THE FAST ESCAPING SET FOR QUASIREGULAR MAPPINGS

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ABSTRACT. The fast escaping set of a transcendental entire function is the set of all points which tend to infinity under iteration as fast as compatible with the growth of the function. We study the analogous set for quasiregular mappings in higher dimensions and show, among other things, that various equivalent definitions of the fast escaping set for transcendental entire functions in the plane also coincide for quasiregular mappings. We also exhibit a class of quasiregular mappings for which the fast escaping set has the structure of a spider's web.

1. INTRODUCTION

Quasiregular mappings in $\mathbb{R}^m$ for $m \geq 2$ form a natural higher dimensional analogue of holomorphic mappings in the plane when $m \geq 3$, see section 2.1 for their definition and basic properties. It is a natural question to ask to what extent the theory of complex dynamics carries over into higher dimensions; cf. the recent survey [4]. The escaping set

$$I(f) = \{x \in \mathbb{R}^m \colon f^n(x) \to \infty\}$$

plays an important role in the dynamics of entire functions ($m = 2$). In [6] it is shown that the escaping set of a quasiregular mapping $f \colon \mathbb{R}^m \to \mathbb{R}^m$ of transcendental type is non-empty, and contains an unbounded component; more recently [14] studied $I(f)$ for quasiregular mappings of polynomial type, and there are more refined results when f is uniformly quasiregular.

While $I(f)$ was introduced in [13], the fast escaping set $A(f)$ of a transcendental entire function first appeared later in [7], and since has been the subject of much recent study, see for example Rippon and Stallard [24]. For certain classes of entire functions ($m = 2$), in particular for all functions which grow slowly enough [25], $A(f)$ has a topological structure called a spider's web. Among papers which present classes of functions for which $A(f)$ has a spider's web structure are [16, 19, 24]. This notion will also be investigated here.

According to [24], there are three equivalent ways of defining $A(f)$ for entire functions. To extend notions of complex dynamics to higher dimensions, we fix one of these to define the fast escaping set for a quasiregular mapping.

Let E be a bounded set in $\mathbb{R}^m$. Its *topological hull* $T(E)$ is the union of E and its bounded complementary components; informally, $T(E)$ is E with the holes filled in. We note that in [24, 25] and other papers on complex dynamics the notation $\widetilde{E}$ has been used instead. The established notation $T(E)$, which appears for example in [1] or [10], is advantageous when working with "complicated" sets E. The set E is called *topologically convex* if $T(E) = E$.

It was shown in [6, Lemma 5.1 (ii)] that if $f \colon \mathbb{R}^m \to \mathbb{R}^m$ is a quasiregular mapping of transcendental type, then there exists $R_0 > 0$ such that if $R > R_0$, there is a sequence

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$(r_n)_{n=1}^{\infty}$ with $r_n \to \infty$ such that

(1.1) $$T(f^n(B(0,R))) \supset B(0, r_n).$$

Here and in the following $B(x, r)$ is the open ball of radius r about $x \in \mathbb{R}^m$. When $x = 0$ we often write $B(0, r)$ as $B(r)$ or, when the specific r is clear, B.

Definition 1.1. Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type and let $R > R_0$ where R_0 is chosen such that (1.1) holds. Then

(1.2) $$A(f) = \{x \in \mathbb{R}^m \colon \exists L \in \mathbb{N} \ \forall n \in \mathbb{N} \colon f^{n+L}(x) \notin T(f^n(B(0,R)))\}$$

is called the *fast escaping set*.

We will see later (Proposition 3.1) that this definition does not depend on R. Our first theorem extends results of [7, 23] to the quasiregular context.

Theorem 1.2. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type. Then $A(f)$ is non-empty and every component of $A(f)$ is unbounded.*

In [24], equivalent definitions of the fast escaping set for transcendental entire functions $f \colon \mathbb{C} \to \mathbb{C}$ are presented in terms of the maximum modulus

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

These definitions are

(1.3) $$A_1(f) = \{z \in \mathbb{C} \colon \exists L \in \mathbb{N} \ \forall n \in \mathbb{N} \colon |f^{n+L}(z)| > M(R, f^n)\},$$

where $R > \min_{z \in J(f)} |z|$ and $J(f)$ is the Julia set, and

(1.4) $$A_2(f) = \{z \in \mathbb{C} \colon \exists L \in \mathbb{N} \ \forall n \in \mathbb{N} \colon |f^{n+L}(z)| \geq M^n(R, f)\},$$

where $M^n(R, f)$ is the nth iterate of $M(R, f)$ with respect to the first variable (for example, $M^2(R, f) = M(M(R, f), f)$), with R so large that $M^n(R, f) \to \infty$ as $n \to \infty$.

For entire functions in the plane, the analogue of (1.3) was the first definition used, whereas now (1.4) has become the standard definition for $A(f)$. We next show that the generalizations of these two alternate definitions also coincide with our initial definition in the quasiregular case.

Theorem 1.3. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type. Then $A(f) = A_1(f) = A_2(f)$, where $A_1(f)$ and $A_2(f)$ are the natural generalizations of (1.3) and (1.4) to quasiregular mappings, with $R > R_0$ and R_0 chosen such that (1.1) holds.*

The main tool in the proof of this theorem is the following result.

Theorem 1.4. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type and let $0 < \eta < 1$. Then there exists $R_1 > 0$ such that*

$$\overline{B(0, M^n(\eta R, f))} \subset T(f^n(B(0, R)))$$

for all $R > R_1$ and all $n \in \mathbb{N}$.

Next, we show that for a certain class of quasiregular mappings, including those constructed by Drasin and Sastry [12], the fast escaping set $A(f)$ has a particular structure called a spider's web.

Definition 1.5. A set $E \subset \mathbb{R}^m$ is a *spider's web* if E is connected and there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of bounded topologically convex domains satisfying $G_n \subset G_{n+1}$, $\partial G_n \subset E$ for $n \in \mathbb{N}$ and such that $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{R}^m$.

This definition in principle allows $\mathbb{R}^m$ itself to be a spider's web, but since a quasiregular mapping of transcendental type has infinitely many periodic points [26], we cannot have $A(f) = \mathbb{R}^m$.

The quasiregular maps constructed by Drasin and Sastry behave like power maps (see [22, p.13]) in large annuli and hence for these maps f the minimum modulus

$$m(r, f) = \min_{|x|=r} |f(x)|$$

is large for most values of r. In particular, these maps satisfy the hypotheses of the next theorem.

Theorem 1.6. *Let* $f \colon \mathbb{R}^m \to \mathbb{R}^m$ *be a quasiregular mapping of transcendental type. Suppose there exist* $\alpha > 1$ *and* $\delta > 0$ *such that for all large* r *there exists* $s \in [r, \alpha r]$ *such that* $m(s, f) \geq \delta M(r, f)$. *Then* $A(f)$ *is a spider's web.*

Remark 1.7. A different situation occurs for maps with a Picard exceptional value, such as Zorich-type mappings considered in [3], where the hypothesis of Theorem 1.6 fail. Quite generally, mappings which are periodic or bounded on a path to infinity cannot satisfy the hypothesis of Theorem 1.6. For the maps studied in [3] the escaping set – as well as the fast escaping set – forms hairs, in analogy with the exponential family in the plane.

Theorem 1.3 shows that the various formulations of $A(f)$ agree for quasiregular mappings of transcendental type. One way to show this for transcendental entire functions in the plane has been to use Wiman-Valiron theory, see [13, 24]. This method shows, in particular, that near most points where the maximum modulus is achieved, a transcendental entire function behaves like a power mapping and maps a neighbourhood of such a point onto a large annulus, a property which may be iterated.

Question. Is there an analagous annulus covering theorem for neighbourhoods of points where the maximum modulus is achieved for quasiregular mappings of transcendental type?

The weaker covering result given by Proposition 5.1 below is sufficient for our purposes.

Wiman-Valiron theory is based on the power series expansion of an entire function, which has no analogue for quasiregular maps. An alternative approach to Wiman-Valiron theory, more in the spirit of Macintyre's theory of flat regions [15], was developed in [9]. Here, as well as in classical Wiman-Valiron theory, one of the key features is the convexity of $\log M(r, f)$ in $\log r$. We construct a variation of the mapping of Drasin and Sastry [12], and generalize unpublished ideas of Dan Nicks, to show that this need not be the case for quasiregular mappings in $\mathbb{R}^m$.

Theorem 1.8. *Let* $\varepsilon > 0$. *There exists a quasiregular mapping* $F \colon \mathbb{R}^m \to \mathbb{R}^m$ *for which* $\log M(r, F) / \log r$ *is decreasing on a collection of intervals whose union has lower logarithmic density at least* $1 - \varepsilon$.

Recall that the lower logarithmic density of a set $A \subset [1, \infty)$ is

$$\liminf_{r \to \infty} \int_{A \cap [1,r]} \frac{dt}{t}.$$

Remark 1.9. On intervals where $\log M(r, f)/\log r$ is decreasing, $\log M(r, f)$ cannot be convex in $\log r$. Note however that by [2, Lemma 3.4], $\log M(r, f)/\log r \to \infty$ as $r \to \infty$ for quasiregular mappings of transcendental type.

This paper is organized as follows. In section 2, we recall some basic material on quasiregular mappings and prove some topological lemmas needed in the sequel. Section 3 contains some basic results on the fast escaping set and then establishes Theorem 1.2 in section 4. In section 5, we prove Theorems 1.3 and 1.4, while section 6 lists properties of $A(f)$ which extend directly to the quasiregular setting from [24]. These results are then used to prove Theorem 1.6. In section 7, we recall the mappings of Drasin and Sastry, and prove that they satisfy the hypotheses of Theorem 1.6, hence giving example of mappings for which that the fast escaping set is a spider's web. Finally, section 8 presents Theorem 1.8.

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2. Preliminaries

2.1. **Quasiregular maps.** A continuous mapping $f \colon U \to \mathbb{R}^m$ defined on a domain $U \subset \mathbb{R}^m$ is *quasiregular* if f belongs to the Sobolev space $W^1_{m,loc}(U)$ and there exists $K \in [1, \infty)$ with

$$(2.1) \qquad\qquad |f'(x)|^m \leq K J_f(x)$$

almost everywhere in G. Here $|f'(x)| = \sup_{|h|=1} |f'(x)|$ is the norm of the derivative $f'(x)$ and $J_f(x) = \det f'(x)$ the Jacobian determinant of f at $x \in G$. The smallest constant $K \geq 1$ for which (2.1) holds is the *outer dilatation* $K_O(f)$. When f is quasiregular we also have

$$(2.2) \qquad\qquad J_f(x) \leq K' \inf_{|h|=1} |f'(x)h|^m$$

almost everywhere in U, for some $K' \in [1, \infty)$. The smallest constant $K' \geq 1$ for which (2.2) holds is the *inner dilatation* $K_I(f)$. The *dilatation* $K(f)$ of f is the maximum of $K_O(f)$ and $K_I(f)$, and we say that f is K-quasiregular if $K(f) \leq K$. Informally, a quasiregular mapping sends infinitesimal spheres to infinitesimal ellipsoids with bounded eccentricity. Quasiregular mappings generalize to higher dimensions the mapping properties of analytic and meromorphic functions in the plane; see Rickman's monograph [22] for many more details. In particular, quasiregular mappings are open and discrete.

Quasiregular mappings share some appropriately-modified value distribution properties with holomorphic functions in the plane. Rickman [20] proved the existence of a constant $q = q(m, K)$ such that if a K-quasiregular mapping $f \colon \mathbb{R}^m \to \mathbb{R}^m$ omits at least q values in $\mathbb{R}^m$, then f is constant. This number q is called Rickman's constant, and this result becomes an extension of Picard's Theorem in the plane; for fixed $m \geq 3$, [11] shows that $q(m, K) \to \infty$ as $K \to \infty$, the case $m = 3$ being due to Rickman [21]. Miniowitz obtained an analogue of Montel's Theorem for quasiregular mappings with poles, i.e. quasiregular mappings $f \colon U \to \overline{\mathbb{R}^m}$ where $\overline{\mathbb{R}^m} = \mathbb{R}^m \cup \{\infty\}$.

Theorem 2.1 ([17], Theorem 5). *Let $\mathcal{F}$ be a family of K-quasimeromorphic mappings in a domain $U \subset \mathbb{R}^m$, $m \geq 2$. Let $q = q(m, K)$ be Rickman's constant. Suppose there exists a positive number ε such that*

(i) *each $f \in \mathcal{F}$ omits $q + 1$ points $a_1(f), \dots, a_{q+1}(f)$ in $\overline{\mathbb{R}^m}$,*
(ii) *$\chi(a_i(f), a_j(f)) \geq \varepsilon$, where χ is the spherical metric on $\overline{\mathbb{R}^m}$.*

Then $\mathcal{F}$ is a normal family.

A quasiregular mapping $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is of *polynomial type* if $|f(x)| \to \infty$ as $|x| \to \infty$, whereas it is said to be of *transcendental type* if this limit does not exist, so that f has an essential singularity at infinity. This is in direct analogy with the dichotomy between polynomials and transcendental entire functions when $m = 2$. The following lemma was proved in [2, Lemma 3.3].

Lemma 2.2. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be quasiregular mapping of transcendental type and $A > 1$. Then*
$$\lim_{r\to\infty} \frac{M(Ar, f)}{M(r, f)} = \infty.$$

The composition of two quasiregular mappings is always quasiregular, but the dilatation typically increases. See [4] for an introduction to the iteration theory of quasiregular mappings, as well as [5, 8], where a Fatou-Julia iteration theory for quasiregular mappings is developed.

2.2. **Some topological lemmas.** For the following lemma we refer to [18, p. 84] and [6, Lemmas 3.1 and 3.2].

Lemma 2.3. *Let E be a continuum in $\overline{\mathbb{R}^m}$ containing ∞. Then:*

  (i) if F is a component of $\mathbb{R}^m \cap E$, then F is unbounded;
  (ii) if $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is a continuous open mapping, then the pre-image
$$f^{-1}(E) = \{x \in \mathbb{R}^m \colon f(x) \in E\}$$

  cannot have a bounded component.

Proposition 2.4. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a continuous open mapping and $U \subset \mathbb{R}^m$ a bounded open set. Then $f(T(U)) \subset T(f(U))$ and $\partial T(f(U)) \subset f(\partial T(U))$.*

Proof. We apply Lemma 2.3 with $E = \overline{\mathbb{R}^m} \backslash T(f(U))$ and $F = \mathbb{R}^m \backslash T(f(U))$. Thus $f^{-1}(E) = f^{-1}(F)$ has no bounded component (in fact, $f^{-1}(F)$ consists of a single unbounded component, but we do not need this fact). Since $f(U) \subset T(f(U))$ we have $F \subset \mathbb{R}^m \backslash f(U)$ and hence
$$f^{-1}(F) \subset f^{-1}(\mathbb{R}^m \backslash f(U)) \subset \mathbb{R}^m \backslash U.$$
Thus every component of $f^{-1}(F)$ is contained in a component of $\mathbb{R}^m \backslash U$. Since $f^{-1}(F)$ has no bounded components, we deduce that $f^{-1}(F)$ is contained in the unique unbounded component of $\mathbb{R}^m \backslash U$. This means that $f^{-1}(F) \subset \mathbb{R}^m \backslash T(U)$. Now
$$f^{-1}(F) = f^{-1}(\mathbb{R}^m \backslash T(f(U))) = \mathbb{R}^m \backslash f^{-1}(T(f(U))).$$
This yields that $T(U) \subset f^{-1}(T(f(U)))$ and hence $f(T(U)) \subset T(f(U))$.

For the second part of the proposition, observe that $\partial f(U) \subset f(\partial U)$ since f is a continuous open mapping. Indeed, let w be in the boundary of $f(U)$. Then w is not in $f(U)$ since $f(U)$ is open, but w is the limit of points $w_k = f(u_k)$ with u_k in U. Without loss of generality u_k tends to a point u. Then $w = f(u)$ and since w is not in $f(U)$, u is not in U. Thus u is in the boundary of U. Hence $\partial f(U) \subset f(\partial U)$ as claimed.

From this it follows that
$$\partial T(f(U)) \subset \partial f(U) \subset f(\partial U).$$

If A is a bounded component of $\mathbb{R}^m \backslash U$, then $A \subset T(U)$ and thus $f(A) \subset f(T(U)) \subset T(f(U))$ by the first part of the proposition. Thus all components of the boundary of U other than $\partial T(U)$ are mapped into $T(f(U))$, that is,

$$f(\partial U \backslash \partial T(U)) \subset T(f(U)).$$

Together with $\partial T(f(U)) \subset f(\partial U)$ this yields $\partial T(f(U)) \subset f(\partial T(U))$. $\qquad\square$

Since non-constant quasiregular mappings are open and discrete, these results apply in particular to quasiregular mappings.

3. BASIC PROPERTIES OF $A(f)$

The following proofs mimic those in [23, 24] for entire functions.

Proposition 3.1. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type. Then:*

(i) $A(f)$ is independent of R as long as $R > R_0$, where R_0 satisfies (1.1);
(ii) for any $p \in \mathbb{N}$, $A(f^p) = A(f)$;
(iii) $A(f)$ is completely invariant under f, i.e. if $x \in A(f)$ then $f(x) \in A(f)$ and vice versa.

Proof. For property (i), suppose $R' > R > R_0$. Of course $T(f^n(B(0,R))) \subset T(f^n(B(0,R')))$ for each $n \in \mathbb{N}$, and so we just need that $T(f^n(B(0,R)))$ covers $B(0,R')$ for some $n \in \mathbb{N}$. However, this follows immediately from (1.1) by choosing n large enough. Since $A(f)$ does not depend on R, we may and do assume that (1.2) is satisfied.

For property (ii), if $x \in A(f)$, there exists $L \in \mathbb{N}$ such that $f^{n+L}(x) \notin T(f^n(B))$ for all $n \in \mathbb{N}$, where $B = B(0,R)$. According to (1.1), $T(f^k(B)) \supset B$ for all $k \in \mathbb{N}$, so Proposition 2.4 yields $T(f^{n+k}(B)) \supset T(f^n(B))$ for $n \in \mathbb{N}$. Hence $f^{pn+L+k}(x) \notin T(f^{pn}(B))$ for $n \in \mathbb{N}$. Choosing k such that p divides $L + k$, we conclude $x \in A(f^p)$. Conversely, if $x \in A(f^p)$, then there exists $\ell \in \mathbb{N}$ with $(f^p)^{n+\ell}(x) \notin T((f^p)^n(B))$ for $n \in \mathbb{N}$. Using Proposition 2.4, $f^{pn+p\ell-k}(x) \notin T(f^{pn-k}(B))$ for $n \in \mathbb{N}$ and $k = 1, 2, \ldots, p-1$. Hence $x \in A(f)$.

Finally, for property (iii), if $x \in A(f)$ and $y \in f^{-1}(x)$, then y satisfies (1.2) with L replaced by $L + 1$. Also, if $z = f(x)$ then $f^{n+L}(z) = f^{n+L+1}(x) \notin T(f^{n+1}(B))$. Then by arguments similar to those for property (ii), Proposition 2.4 implies that $f^{n+L}(x) \notin T(f^n(B))$. $\qquad\square$

4. PROOF OF THEOREM 1.2

This was essentially proved, but not stated, in [6]. By Proposition 3.1 (i), we may choose R such that (1.1) holds. We first show that $A(f) \neq \emptyset$. For $n \in \mathbb{N}$ let $\gamma_n = \partial T(f^n(B))$, where $B = B(0,R)$. Then $\gamma_{n+1} \subset f(\gamma_n)$ by the second part of Proposition 2.4. By [6, Lemma 5.2] there is a point $x_0 \in \mathbb{R}^m$ such that

$$f^n(x_0) \in \gamma_n,$$

for each $n \in \mathbb{N}$. In particular, $x_0 \in I(f)$. However, since each $T(f^n(B))$ is open, we have

$$x_0 \notin T(f^n(B))$$

for $n \in \mathbb{N}$, and so $x_0 \in A(f)$; thus $A(f) \neq \emptyset$.

To prove that the components of $A(f)$ are unbounded, write $B_n = f^n(B)$ and $E_n = \mathbb{R}^m \backslash T(B_n)$. Suppose that $x_0 \in \mathbb{R}^m$ satisfies

(4.1) $$f^n(x_0) \in E_n$$

6

for all $n \in \mathbb{N}$. Let L_n be the component of $f^{-n}(E_n)$ which contains x_0. Then L_n is obviously closed, and is unbounded by Lemma 2.3 (ii). Further, for $n \in \mathbb{N}$,

$$L_{n+1} \subset L_n.$$

To see this, note from Proposition 2.4 that $f^{n+1}(x) \in E_{n+1}$ implies that $f^n(x) \in E_n$ so $L_{n+1} \subset f^{-n}(E_n)$. Therefore, by [18, Theorem 5.3, p. 81],

$$K := \bigcap_{n \in \mathbb{N}} (L_n \cup \{\infty\})$$

is a closed connected subset of $\overline{\mathbb{R}^m}$ which contains x_0 and ∞. Now let K_0 be the component of $K \backslash \{\infty\}$ containing x_0. Then K_0 is closed in $\mathbb{R}^m$, and unbounded by Lemma 2.3 (i). We claim that $K_0 \subset A(f)$. To see this, observe that if $x \in K_0$, then $f^n(x) \in E_n$ for $n \in \mathbb{N}$ and so

$$f^n(x) \notin T(B_n) = T(f^n(B(0,R))),$$

for $n \in \mathbb{N}$. Hence $x \in A(f)$. We have shown that if x_0 satisfies (4.1), then x_0 is contained in an unbounded component K_0 of $A(f)$.

Next, suppose that $x \in A(f)$. Then by (1.2), there exists $L \in \mathbb{N}$ such that

$$f^{n+L}(x) \in E_n,$$

for $n \in \mathbb{N}$ and so $y = f^L(x)$ satisfies

$$f^n(y) \in E_n,$$

for $n \in \mathbb{N}$. By the argument above, y lies in an unbounded closed connected subset K' of $A(f)$. Lemma 2.3 (ii) implies that if K'' is the component of $f^{-L}(K')$ containing x, then K'' is closed and unbounded. Since $A(f)$ is completely invariant by Proposition 3.1 (iii), it follows that $K'' \subset A(f)$. This completes the proof of Theorem 1.2.

5. PROOF OF THEOREMS 1.3 AND 1.4

We begin with the following proposition where

$$A(R_1, R_2) = \{y \in \mathbb{R}^m \colon R_1 < |y| < R_2\}$$

is the annulus centred at 0 with radii R_1, R_2.

Proposition 5.1. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type and let $\alpha, \beta > 1$. Then, for all large enough r, there exists $R > M(r, f)$ such that*

$$f(A(r, \alpha r)) \supset A(R, \beta R).$$

Proof. Choose r large and a point $a_r \in \mathbb{R}^m$ with $|a_r| = (1 + \alpha)r/2$ and

$$M\left((1 + \alpha)r/2, f\right) = |f(a_r)|.$$

Define $g_r \colon B(0, 1) \to \mathbb{R}^m$ by

$$g_r(x) = \frac{f(a_r + (\alpha - 1)rx/2)}{|f(a_r)|}.$$

For a fixed $\rho \in (0, 1)$ put $b_r = -2\rho a_r/((1 + \alpha)r)$. Then $|b_r| = \rho$ and

$$|g_r(b_r)| = \frac{|f(a_r - 2\rho a_r/(1 + \alpha))|}{|f(a_r)|} \le \frac{M((1 - 2\rho/(1 + \alpha))(1 + \alpha)r/2, f)}{M((1 + \alpha)r/2, f)}.$$

7

Combining this with Lemma 2.2, we see that $|g_r(b_r)| \to 0$ as $r \to \infty$ and so

$$\min_{|x|=\rho} |g_r(x)| \to 0$$

as $r \to \infty$ for all $\rho \in (0,1)$. As $|g_r(0)| \equiv 1$, this implies that the family of K-quasiregular mappings $\{g_r : r > 0\}$ is not normal. In fact, for any sequence (r_k) tending to ∞, the family $\{g_{r_k} : k \in \mathbb{N}\}$ is not normal.

Let $q = q(m, K)$ be Rickman's constant and $\beta > 1$. It follows from Miniowitz's version of Montel's Theorem, i.e. Theorem 2.1, that if r is large enough then there exists $p = p_r \in \{0, 1, \ldots, q\}$ such that $g_r(B(0,1)) \supset A(\beta^{2p}, \beta^{2p+1})$. This implies that

$$f(A(r, \alpha r)) \supset f(B(a_r, (\alpha - 1)r/2)) \supset A(\beta^{2p}|f(a_r)|, \beta^{2p+1}|f(a_r)|).$$

Since $\beta^{2p}|f(a_r)| \geq |f(a_r)| = M((1+\alpha)r/2, f) > M(r, f)$, the conclusion follows. $\qquad\square$

Proof of Theorem 1.4. Put $\alpha = 1/\eta$, let $\beta \geq \alpha$ and choose r large enough so we may apply Proposition 5.1. Then

$$\overline{B(0, M(r, f))} \subset B(0, \beta M(r, f)) \subset T(f(B(0, \alpha r))).$$

Further, by Proposition 2.4, Proposition 5.1 and the fact that if $U \subset V$ then $T(U) \subset T(V)$, we have

$$\begin{aligned}
\overline{B(0, M^2(r, f))} &= \overline{B(0, M(M(r, f), f))} \\
&\subset T(f(B(0, \alpha M(r, f)))) \\
&\subset T(f(B(0, \beta M(r, f)))) \\
&\subset T(f^2(B(0, \alpha r))).
\end{aligned}$$

Continuing by induction and replacing r with $\eta R = R/\alpha$ yields the conclusion. $\qquad\square$

Proof of Theorem 1.3. As in the proof of Proposition 3.1, we can show that the definitions of $A_1(f)$ and $A_2(f)$ also do not depend on R as long as $R > R_0$. Given $\eta \in (0,1)$, Theorem 1.4 implies that

$$\begin{aligned}
\overline{B(0, M^n(\eta R, f))} &\subset T(f^n(B(0, R))) \\
&\subset \overline{B(0, M(R, f^n))} \\
&\subset \overline{B(0, M^n(R, f))} \\
&\subset B(0, M^{n+1}(R, f))
\end{aligned}$$

for large R, from which the conclusion easily follows. $\qquad\square$

6. Further properties of $A(f)$

Many results on the structure of the fast escaping set for entire functions from Rippon and Stallard's paper [24] hold in this context. In this section, we state these results and refer to [24] for the proofs, where they go through almost word for word.

Definition 6.1. Let $R > R_0$ with R_0 as in (1.1) and let $L \in \mathbb{Z}$. Then

$$A_R(f) = \{x \in \mathbb{R}^m : |f^n(x)| \geq M^n(R, f) \text{ for all } n \in \mathbb{N}\},$$

8

is the *fast escaping set with respect to R*, and its L'th *level* is

$$A_R^L(f) = \{x \in \mathbb{R}^m \colon |f^n(x)| \geq M^{n+L}(R, f) \text{ for all } n \in \mathbb{N} \text{ with } n \geq -L\}.$$

Note that $A_R(f) = A_R^0(f)$ and that $A_R^L(f)$ is a closed set. Since $M^{n+1}(R, f) > M^n(R, f)$, we have

$$A_R^L(f) \subset A_R^{L-1}(f),$$

for $L \in \mathbb{Z}$. Therefore $A(f)$ as defined in (1.2) is an increasing union of closed sets

$$A(f) = \bigcup_{L \in \mathbb{N}} A_R^{-L}(f).$$

We also note

$$A_R^L(f) \subset \{x \in \mathbb{R}^m \colon |x| \geq M^L(R, f)\},$$

for $L \geq 0$ and that

$$f(A_R^L(f)) \subset A_R^{L+1}(f) \subset A_R^L(f),$$

for $L \in \mathbb{Z}$.

Proposition 6.2. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type. Then:*
   (i) if $p \in \mathbb{N}$, $0 < \eta < 1$ and R is sufficiently large, then
$$A_R(f) \subset A_R(f^p) \subset A_{\eta R}(f);$$
   (ii) if $\mathbb{R}^m \backslash A_R(f)$ has a bounded component, then $A_R(f)$ and $A(f)$ are spider's webs;
   (iii) if G is a bounded component of $\mathbb{R}^m \backslash A_R^L(f)$, then $\partial G \subset A_R^L(f)$ and f^n is a proper map of G onto a bounded component of $\mathbb{R}^m \backslash A_R^{n+L}(f)$ for $n \in \mathbb{N}$;
   (iv) if $\mathbb{R}^m \backslash A_R^L(f)$ has a bounded component, then $A_R^L(f)$ is a spider's web and hence every component of $\mathbb{R}^m \backslash A_R^L(f)$ is bounded;
   (v) $A_R(f)$ is a spider's web if and only if for each L, $A_R^L(f)$ is a spider's web;
   (vi) if $R' > R > R_0$, then $A_R(f)$ is a spider's web if and only if $A_{R'}(f)$ is a spider's web.

Proof. Part (i) is [24, Theorem 2.6], and the proof carries over with [24, Lemma 2.4] replaced by Theorem 1.4. Part (ii) is [24, Theorem 1.4], parts (iii)-(vi) are [24, Lemma 7.1 (a)-(d)]. $\square$

Note that (i) also gives $A(f) = A(f^p)$ for $p \in \mathbb{N}$, as was already proved in Proposition 3.1, (ii).

We next define two sequences which in dimension 2 are called the sequences of fundamental holes and loops for $A_R(f)$.

Definition 6.3. *If $A_R(f)$ is a spider's web, then for $n \geq 0$ we denote by H_n the component of $\mathbb{R}^m \backslash A_R^n(f)$ that contains 0 and by L_n the boundary of H_n.*

Proposition 6.4. *Let $f \colon \mathbb{R}^m \to \mathbb{R}^m$ be a K-quasiregular mapping of transcendental type. Then we have the following:*
   (i) for $n \geq 0$, we have $B(0, M^n(R, f)) \subset H_n$, $L_n \subset A_R^n(f)$ and $H_n \subset H_{n+1}$;
   (ii) for $n \in \mathbb{N}$ and $k \geq 0$, we have $f^n(H_k) = H_{k+n}$ and $f^n(L_k) = L_{k+n}$;
   (iii) there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $k \geq 0$, we have $L_k \cap L_{k+n} = \emptyset$;
   (iv) if $L \in \mathbb{Z}$ and G is a component of $\mathbb{R}^m \backslash A_R^L(f)$, then for n sufficiently large, we have $f^n(G) = H_{n+L}$ and $f^n(\partial G) = L_{n+L}$;
   (v) all components of $\mathbb{R}^m \backslash A(f)$ are compact if $A_R(f)$ is a spider's web;
   (vi) $A_R(f^n)$ is a spider's web if and only if $A_R(f)$ is a spider's web.

9

Proof. Parts (i)-(iv) are [24, Lemma 7.2 (a)-(e)], part (v) is [24, Theorem 1.6] and part (vi) is [24, Theorem 8.4]. $\qquad\square$

The following characterization of the spider's web structure for $A_R(f)$ was proved for dimension 2 in [24, Theorem 8.1].

Proposition 6.5. *Let* $f\colon \mathbb{R}^m \to \mathbb{R}^m$ *be a quasiregular mapping of transcendental type,* R_0 *as in* (1.1) *and* $R > R_0$*. Then* $A_R(f)$ *is a spider's web if and only if there exists a sequence* $(G_n)_{n=1}^{\infty}$ *of bounded topologically convex domains such that, for all* $n \in \mathbb{N}$,

$$B(0, M^n(R, f)) \subset G_n,$$

and G_{n+1} *is contained in a bounded component of* $\mathbb{R}^m \backslash f(\partial G_n)$.

This proposition has the following corollary, see [24, Corollary 8.2].

Corollary 6.6. *Let* $f\colon \mathbb{R}^m \to \mathbb{R}^m$ *be a quasiregular mapping of transcendental type,* R_0 *as in* (1.1)*,* $R > R_0$ *and recall the minimum modulus function* $m(r, f)$*. Then* $A_R(f)$ *is a spider's web if there exists a sequence* $(\rho_n)_{n=1}^{\infty}$ *such that*

$$\rho_n > M^n(R, f),$$

and

$$m(\rho_n, f) \geq \rho_{n+1}.$$

Proof of Theorem 1.6. By Proposition 6.2 (vi), we may restrict to large values of R. Note then that $M^n(R, f)$ is also large, for all $n \in \mathbb{N}$. Thus we may assume that for all $n \in \mathbb{N}$ there exists $\rho_n \in [2M^n(R, f), 2\alpha M^n(R, f)]$ such that

$$m(\rho_n, f) \geq \delta M(2M^n(R, f), f).$$

By Lemma 2.2 we have

$$\delta M(2M^n(R, f), f) \geq 2\alpha M^{n+1}(R, f) \geq \rho_{n+1}$$

for all $n \in \mathbb{N}$, provided R is large enough. The conclusion now follows from Corollary 6.6. $\qquad\square$

7. A SPIDER'S WEB EXAMPLE

In this section, we briefly outline the salient points of the class of quasiregular mappings $f\colon \mathbb{R}^m \to \mathbb{R}^m$ constructed by Drasin and Sastry and then show such mappings have a spider's web structure for $A(f)$.

Drasin and Sastry [12] build quasiregular mappings of transcendental type with prescribed (slow) growth. This is achieved by starting with a positive continuous increasing function ν which is almost flat, that is, that

$$r\nu'(r) < \nu(r)/2, \quad r\nu'(r) = o(\nu(r)),$$

as $r \to \infty$. Then

(7.1) $$M(r, f) = \exp \int_1^r \frac{\nu(t)}{t}\, dt$$

for $r \geq 1$. We remark that [12] only states asymptotic equality in (7.1). However, it is not hard to see that equality is achieved for $x = (r, 0, \dots, 0)$ where $r \geq 1$.

For $0 < r < s$, define

$$A_{\infty}(r, s) = \{x \in \mathbb{R}^m \colon r < \|x\|_{\infty} < s\}.$$

10

There exist $n_0 \in \mathbb{N}$ and a sequence of closed sets
$$V_n = \overline{A_\infty}(r_n, s_n)$$
for $n \geq n_0$, so that the mapping f restricted to the sets V_n satisfies:

(i) $\nu(r_n) = n$, $\nu(s_n) \in [n + 1/(m+1), n+1)$ and $s_n/r_n \to \infty$;
(ii) in V_n, f coincides with a quasiregular power-type mapping of degree j_n (see [22, p.13] and [12]), where $(j_n)_{n=1}^\infty$ is an increasing sequence in $\mathbb{N}$;
(iii) if $\partial B(0, r) \subset V_n$, there exists $\delta > 0$ independent of n such that
$$m(r, f) \geq \delta M(r, f)$$
since f behaves like a power mapping on V_n;
(iv) by [12, (2.10)], for $n \geq n_0$ we have
$$\log\left(\frac{r_{n+1}}{s_n}\right) = C(m)\log\left(\frac{n+1}{n}\right),$$
where $C(m)$ is a constant depending only on m. This condition says that the region where f does not behave like a power mapping gets relatively smaller as n increases;
(v) for $n \geq n_0$ there exists a constant $\alpha = \alpha(m) > 1$ depending only on m such that if $\partial B(0, r)$ intersects $A_\infty(s_n, r_{n+1})$ then $\partial B(0, \alpha r) \subset V_{n+1}$.

We remark that the delicate part of the construction of f is to interpolate between the V_n to increase the degree whilst keeping f quasiregular.

Let $E \subset (0, \infty)$ be defined as follows: $r \in E$ if and only if $\partial B(0, R)$ is contained in the region where f behaves like a power mapping. By property (v) above, if we choose n_0 large enough, there exists $\alpha > 1$ such that for $r \geq r_{n_0}$ there is a corresponding $s \in [r, \alpha r]$ with $s \in E$. Then by property (iii) and the fact that $M(r, f)$ is increasing in r,
$$m(s, f) \geq \delta M(s, f) \geq \delta M(r, f).$$

Therefore f satisfies the hypotheses of Theorem 1.6 and hence $A(f)$ is a spider's web.

8. Failure of log-convexity of the maximum modulus

In this section, we prove Theorem 1.8. As remarked in the introduction, this is a generalization of an idea of Dan Nicks, who considered compositions of quasiconformal radial mappings analogous to h below, and certain transcendental entire functions in the plane. We will replace the transcendental entire function with a mapping which is similar to the type considered in the previous section, but now the growth function ν is allowed to be discontinuous. This still yields a quasiregular mapping, but allows greater flexibility in its properties.

Let $\rho \in (0, 1)$. Fix $r_1 > 1$ and define $r_{n+1} = r_n^2$ for $n \in \mathbb{N}$. Define a radial quasiconformal map $h \colon \mathbb{R}^m \to \mathbb{R}^m$ by
$$h(x) = \begin{cases} r_1 x, & |x| \in [0, r_1] \\ |x|^{1/\rho} x r_n^{1-1/\rho}, & |x| \in [r_n, r_n^{1+\rho}] \\ r_n^2 x, & |x| \in [r_n^{1+\rho}, r_n^2]. \end{cases}$$

Next define
$$\nu(r) = \begin{cases} 1, & r \in [0, r_1] \\ n, & r \in [r_{n-1}, r_n], \quad n \geq 2. \end{cases}$$

11

We define a function f in a similar way to those constructed in [12], but now using the intervals $[r_n, r_{n+1}]$ (recall that $s_n \in (r_n, r_{n+1})$ and f behaves like a power mapping on $\overline{A_\infty(r_n, s_n)}$) and subject to the growth condition

$$M(r, f) = \exp \int_1^r \frac{\nu(t)}{t} \, dt.$$

We reiterate that this positive increasing function does not satisfy the conditions for ν considered in [12] since it is not continuous. However, using the same method as [12, Lemma 3.7], one can see that this mapping is indeed quasiregular. One can calculate that if $r \in [r_{n-1}, r_n]$, then

$$\int_1^r \frac{\nu(t)}{t} \, dt = n \log r - \log r_{n-1} - \ldots - \log r_2 - \log r_1$$
$$= n \log r - (2^n - 1) \log r_1.$$

Hence

$$\log M(r, f) = \psi(\log r),$$

where ψ is a positive continuous piecewise linear function with

$$\psi(t) = nt + d_n, \quad t \in [\log r_{n-1}, \log r_n],$$

and

$$d_n = (1 - 2^n) \log r_1.$$

Define the quasiregular mapping $F \colon \mathbb{R}^m \to \mathbb{R}^m$ by $F = f \circ h$. Then if $r \in [r_n^{1+\rho}, r_n^2]$, the construction of h and the fact that $r_n^2 r \in [r_n^{3+\rho}, r_n^4] \subset [r_{n+1}, r_{n+2}]$ yield that

$$\log M(r, F) = \log M(r, f \circ h) = \log M(r_n^2 r, f)$$
$$= \psi(\log r_n^2 r)$$
$$= (n + 2) \log r + (n + 2) \log r_n^2 + d_{n+2}.$$

Hence for $r \in [r_n^{1+\rho}, r_n^2]$ we have

$$\frac{\log M(r, F)}{\log r} = (n + 2) + \frac{\psi(\log r_n^2)}{\log r}.$$

Since ψ is a positive function, $\log M(r, F)/\log r$ decreases on $[r_n^{1+\rho}, r_n^2]$. The union of these intervals has lower logarithmic density at least $(1 - \rho)/(1 + \rho)$, since $r_n = r_1^{2^{n-1}}$. Choosing ρ close enough to zero implies the result.

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