\pi^2}.$$

The other part which comes from $\psi^{-1}(G^-\setminus Y)$ is similar, and the contribution of Y' is negligible in view of (2.9). So $m(r, 1/E) = m(r, F'/F)$ has order ρ.

Now (1.3) says that

$$4A = -2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2}.$$

It follows from the lemma on the logarithmic derivative that

$$m(r, A) = 2m\left(r, \frac{1}{E}\right) + O(\log r).$$

Thus A also has order ρ.

3 Remarks

Remark 1. To prove that $\rho(A) = \rho$ it was sufficient to determine the growth of $m(r, 1/E)$. To show that $\rho(E) = \rho$ we also have to estimate the counting function of the zeros of E. In order to do so we note that $N(r, 1/g_m) = O(r)$ and $N(r, 1/g_n) = O(r)$. Hence $N(r, 1/U) = O(r^\rho)$ and thus (2.7) implies that

$$N\left(r, \frac{1}{E}\right) = N(r, F) = O(r^\rho).$$

Altogether we see that $\rho(E) = \rho = \rho(A)$, as stated in the introduction.

We note that $\rho(A) < 1$ implies that $\rho(E) > 1$, as follows from any of the following inequalities [13, Theorem 12.3.1]:

$$\rho(A) + \rho(E) \geq 2, \quad \frac{1}{\rho(A)} + \frac{1}{\rho(E)} \leq 2 \quad \text{and} \quad \rho(A)\rho(E) \geq 1.$$

Moreover, it can be deduced from (1.3) that if $\rho(A) < 1$, then $\lambda(E) = \rho(E)$; see [13, p. 442].

As our method yields examples with $\rho(E) = \rho(A)$, it does not seem suitable to give examples with $\rho(A) < 1$. The question whether $\rho(A) \in (\frac{1}{2}, 1)$ implies that $\max\{\lambda(w_1), \lambda(w_2)\} = \infty$ for linearly independent solutions w_1 and w_2 of (1.1) remains open.

Remark 2. We started our construction with two periodic locally univalent functions g_m and g_n and obtained a set of orders ρ which is dense in $[1, +\infty)$. By using almost periodic building blocks instead of g_m and g_n, one can probably achieve any prescribed order greater than 1; cf. [7, Chapter 7, Section 6]. In this case g_m and g_n will not be explicitly known, but their asymptotic behavior can be obtained.

Remark 3. The Bank–Laine functions we have constructed actually satisfy $E(z) = 1$ whenever $E'(z) = 0$. Equivalently, one of the two solutions of (1.1) whose product is E has no zeros while the other one has a finite exponent of convergence.

References

[1] A. Alotaibi, On Bank–Laine functions, Comput. Methods Funct. Theory 9 (2009), 335–345.

[2] S. Bank and I. Laine, On the oscillation theory of $f'' + Af = 0$ where A is entire, Bull. Amer. Math. Soc. 6 (1982), 95–98.

[3] S. Bank and I. Laine, On the oscillation theory of $f'' + Af = 0$ where A is entire, Trans. Amer. Math. Soc. 273 (1982), 351–363.

[4] D. Drasin and J. Langley, Bank–Laine functions via quasiconformal surgery, in: "Transcendental dynamics and complex analysis", 165–178, Cambridge Univ. Press, Cambridge, 2008.

[5] S. M. Elzaidi and J. K. Langley, Bank–Laine functions with periodic zero-sequences, Results Math. 48 (2005), 34–43.

[6] A. Fletcher, On Bank–Laine Functions, Comput. Methods Funct. Theory 9 (2009), 227–238

[7] A. Goldberg and I. Ostrovskii, Distribution of values of meromorphic functions, AMS, Providence, RI, 2008.

[8] G. Gundersen, The mathematical work of Ilpo Laine, in "Proceedings of the Workshop on Complex Analysis and its Applications to Differential and Functional Equations", J. Gröhn, J. Heittokangas, R. Korhonen and J. Rättyä (editors), University of Eastern Finland, 2014.

[9] J. Heittokangas and K. Tohge, A unit disc analogue of the Bank-Laine conjecture does not hold, Ann. Acad. Sci. Fenn. Math. 36 (2011), 341–351.

[10] E. Hille, Oscillation theorems in the complex domain, Trans. Amer. Math. Soc. 23 (1922), 350–385.

[11] E. Hille, Ordinary differential equations in the complex domain, John Wiley & Sons, New York–London–Sydney, 1976; reprint: Dover, Mineola, NY, 1997.

[12] I. Laine, Nevanlinna theory and complex differential equations, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.

[13] I. Laine and K. Tohge, The Bank–Laine conjecture – a survey, in: "Some topics on value distribution and differentiability in complex and p-adic analysis", 398–417, Sci. Press, Beijing, 2008.

[14] J. K. Langley, Quasiconformal modifications of Bank–Laine functions, Arch. Math. 71 (1998), 233–239.

[15] J. K. Langley, Bank–Laine functions with sparse zeros, Proc. Amer. Math. Soc. 129 (2001), 1969–1978.

[16] J. K. Langley, Composite Bank–Laine functions and a question of Rubel, Trans. Amer. Math. Soc. 354 (2002), 1177–1191.

[17] O. Lehto and K. Virtanen, Quasiconformal mappings in the plane, Springer-Verlag, New York-Heidelberg, 1973.

[18] J. Meyer, The distribution of zeros of finite order Bank–Laine functions, Comput. Methods Funct. Theory 7 (2007), 563–571.

[19] J. Rossi, Second order differential equations with transcendental coefficients, Proc. Amer. Math. Soc. 97 (1986), 61–66.

[20] L. Shen, Proof of a conjecture of Bank and Laine regarding the product of two linearly independent solutions of $y'' + Ay = 0$, Proc. Amer. Math. Soc. 100 (1987), 301–308.

Mathematisches Seminar
Christian-Albrechts-Universität zu Kiel
Ludewig–Meyn–Str. 4
24098 Kiel
Germany

Purdue University
West Lafayette, IN 47907
USA