THE JULIA SET AND THE FAST ESCAPING SET OF A QUASIREGULAR MAPPING

WALTER BERGWEILER, ALASTAIR FLETCHER, AND DANIEL A. NICKS

Dedicated to the memory of Professor Frederick W. Gehring

ABSTRACT. It is shown that for quasiregular maps of positive lower order the Julia set coincides with the boundary of the fast escaping set.

1. INTRODUCTION

Quasiregular maps are a natural generalisation of holomorphic functions to higher dimensions, see section 2 for the definition and basic properties. A quasiregular self-map f of the d-sphere S^d is called *uniformly quasiregular* if there is a uniform bound on the dilatation of the iterates f^n of f. As in the case of rational functions, the *Julia set* of such a map is defined as the set of all points in S^d where the iterates of f fail to be normal. Many results of the Fatou-Julia iteration theory of rational functions continue to hold in this more general setting; see [2, Section 4] and [15, Chapter 21] for surveys. In principle the corresponding iteration theory of transcendental entire functions could also be extended to uniformly quasiregular self-maps of $\mathbb{R}^d$ with an essential singularity at ∞, but so far no examples of such maps are known for $d \geq 3$. And for $d = 2$ uniformly quasiregular maps are conjugate to holomorphic maps; see [2, Section 4.1] for a discussion of this result.

A Fatou-Julia theory for quasiregular self-maps of S^d which are not uniformly quasiregular was developed in [3], extending work of Sun and Yang [20, 21, 22] dealing with the case $d = 2$. The theory was carried over to the case of quasiregular self-maps of $\mathbb{R}^d$ with an essential singularity at ∞ in [8]. Such maps are said to be of *transcendental type*. In [3, 8] the Julia set $J(f)$ of f is defined as the set of all x such that the

complement of $\bigcup_{k=1}^{\infty} f^k(U)$ has capacity zero for every neighbourhood U of x; see section 2 for the definition and a discussion of capacity.

The *escaping set* $I(f)$ of a quasiregular self-map f of $\mathbb{R}^d$ is defined by

$$I(f) = \{x \in \mathbb{R}^d \colon f^n(x) \to \infty\},$$

and it plays a major role in the dynamics of entire functions. The escaping set was first considered by Eremenko [10] for transcendental entire functions. He showed that it is always non-empty. The result was extended to quasiregular maps of transcendental type in [6]. Moreover, Eremenko proved that $J(f) = \partial I(f)$ for every transcendental entire function. This key result in the dynamics of entire functions fails to be true in general for quasiregular maps of transcendental type. In this setting we still have $J(f) \subset \partial I(f)$, but strict inclusion is possible [8, Theorem 1.3, Example 7.3].

Besides the escaping set $I(f)$, the *fast escaping set* $A(f)$ introduced in [7] has become increasingly important in recent years, see [18, 19]. In order to define it, recall that the *maximum modulus* is given by

$$M(r, f) = \max_{|x|=r} |f(x)|,$$

where $|x|$ denotes the Euclidean norm of a point $x \in \mathbb{R}^d$. If f is a quasiregular map of transcendental type, then there exists $R_0 \geq 0$ such that $M(R, f) > R$ for $R > R_0$. Denote by $M^n(R, f)$ the iteration of $M(R, f)$ with respect to the first variable; that is, $M^1(R, f) = M(R, f)$ and $M^n(R, f) = M(M^{n-1}(R, f), f)$ for $n \geq 2$. For $R > R_0$ we then have $M^n(R, f) \to \infty$ as $n \to \infty$. For such R we put

$$(1.1) \qquad A(f) = \{x \in \mathbb{R}^d \colon \exists L \in \mathbb{N} \ \forall n \in \mathbb{N} \colon |f^{n+L}(x)| \geq M^n(R, f)\}.$$

It can be shown that this definition does not depend on R as long as $M^n(R, f) \to \infty$, see [19] for transcendental entire functions and [4] for quasiregular maps. These papers also contain some equivalent definitions of $A(f)$.

The methods used in [8] to show that $J(f) \subset \partial I(f)$ also yield the following result.

Theorem 1.1. *Let $f \colon \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map of transcendental type. Then $J(f) \subset \partial A(f)$.*

For transcendental entire functions we have not only $J(f) = \partial I(f)$, but also $J(f) = \partial A(f)$, see [18, p. 1125, Remark 1]. While, as mentioned, the first equation does not hold for quasiregular maps in general, we conjecture that the second equation remains valid for quasiregular maps of transcendental type. We can show that this is the case for maps which do not grow too slowly.

Theorem 1.2. *Let* $f \colon \mathbb{R}^d \to \mathbb{R}^d$ *be a quasiregular map satisfying*

$$(1.2) \qquad \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log \log r} = \infty.$$

Then $J(f) = \partial A(f)$.

Recall that the *lower order* of a quasiregular self-map of $\mathbb{R}^d$ is given by [17, Section V.8]

$$(d - 1) \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

Thus Theorem 1.2 applies in particular to functions of positive lower order. For example, the result applies to quasiregular analogues of the exponential function and the trigonometric functions whose dynamics where studied in [1, 5, 12]. On the other hand, quasiregular maps of transcendental type which have lower order zero were constructed in [9].

One consequence of Theorem 1.2 is that if $f \colon \mathbb{R}^d \to \mathbb{R}^d$ is a quasiregular map satisfying (1.2) and if $n \in \mathbb{N}$, then $J(f^n) = J(f)$. This follows since $A(f^n) = A(f)$ for all quasiregular maps of transcendental type [4, Proposition 3.1, (ii)].

A quasiregular map $f \colon \mathbb{R}^d \to \mathbb{R}^d$ which is not of transcendental type is said to be of *polynomial type*. It was shown in [11] that if f is such a map satisfying $\deg f > K_I(f)$, then $I(f) \neq \emptyset$ and $\partial I(f)$ is perfect. Moreover, $I(f) = A(f)$ for such maps [13]. The inclusion $J(f) \subset \partial I(f)$ also holds for such maps, and again it may be strict; cf. [3, 16].

2. Quasiregular maps

We introduce quasiregular maps only briefly and refer to [17] for more details. Let $\Omega \subset \mathbb{R}^d$ be a domain and $1 \leq p < \infty$. The *Sobolev space* $W^1_{p,\text{loc}}(\Omega)$ consists of all functions $f = (f_1, \ldots, f_d) \colon \Omega \to \mathbb{R}^d$ for which all first order weak partial derivatives $\partial_k f_j$ exist and are locally in L^p. A continuous map $f \in W^1_{d,\text{loc}}(\Omega)$ is called *quasiregular* if there exists a constant $K_O \geq 1$ such that

$$(2.1) \qquad |Df(x)|^d \leq K_O J_f(x) \qquad \text{a.e.}$$

Here $Df(x)$ is the derivative of f at x,

$$|Df(x)| = \sup_{|h|=1} |Df(x)(h)|$$

its norm, and $J_f(x)$ the Jacobian determinant. Put

$$\ell(Df(x)) = \inf_{|h|=1} |Df(x)(h)|.$$

It turns out that, for $f \in W^1_{d,\mathrm{loc}}(\Omega)$, the condition that (2.1) holds for some $K_O \geq 1$ is equivalent to the existence of a constant $K_I \geq 1$ such that

$$(2.2) \qquad J_f(x) \leq K_I \ell(Df(x))^d \quad \text{a.e.}$$

The smallest constants K_O and K_I such that (2.1) and (2.2) hold are denoted by $K_O(f)$ and $K_I(f)$ and called the *outer and inner dilatation* of f.

For quasiregular maps $f, g \colon \mathbb{R}^d \to \mathbb{R}^d$ we have [17, Theorem II.6.8]

$$K_O(f \circ g) \leq K_O(f) K_O(g) \quad \text{and} \quad K_I(f \circ g) \leq K_I(f) K_I(g)$$

and thus

$$(2.3) \qquad K_O(f^n) \leq K_O(f)^n \quad \text{and} \quad K_I(f^n) \leq K_I(f)^n$$

for all $n \in \mathbb{N}$.

Let $G \subset \mathbb{R}^d$ be open and $C \subset G$ compact. The pair (G, C) is called a *condenser* and the quantity

$$\mathrm{cap}(G, C) = \inf_u \int_G |\nabla u|^d \, dm,$$

with the infimum taken over all non-negative functions $u \in C_0^\infty(G)$ satisfying $u(x) \geq 1$ for $x \in C$, is called the *capacity* of the condenser. If $\mathrm{cap}(G, C) = 0$ for some bounded open set G containing C, then $\mathrm{cap}(G', C) = 0$ for every bounded open set G' containing C; see [17, Lemma III.2.2]. In this case we say that C is of *capacity zero*, while otherwise C has *positive capacity*. A closed subset C of $\mathbb{R}^d$ is said to have capacity zero if this is the case for all compact subsets of C.

An important concept used in the study of quasiregular maps is the modulus of a curve family; see [17, Chapter II] and [24, Chapter 2] for a detailed discussion. For a family Γ of paths in $\mathbb{R}^d$, a non-negative Borel function $\rho \colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is called *admissible* if $\int_\gamma \rho \, ds \geq 1$ for all locally rectifiable paths $\gamma \in \Gamma$. Let $\mathcal{F}(\Gamma)$ be the family of all admissible Borel functions. Then

$$M(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^d} \rho^d \, dm$$

is called the *modulus* of Γ.

For a domain $G \subset \mathbb{R}^d$ and subsets E, F of $\overline{G}$, let $\Delta(E, F; G)$ be the family of all paths which have one endpoint in E and one endpoint in F, and which are in G otherwise. A connection between capacity and the modulus of a path family was first noted by Gehring [14] and

this has since been extended to the general equation [17, Proposition II.10.2]

$$\mathrm{cap}(G, C) = M(\Delta(C, \partial G; G)).$$

We denote by $B(a, r)$ the open ball and by $\overline{B}(a, r)$ the closed ball of radius r around a point $a \in \mathbb{R}^d$ and by $\mathrm{diam}\, X$ the Euclidean diameter of a subset X of $\mathbb{R}^d$. We need the following estimate of Vuorinen ([23, Lemma 2.18], [24, Lemma 5.42]).

Lemma 2.1. *Let $a \in \mathbb{R}^d$, $r > 0$ and E, F continua in $\overline{B}(a, r)$. Then*

$$M(\Delta(E, F; B(a, r))) \geq c_0 \frac{\min\{\mathrm{diam}\, E, \mathrm{diam}\, F\}}{r}$$

for some constant c_0 depending only on d.

3. Averages of counting functions

For a quasiregular map $f \colon \Omega \to \mathbb{R}^d$ and $x \in \Omega$ the *local index* $i(x, f)$ of f at x is defined by

$$i(x, f) = \inf_U \sup_{y \in \mathbb{R}^d} \mathrm{card}\left(f^{-1}(y) \cap U\right),$$

the infimum being taken over all neighborhoods $U \subset \Omega$ of x. For $y \in \mathbb{R}^d$ and $E \subset \Omega$ compact we put

$$n(E, y) = \sum_{x \in f^{-1}(y) \cap E} i(x, f).$$

Thus $n(E, y)$ is the number of y-points of f in E, counted according to multiplicity.

The cardinality of $f^{-1}(y) \cap E$ is denoted by $N(y, f, E)$. Clearly (cf. [17, Proposition I.4.10]) we have

$$(3.1) \qquad N(y, f, E) \leq n(E, y).$$

The average value of $n(E, y)$ over the sphere $\partial B(0, t)$ is denoted by $\nu(E, t)$. With the normalized d-dimensional Hausdorff measure $\mathcal{H}^d$ and $\omega_d = \mathcal{H}^d(S^d(1))$ we thus have

$$(3.2) \qquad \nu(E, t) = \frac{1}{\omega_{d-1} t^{d-1}} \int_{\partial B(0,t)} n(E, y)\, d\mathcal{H}^{d-1}(y).$$

We will consider the case where E is a ball and put $\nu(r, t) = \nu\big(\overline{B}(0, r), t\big)$. The following result is from [17, Lemma IV.1.1].

Lemma 3.1. *Let $r, s, t > 0$, $\theta > 1$ and let $f \colon B(0, \theta r) \to \mathbb{R}^d$ be quasiregular. Then*

$$\nu(r, t) \leq \nu(\theta r, s) + K_I(f) \frac{|\log(t/s)|^{d-1}}{(\log \theta)^{d-1}}.$$

The next result is from [3, Theorem 3.1].

Lemma 3.2. *Let $E \subset \mathbb{R}^d$ be a set of positive capacity and $\theta > 1$. Then there exists a constant c_1 such that if $r > 0$ and $f \colon B(0, \theta r) \to \mathbb{R}^d \backslash E$ is quasiregular, then $\nu(r, 1) \leq c_1 K_I(f)$.*

In [3, Theorem 3.1] this result is stated with $\nu(r, 1)$ replaced by $\nu(r, t)$ where $E \subset B(0, t/2)$, but the above version follows easily from this using Lemma 3.1.

4. Proof of the theorems

Proof of Theorem 1.1. Let $f \colon \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map of transcendental type. As already mentioned, the argument is essentially the same as in [8, Theorem 1.3] where it was proved that $J(f) \subset \partial I(f)$.

Let
$$BO(f) = \left\{ x \in \mathbb{R}^d \colon (f^n(x)) \text{ is bounded} \right\}$$
be the set of points with bounded orbits. It was shown in [8, Theorem 1.4] that $BO(f)$ has positive capacity.

Also, it was shown in [4, Theorem 1.2] that $A(f)$ is non-empty and that every component of $A(f)$ is unbounded. This implies that $A(f)$ contains continua and thus $A(f)$ has positive capacity; see [17, Corollary III.2.5].

Let now $x_0 \in J(f)$ and let U be a neighborhood of x_0. By the definition of $J(f)$, the complement of $\bigcup_{n=1}^{\infty} f^n(U)$ has capacity zero. As both $A(f)$ and $BO(f)$ have positive capacity, each of these sets intersects $\bigcup_{n=1}^{\infty} f^n(U)$. Since both $A(f)$ and $BO(f)$ are completely invariant, we deduce that each of the two sets in fact intersects U. As $BO(f) \subset \mathbb{R}^d \backslash A(f)$ it follows that $\partial A(f)$ intersects U. This holds for every neighborhood U of x_0. Hence $x_0 \in \partial A(f)$. $\square$

Proof of Theorem 1.2. In view of Theorem 1.1 it suffices to prove that $\partial A(f) \subset J(f)$. Suppose that $x_0 \in \partial A(f)$ but $x_0 \notin J(f)$. Since $J(f)$ is closed there exists $r > 0$ such that $B(x_0, 4r) \cap J(f) = \emptyset$. Thus the iterates of f omit a set C of positive capacity in $B(x_0, 4r)$. Since $x_0 \in \partial A(f)$ there exist $x_1 \in B(x_0, r) \backslash A(f)$ and $x_2 \in B(x_0, r) \cap A(f)$. Let $R > 2$ be such that $B(0, R) \cap J(f) \neq \emptyset$. For $k \in \mathbb{N}$ we put $F = f^k$, $R_1 = \max\{|F(x_1)|, R\}$ and $R_2 = |F(x_2)|$. For large k we then have $R_2 > R_1$.

For such k we consider the sets $X_1 = \{x \in B(x_0, 2r) \colon |F(x)| \leq R_1\}$ and $X_2 = \{x \in B(x_0, 2r) \colon |F(x)| \geq R_2\}$. Denote by Y_j the component of X_j that contains x_j, for $j \in \{1, 2\}$. By the maximum principle, Y_2 connects x_2 to $\partial B(x_0, 2r)$. We claim that, analogously, Y_1 connects x_1 to $\partial B(x_0, 2r)$. Suppose that this is not the case. Then Y_1 is a

compact subset of $B(x_0, 2r)$. This implies that $F(Y_1) = \overline{B}(0, R_1)$. Hence $F(B(x_0, 2r)) \supset \overline{B}(0, R_1) \supset B(0, R)$. Since $J(f)$ is completely invariant and $B(0, R) \cap J(f) \neq \emptyset$, we deduce that $B(x_0, 2r) \cap J(f) \neq \emptyset$, contradicting the choice of r.

Let $\Gamma = \Delta(X_1, X_2; B(x_0, 2r))$. Using Lemma 2.1 we find that

$$M(\Gamma) \geq M(\Delta(Y_1, Y_2; B(x_0, 2r)))$$

(4.1)
$$\geq c_0 \frac{\min\{\operatorname{diam} Y_1, \operatorname{diam} Y_2\}}{2r} \geq \frac{c_0}{2}.$$

On the other hand, it follows from the proof of the K_O-inequality [17, Remark II.2.5] that

$$M(\Gamma) \leq K_O(F) \int_{\mathbb{R}^d} \rho(y)^d N\big(y, F, \overline{B}(x_0, 2r)\big) \, dy$$

for every $\rho \in \mathcal{F}(\Gamma)$. We consider functions ρ of the form $\rho(y) = \sigma(|y|)$ for some continuous function $\sigma \colon [R_1, R_2] \to [0, \infty)$. Then $\rho \in \mathcal{F}(\Gamma)$ if

$$\int_{R_1}^{R_2} \sigma(t) dt \geq 1.$$

Using (3.1) and (3.2) we obtain

$$M(\Gamma) \leq K_O(F) \omega_{d-1} \int_{R_1}^{R_2} \sigma(t)^d \nu\big(\overline{B}(x_0, 2r), t\big) \, t^{d-1} dt.$$

For simplicity and without loss of generality we assume that $x_0 = 0$ so that $\nu\big(\overline{B}(x_0, 2r), t\big) = \nu(2r, t)$. Then the last inequality takes the form

(4.2)
$$M(\Gamma) \leq K_O(F) \omega_{d-1} \int_{R_1}^{R_2} \sigma(t)^d \nu(2r, t) t^{d-1} dt.$$

Lemmas 3.1 and 3.2 yield

$$\nu(2r, t) \leq \nu(3r, 1) + K_I(F) \left(\frac{\log t}{\log \frac{3}{2}}\right)^{d-1} \leq K_I(F) \left(c_1 + \left(\frac{\log t}{\log \frac{3}{2}}\right)^{d-1}\right)$$

and thus

$$\nu(2r, t) \leq c_2 K_I(F) \left(\log t\right)^{d-1}.$$

for some constant c_2 and $t \geq R_1 \geq 2$. Choosing

$$\sigma(t) = \frac{1}{t \log t \log \dfrac{\log R_2}{\log R_1}}.$$

we deduce from (4.2) that

$$M(\Gamma) \leq c_2 \omega_{d-1} K_O(F) K_I(F) \left(\log \frac{\log R_2}{\log R_1}\right)^{1-d}.$$

Combining this with (4.1) yields

$$\left(\log \frac{\log R_2}{\log R_1}\right)^{d-1} \leq \frac{2c_2\omega_{d-1}}{c_0} K_O(F)K_I(F).$$

Recall that $F = f^k$ and hence $K_O(F) \leq K_O(f)^k$ and $K_I(F) \leq K_I(f)^k$ by (2.3). With $K = (K_O(f)K_I(f))^{1/(d-1)}$ we deduce that

$$\log \frac{\log R_2}{\log R_1} \leq c_3 K^k,$$

with $c_3 = (2c_2\omega_{d-1}/c_0)^{1/(d-1)}$. Recall also that $R_1 = \max\{|f^k(x_1)|, 2\}$ and $R_2 = |f^k(x_2)|$. Since $x_2 \in A(f)$ there exists $L \in \mathbb{N}$ such that $R_2 \geq M^{k-L}(R, f)$ for $k \geq L$, with R as in the definition of $A(f)$. On the other hand, $R_1 \leq M^{k-L-1}(R, f)$ for large k. Putting $S_k = M^{k-L-1}(R, f)$ we deduce that

$$\log \frac{\log M(S_k, f)}{\log S_k} \leq c_3 K^k.$$

By hypothesis, we have $\log M(S_k, f) \geq (\log S_k)^2$ for large k. This yields

(4.3) $$\log \log S_k \leq c_3 K^k.$$

On the other hand, using again the hypothesis (1.2),

$$\log \log S_k = \log \log M(S_{k-1}, f) \geq 2K \log \log S_{k-1}$$

for large k and hence

$$\log \log S_k \geq c_4 (2K)^k$$

for some positive constant c_4. This contradicts (4.3). $\square$

Remark 4.1. The proof shows that (1.2) can be replaced by

$$\liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log \log r} > (K_I(f)K_O(f))^{1/(d-1)}.$$

Remark 4.2. It does not seem possible to improve the argument by choosing a different function σ. In fact, choosing $\rho(x) = cg(|x|)$ with $c = 1/\int_{R_1}^{R_2} g(t)dt$ yields the best estimate when

$$\frac{\int_{R_1}^{R_2} g(t)^d (t \log t)^{d-1} \, dt}{\left(\int_{R_1}^{R_2} g(t)dt\right)^d}$$

is minimal. But by Hölder's inequality we have

$$
\begin{aligned}
\int_{R_1}^{R_2} g(t)dt &= \int_{R_1}^{R_2} g(t)(t\log t)^{(d-1)/d}\frac{1}{(t\log t)^{(d-1)/d}}dt \\
&\leq \left(\int_{R_1}^{R_2} g(t)^d(t\log t)^{d-1}dt\right)^{1/d}\left(\int_{R_1}^{R_2}\frac{1}{t\log t}dt\right)^{(d-1)/d} \\
&= \left(\int_{R_1}^{R_2} g(t)^d(t\log t)^{d-1}dt\right)^{1/d}\left(\log\frac{\log R_2}{\log R_1}\right)^{(d-1)/d}.
\end{aligned}
$$

This shows that $g(t) = 1/(t\log t)$, which corresponds to our choice of σ, minimizes the expression in question.

References

[1] W. Bergweiler, Karpińska's paradox in dimension 3, *Duke Math. J.*, **154** (2010), no. 3, 599–630.

[2] W. Bergweiler, Iteration of quasiregular mappings, *Comput. Methods Funct. Theory*, **10** (2010), no. 2, 455–481.

[3] W. Bergweiler, Fatou-Julia theory for non-uniformly quasiregular maps. *Ergodic Theory Dynam. Systems*, **33** (2013), no. 1, 1–23.

[4] W. Bergweiler, D. Drasin, A. Fletcher, The fast escaping set for quasiregular mappings, preprint, arXiv: 1308.2860.

[5] W. Bergweiler, A. Eremenko, Dynamics of a higher dimensional analog of the trigonometric functions, *Ann. Acad. Sci. Fenn. Math.*, **36** (2011), no. 1, 165–175.

[6] W. Bergweiler, A. Fletcher, J. K. Langley, J. Meyer, The escaping set of a quasiregular mapping, *Proc. Amer. Math. Soc.*, **137** (2009), no. 2, 641–651.

[7] W. Bergweiler, A. Hinkkanen, On semiconjugation of entire functions, *Math. Proc. Cambridge Philos. Soc.*, **126** (1999), no. 3, 565–574.

[8] W. Bergweiler, D. A. Nicks, Foundations for an iteration theory of entire quasiregular maps, to appear in *Israel J. Math.*, arXiv: 1210.3972.

[9] D. Drasin, S. Sastry, Periodic quasiregular mappings of finite order, *Rev. Mat. Iberoamericana*, **19** (2003), no. 3, 755–766.

[10] A. E. Eremenko, On the iteration of entire functions, *Dynamical systems and ergodic theory*, Banach Center Publications 23, Polish Scientific Publishers, Warsaw, 1989, 339–345.

[11] A. Fletcher, D. A. Nicks, Quasiregular dynamics on the n-sphere, *Ergodic Theory and Dynamical Systems*, **31** (2011), 23–31.

[12] A. Fletcher, D. A. Nicks, Chaotic dynamics of a quasiregular sine mapping, *J. Difference Equ. Appl.*, **19** (2013), no. 8, 1353–1360.

[13] A. Fletcher, D. A. Nicks, Superattracting fixed points of quasiregular mappings, preprint.

[14] F. W. Gehring, Extremal length definitions for the conformal capacity of rings in space, *Michigan Math. J.*, **9** (1962), 137–150.

[15] T. Iwaniec, G. J. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. Oxford University Press, New York, 2001.

[16] D. A. Nicks, Wandering domains in quasiregular dynamics, *Proc. Amer. Math. Soc.*, **141** (2013), no. 4, 1385–1392.

[17] S. Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 26, Springer-Verlag, Berlin, 1993.

[18] P. Rippon, G. Stallard, On questions of Fatou and Eremenko, *Proc. Amer. Math. Soc.*, **133** (2005), no. 4, 1119–1126.

[19] P. Rippon, G. Stallard, Fast escaping points of entire functions, *Proc. London Math. Soc.*, **105** (2012), no. 4, 787–820.

[20] D. Sun, L. Yang, Quasirational dynamical systems (Chinese). *Chinese Ann. Math. Ser. A*, **20** (1999), no. 6, 673–684.

[21] D. Sun, L. Yang, Quasirational dynamic system. *Chinese Science Bull.*, **45** (2000), no. 14, 1277–1279.

[22] D. Sun, L. Yang, Iteration of quasi-rational mapping. *Progr. Natur. Sci. (English Ed.)*, **11** (2001), no. 1, 16–25.

[23] M. Vuorinen, Some inequalities for the moduli of curve families. *Michigan Math. J.*, **30** (1983), no. 3, 369–380.

[24] M. Vuorinen, *Conformal geometry and quasiregular mappings.* Lecture Notes in Mathematics 1319, Springer-Verlag, Berlin, 1988.

MATHEMATISCHES SEMINAR, CHRISTIAN–ALBRECHTS–UNIVERSITÄT ZU KIEL, LUDEWIG–MEYN–STR. 4, D–24098 KIEL, GERMANY
  E-mail address: `bergweiler@math.uni-kiel.de`

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, IL 60115-2888, USA
  E-mail address: `fletcher@math.niu.edu`

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, NOTTINGHAM NG7 2RD, UK
  E-mail address: `Dan.Nicks@nottingham.ac.uk`