THE SIZE OF JULIA SETS OF QUASIREGULAR MAPS

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Abstract. Sun Daochun and Yang Lo have shown that many results of the Fatou-Julia iteration theory of rational functions extend to quasiregular self-maps of the Riemann sphere for which the degree exceeds the dilatation. We show that in this context, in contrast to the case of rational functions, the Julia set may have Hausdorff dimension zero. On the other hand, we exhibit a gauge function depending on the degree and the dilatation such that the Hausdorff measure with respect to this gauge function is always positive, but may be finite.

1. Introduction

Sun Daochun and Yang Lo [15, 16, 17] have extended many results of the Fatou-Julia iteration theory of rational functions to quasiregular maps $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ for which the degree $\deg(f)$ exceeds the dilatation $K(f)$. Here $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The key idea is to define the Julia set $J(f)$ of such a map f not via non-normality but as the set of all points z such that for all neighborhoods U of z the forward orbit

$$O_f^+(U) = \bigcup_{k\geq 0} f^k(U)$$

misses at most two points of the sphere; that is,

$$J(f) = \left\{z \in \overline{\mathbb{C}}\colon \operatorname{card}\left(\overline{\mathbb{C}} \backslash O_f^+(U)\right) \leq 2 \text{ for all neighborhoods } U \text{ of } z\right\}.$$

Here $\operatorname{card} X$ denotes the cardinality of a set X.

For example, Sun and Yang [16, Theorem 9] proved that if $z \in J(f)$, then the backward orbit

$$O_f^-(z) = \bigcup_{k\geq 0} f^{-k}(z) = \bigcup_{k\geq 0}\{\zeta \in \overline{\mathbb{C}}\colon f^k(\zeta) = z\}$$

is dense in $J(f)$; that is, $J(f) = \overline{O_f^-(z)}$. Also, the exceptional set

$$E(f) = \left\{z \in \overline{\mathbb{C}}\colon O_f^-(z) \text{ is finite}\right\}$$

contains at most two points, and we have $J(f) \cap E(f) = \emptyset$ and $J(f) \subset \overline{O_f^-(z)}$ for all $z \in \overline{\mathbb{C}} \backslash E(f)$. Many other results of complex dynamics have been extended by Sun and Yang to quasiregular self-maps of the Riemann sphere $\overline{\mathbb{C}} = S^2$ satisfying $\deg(f) > K(f)$; see also [3, §5] for an exposition of some of their results.

1991 *Mathematics Subject Classification.* Primary 37F10; Secondary 30C65, 30D05.
Supported by the ESF Networking Programme HCAA.

An extension of the theory to quasiregular maps $f\colon S^n \to S^n$, where $n \geq 2$ and S^n is the n-sphere, was given in [4]. Here the Julia set consists of all points such that $S^n \backslash O_f^+(U)$ has capacity zero for all neighborhoods U and the essential hypothesis is that the degree exceeds the inner dilatation $K_I(f)$. It turns out that for $n = 2$ these two definitions yield the same set; cf. the remark at the end of section 3, as well as [5, Section 6].

A well-known result of Garber [9] says that the Julia set of a rational function has positive Hausdorff dimension. This result was extended by Fletcher and Nicks [8] to uniformly quasiregular maps $f\colon S^n \to S^n$. These are, by definition, maps such that all iterates are K-quasiregular for some common K. Also, if a quasiregular map $f\colon S^n \to S^n$ with $\deg(f) > K_I(f)$ is Lipschitz continuous, then $J(f)$ has positive Hausdorff dimension [4, Theorem 1.7].

The main purpose of this note is to show that Garber's result does not extend to the quasiregular setting without additional hypotheses like uniform quasiregularity or Lipschitz continuity. We will actually estimate the Hausdorff measure of the Julia set with respect to certain gauge functions. We introduce this concept only briefly and refer to Falconer's book [6] for more details. For $\varepsilon > 0$ a continuous, non-decreasing function $h\colon (0, \varepsilon] \to (0, \infty)$ satisfying $\lim_{t\to 0} h(t) = 0$ is called a gauge function (or dimension function). The (Euclidean) diameter of a subset X of $\mathbb{R}^n$ is denoted by $\operatorname{diam} X$. The Hausdorff measure $H_h(A)$ is then defined by

$$H_h(A) = \lim_{\delta\to 0} \inf_{(A_i)} \left\{ \sum_{i=1}^{\infty} h(\operatorname{diam} A_i)\colon \bigcup_{i=1}^{\infty} A_i \supset A, \operatorname{diam}(A_i) < \delta \right\}.$$

It was shown in [4, Theorem 1.8] that if $f\colon S^n \to S^n$ is a quasiregular map satisfying $\deg(f) > K_I(f)$ such that the branch set does not intersect the Julia set, then $J(f)$ has positive capacity; see section 2 for the definition of the branch set. In the proof it was actually shown that

$$H_h(J(f)) > 0 \quad \text{for} \quad h(t) = \left(\log \frac{1}{t}\right)^{\frac{(1-n)\log\deg(f)}{\log K_I(f)}}. \tag{1.1}$$

A result of Wallin [18] implies that then $J(f)$ has positive capacity.

First we show that in dimension 2, which is the case considered by Sun and Yang, the conclusion (1.1) holds without an additional hypothesis on the branch set. Note that in the 2-dimensional case the branch set of a quasiregular map is discrete. As $\overline{\mathbb{C}}$ is compact, the branch set of a quasiregular map $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is actually finite. This simplifies certain aspects considerably; cf. [5, Section 6].

Theorem 1. *Let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiregular map satisfying $\deg(f) > K(f)$. If $\xi \in \overline{\mathbb{C}} \backslash E(f)$, then*

$$H_h\left(\overline{O_f^-(\xi)}\right) > 0 \quad \textit{for} \quad h(t) = \left(\log \frac{1}{t}\right)^{-\frac{\log\deg(f)}{\log K(f)}}. \tag{1.2}$$

In particular, $H_h(J(f)) > 0$. Moreover, $\overline{O_f^-(\xi)}$ and $J(f)$ have positive capacity.

A quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be of polynomial type if

$$\lim_{x\to\infty} |f(x)| = \infty.$$

Identifying S^n with $\mathbb{R}^n \cup \{\infty\}$ by stereographic projection, a quasiregular map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of polynomial type extends to a quasiregular self-map of S^n by putting $f(\infty) = \infty$. In particular, quasiregular maps $f\colon \mathbb{C} \to \mathbb{C}$ of polynomial type extend to quasiregular self-maps of $\overline{\mathbb{C}}$. Fletcher and Nicks [7] have studied the dynamics of quasiregular self-maps of $\mathbb{R}^n$ of polynomial type and shown that if the degree exceeds the inner dilatation, then ∞ is an attracting fixed point and the boundary of its attracting basin has many properties usually associated with Julia sets.

Here we only note that for such maps the Julia set is contained in the set

$$BO(f) = \{x \in \mathbb{R}^n \colon (f^k(x)) \text{ is bounded}\}$$

of points with bounded orbits. (In complex dynamics this set is called the filled Julia set and usually denoted by $K(f)$, but we reserve the notation $K(f)$ for the dilatation.) We show that the estimate in Theorem 1 is sharp.

Theorem 2. *For all $K \in (1, 2)$ there exists a quasiregular map $f\colon \mathbb{C} \to \mathbb{C}$ of polynomial type with $\deg(f) = 2$ and $K(f) = K$ such that*

$$H_h(J(f)) \leq H_h(BO(f)) < \infty \quad \text{for} \quad h(t) = \left(\log\frac{1}{t}\right)^{-\frac{\log 2}{\log K}}. \tag{1.3}$$

In particular, $J(f)$ and $BO(f)$ have Hausdorff dimension 0.

With some more effort one could obtain analogous examples of any given degree. For degrees of the form 2^k with $k \in \mathbb{N}$ we only have to replace f by f^k.

2. Preliminaries for the proof of Theorem 1

We denote the open disk of radius r around a point $a \in \mathbb{C}$ by $D(a, r)$ and the closed disk by $\overline{D}(a, r)$. The same notation will be used for balls in $\mathbb{R}^n$. The disk around $a \in \overline{\mathbb{C}}$ with respect to the chordal metric χ is denoted by $D_\chi(a, r)$ and the diameter of a subset A of $\overline{\mathbb{C}}$ with respect to χ is denoted by $\operatorname{diam}_\chi A$.

An important tool to obtain lower bounds for the Hausdorff measure and the Hausdorff dimension is the mass distribution principle. We will use the following version; see [12, Theorem 7.6.1].

Lemma 1. *Let $A \subset \mathbb{R}^n$ be compact and let h be a gauge function. Suppose that there exist a probability measure μ supported on A and a positive constant C such that $\mu(D(x, r)) \leq C\, h(r)$ for $0 < r \leq \varepsilon$ and all $x \in A$. Then $H_h(A) > 0$.*

For the definition and basic properties of quasiregular maps we refer to Rickman's book [13]. A standard book for the the 2-dimensional case is the book by Lehto and Virtanen [10]. Note that their book, except for the last chapter, deals with quasiconformal maps, i.e., injective quasiregular maps. However, since every quasiregular map can be written as the composition of an analytic map with a quasiconformal one (cf. [10, Chapter VI]), many properties of quasiconformal maps extend to quasiregular ones.

Let Ω be a domain in $\mathbb{R}^n$ and let $f\colon \Omega \to \mathbb{R}^n$ be a (non-constant) quasiregular map. The *local index* $i(x,f)$ at a point $x \in \Omega$ is defined by

$$i(x,f) = \inf_U \sup_{y\in\mathbb{R}^n} \operatorname{card}\left(f^{-1}(y)\cap U\right),$$

where the infimum is taken over all neighborhoods $U \subset \Omega$ of x. Thus $i(x,f)=1$ if and only if f is injective in a neighborhood of x. The *branch set* consists of all $x \in \Omega$ for which $i(x,f) \geq 2$.

As already mentioned, the 2-dimensional case (i.e. the case $n=2$) is somewhat easier to deal with since then the branch set is a discrete subset of Ω. Its elements are called critical points. For a critical point c we call $i(x,f)-1$ the *multiplicity* of c. An important tool is the following result known as the Riemann-Hurwitz Formula; see [2, §5.4], [11, p. 68] or [14, §1.3]. Here $\chi(\Omega)$ denotes the Euler characteristic of a domain Ω.

Lemma 2. *Let Ω_1 and Ω_2 be domains in $\overline{\mathbb{C}}$ and let $f\colon \Omega_1 \to \Omega_2$ be a proper quasiregulars map of degree d. Denote by s the number of critical points of f, counting multiplicity; that is,*

$$s = \sum_{x\in B_f} (i(x,f)-1).$$

Then

$$\chi(\Omega_1) + s = d\chi(\Omega_2). \tag{2.1}$$

Since $\chi(\overline{\mathbb{C}}) = 2$, the equation (2.1) takes the form $s = 2d-2$ if $\Omega_1 = \Omega_2 = \overline{\mathbb{C}}$. Thus, counting multiplicities, the number of critical points of a quasiregular map $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is equal to $2\deg(f)-2$, as in the case of rational functions.

If Ω_j is a domain of connectivity c_j, then $\chi(\Omega_j) = 2-c_j$ and (2.1) takes the form

$$c_1 - 2 = d(c_2-2) + r. \tag{2.2}$$

We shall only need the case that $c_1 = c_2 = 1$. Then (2.2) simplifies to

$$s = d-1. \tag{2.3}$$

A consequence is the following result.

Lemma 3. *Let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a non-constant quasiregular map and $V \subset \overline{\mathbb{C}}$ a simply connected domain. Denote by n the number of components of $f^{-1}(U)$ and by s the number of critical points in $f^{-1}(U)$, counting multiplicities. If all components of $f^{-1}(U)$ are simply connected, then $n = \deg(f) - s$.*

Proof. Denote by $V_1,\dots,V_n$ the components of $f^{-1}(U)$, by s_j the number of critical points in V_j and by d_j the degree of the proper map $f\colon V_j \to U$. Then

$$\sum_{j=1}^n s_j = s \quad \text{and} \quad \sum_{j=1}^n d_j = \deg(f).$$

By (2.3) we have $s_j = d_j - 1$. Hence

$$n = \sum_{j=1}^{n}(d_j - s_j) = \sum_{j=1}^{n} d_j - \sum_{j=1}^{n} s_j = \deg(f) - s. \qquad \square$$

Lemma 4. *Let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be quasiregular. Then there exists $\eta > 0$ such that if $U \subset \overline{\mathbb{C}}$ is a simply connected domain satisfying $\operatorname{diam}_\chi U < \eta$, then all components of $f^{-1}(U)$ are simply connected.*

Proof. As f is continuous, there exists $\delta > 0$ such that $\chi(f(z), f(w)) < 1$ for $z, w \in \overline{\mathbb{C}}$ with $\chi(z, w) < 1$. We may assume that f is non-constant. This implies that there exists $\eta > 0$ such that if $z \in \overline{\mathbb{C}}$, then all components of $f^{-1}(D_\chi(z, \eta))$ have chordal diameter less than δ. We may assume that $\eta < 1$.

Let now $U \subset \overline{\mathbb{C}}$ be a simply connected domain satisfying $\operatorname{diam}_\chi U < \eta$ and let V be a component of $f^{-1}(U)$. Then $\operatorname{diam}_\chi V < \delta$. Since $f\colon V \to U$ is proper, we have $f(\partial V) = \partial U$. Suppose that V is multiply connected. Then V contains a Jordan curve γ such that both complementary components of γ intersect ∂V. Since $\operatorname{diam}_\chi \gamma \le \operatorname{diam}_\chi V < \delta$, one of these two complementary components of γ has chordal diameter less than δ. Denote this component by W; that is, W is a component of $\overline{\mathbb{C}}\backslash\gamma$ with $\operatorname{diam}_\chi W < \delta$. Then $\operatorname{diam}_\chi f(W) < 1$ by the choice of δ. Moreover, $W \cap \partial V \ne \emptyset$ and $\partial W = \gamma \subset V$. This implies that $f(W) \cap \partial U = f(W \cap \partial V) \ne \emptyset$ and $\partial f(W) \subset f(\partial W) \subset f(V) = U$. We deduce that $f(W) \supset \overline{\mathbb{C}}\backslash U$. Since $\operatorname{diam}_\chi U < \eta < 1$, but also $\operatorname{diam}_\chi f(W) < 1$, this is a contradiction. $\square$

The following estimate is far from sharp, but suffices for our purposes.

Lemma 5. *Let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiregular map of degree at least 2 and let $z \in \overline{\mathbb{C}}\backslash E(f)$. Then* $\operatorname{card} f^{-6}(z) \ge 3$.

Proof. Suppose that $a \in \overline{\mathbb{C}}$ satisfies $\operatorname{card} f^{-1}(a) = 1$, say $f^{-1}(a) = \{b\}$. Then $i(x, b) = \deg(f)$. Thus b is a critical point of multiplicity $\deg(f) - 1$, which is the maximal multiplicity a critical point can have. As the number of critical points, counting multiplicities, is equal to $2\deg(f) - 2$, there are at most two critical points of this maximal multiplicity, and thus at most two such values of a.

Suppose now that $\operatorname{card} f^{-6}(z) \le 2$. Then at least five of the six sets $f^{-k}(z)$, where $k \in \{1, \dots, 6\}$, consist only of critical points of maximal multiplicity. Thus some $f^{-k}(z)$ contains a critical point b of maximal multiplicity such that $f(b)$ and $f^2(b)$ are also critical points of maximal multiplicity. As there are at most two such critical points, we see that the union of the sets $f^{-k}(z)$ contains a periodic orbit consisting only of critical points of maximal multiplicity. Hence z is also in this orbit, and $O^-(z)$ is equal to this orbit, contradicting the assumption that $z \in E(f)$. $\square$

3. Proof of Theorem 1

We put $K = K(f)$ and $d = \deg(f)$. We may assume that $K > 1$ since otherwise f is a rational function so that the conclusion follows from the result of Garber already mentioned in the introduction. We note that a K-quasiregular map is Hölder continuous with exponent $\alpha = 1/K$; see, e.g., [10, §II.3.4]. Thus there exists $M > 0$ such that $\chi(f(z), f(w)) \leq M\chi(z,w)^\alpha$ for all $z, w \in \overline{\mathbb{C}}$. Induction shows that

$$\chi(f^k(z), f^k(w)) \leq M^{1+\alpha+\cdots+\alpha^{k-1}} \chi(z,w)^{\alpha^k}$$

for $k \in \mathbb{N}$ and $z, w \in \overline{\mathbb{C}}$. With $L = M^{1/(1-\alpha)}$ we thus have

$$\chi(f^k(z), f^k(w)) \leq L\chi(z,w)^{\alpha^k} \tag{3.1}$$

for $k \in \mathbb{N}$ and $z, w \in \overline{\mathbb{C}}$.

Choose η according to Lemma 4 and let $0 < \varepsilon < \min\{L, \eta/2\}$. For $w \in \overline{\mathbb{C}}\backslash E(f)$ we will inductively define a sequence $(N_m(w))_{m\geq 0}$ of positive integers and, for $j \in \{1, \dots, N_m(w)\}$, we will also define domains $U_{m,j}(w)$ and $V_{m,j}(w)$ and points $a_{m,j}(w)$ satisfying $a_{m,j}(w) \in U_{m,j}(w) \subset V_{m,j}(w)$. First we put $N_0(w) = 1$, $U_{0,1}(w) = V_{0,1}(w) = D_\chi(w, \varepsilon)$ and $a_{0,1}(w) = w$. Assuming that $N_{m-1}(w)$, the domains $U_{m-1,j}(w)$ and $V_{m-1,j}(w)$ and the points $a_{m-1,j}(w)$ have been defined, we define $N_m(w)$ as the number of components of

$$f^{-1}\left(\bigcup_{i=1}^{N_{m-1}(w)} U_{m-1,i}(w)\right)$$

and we denote these components by $V_{m,1}(w), \dots, V_{m,N_m(w)}(w)$. Then we choose

$$a_{m,j}(w) \in V_{m,j}(w) \cap f^{-1}(\{a_{m-1,i}(w) \colon 1 \leq i \leq N_{m-1}(w)\})$$

and we define $U_{m,j}(w)$ as the component of $V_{m,j}(w) \cap D_\chi(a_{m,j}(w), \varepsilon)$ that contains $a_{m,j}(w)$. It follows from Lemma 4 and the choice of η and ε that the $V_{m,j}(w)$ and hence the $U_{m,j}(w)$ are simply connected.

If $z \in \partial U_{m,j}(w)$, then $\chi(f^l(z), f^l(a_{m,j}(w))) = \varepsilon$ for some l satisfying $0 \leq l \leq m$. Hence

$$\chi(z, a_{m,j}) \geq \left(\frac{\chi(f^l(z), f^l(a_{m,j}(w)))}{L}\right)^{1/\alpha^l} = \left(\frac{\varepsilon}{L}\right)^{K^l} \geq \left(\frac{\varepsilon}{L}\right)^{K^m}$$

for $z \in \partial U_{m,j}(w)$ by (3.1). With $r_m = (\varepsilon/L)^{K^m}$ we thus find that

$$D_\chi(a_{m,j}(w), r_m) \subset U_{m,j}(w) \tag{3.2}$$

for $w \in \overline{\mathbb{C}}\backslash E(f)$, $m \geq 0$ and $1 \leq j \leq N_m(w)$.

We now fix a point $\xi \in \overline{\mathbb{C}}\backslash E(f)$ and put $N_m = N_m(\xi)$, $U_{m,j} = U_{m,j}(\xi)$ and $a_{m,j} = a_{m,j}(\xi)$. Let $s_{m,j}$ be the number of critical points in $f^{-1}(U_{m,j})$ and let $n_{m,j}$ be the number of components of $f^{-1}(U_{m,j})$. Then $n_{m,j} = d - s_{m,j}$ by Lemma 3. Thus

$$N_{m+1} = \sum_{j=1}^{N_m} n_{m,j} = \sum_{j=1}^{N_m} (d - s_{m,j}) = dN_m - \sum_{j=1}^{N_m} s_{m,j} \geq dN_m - (2d-2).$$

Writing this inequality in the form $N_{m+1} - 2 \geq d(N_m - 2)$ we see by induction that

$$N_{m+l} - 2 \geq d^l(N_m - 2)$$

for $l \in \mathbb{N}$.

By Lemma 5 we have card $f^{-6}(\xi) \geq 3$. Choosing ε sufficiently small we may thus achieve that $N_6 \geq 3$. Hence

$$N_m \geq d^{m-6}(N_6 - 2) + 2 \geq d^{m-6} \tag{3.3}$$

for $m \geq 6$. In the opposite direction, we clearly have $N_m \leq d^m$.

For $m \in \mathbb{N}$ we put $A_m = \{a_{m,j} \colon 1 \leq j \leq N_m\}$ and define a probability measure μ_m on $\overline{\mathbb{C}}$ by

$$\mu_m = \frac{1}{N_m} \sum_{z \in A_m} \delta_z,$$

where δ_z denotes the Dirac measure. By [19, Theorem 6.5], the sequence (μ_m) has a subsequence which converges with respect to the weak*-topology, say $\mu_{m_j} \to \mu$. By construction, the supports of the measures μ_m, and hence the support of μ, are contained in $\overline{O_f^-(\xi)}$.

In order to apply Lemma 1, we shall estimate $\mu(D_\chi(z,r))$ for $z \in \overline{\mathbb{C}}$ and $0 < r \leq \varepsilon/2L$. (While Lemma 1 is stated in terms of Euclidean balls, we may also use the chordal metric, as this is the restriction of the Euclidean metric in $\mathbb{R}^3$ to $S^2 = \overline{\mathbb{C}}$.) We choose $l \in \mathbb{N}$ such that $r_{l+1}/2 < r \leq r_l/2$. Then

$$K^{l+1} \log \frac{L}{\varepsilon} = \log \frac{1}{r_{l+1}} \geq \log \frac{1}{r} - \log 2. \tag{3.4}$$

Suppose that $\mu_{k+l}(D_\chi(z,r)) \neq 0$ for some $k \in \mathbb{N}$. Then $A_{k+l} \cap D_\chi(z,r) \neq \emptyset$. We choose $a \in A_{k+l} \cap D_\chi(z,r)$ and put $w = f^l(a)$. Then $w \in A_k$ and $a = a_{l,j}(w)$ for some $j \in \{1, \ldots, N_l(w)\}$. Since $r \leq r_l/2$ we deduce from (3.2) that

$$D_\chi(z,r) \subset D_\chi(a,2r) \subset D_\chi(a,r_l) \subset U_{l,j}(w).$$

This implies that a is the only point in $A_{k+l} \cap D_\chi(z,r)$ which is mapped onto w by f^l. Hence

$$\operatorname{card}(A_{k+l} \cap D_\chi(z,r)) \leq \operatorname{card} A_k = N_k.$$

Using (3.3) we deduce that

$$\mu_{k+l}(D_\chi(z,r)) \leq \frac{N_k}{N_{k+l}} \leq \frac{d^k}{d^{k+l-6}} = \frac{d^7}{d^{l+1}} = d^7 \left(K^{l+1}\right)^{-\frac{\log d}{\log K}}.$$

Using (3.4) we see that

$$\mu_{k+l}(D_\chi(z,r)) \leq C \left(\log \frac{1}{r}\right)^{-\frac{\log d}{\log K}}$$

for some constant C. Clearly, the same estimate is also satisfied by the limit measure μ. Now (1.2) follows from Lemma 1. Since $J(f) = \overline{O_f^-(z)}$ for all $z \in J(f)$, we also have $H_h(J(f)) > 0$. Finally the conclusion about the capacity of $\overline{O_f^-(z)}$ and $J(f)$ follows from the result of Wallin [18] already quoted. $\square$

Remark. Let $f \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiregular map with $\deg(f) > K(f)$, let U be an open set such $\overline{\mathbb{C}} \backslash O_f^+(U)$ has capacity zero and let $\xi \in \overline{\mathbb{C}} \backslash E(f)$. By Theorem 1, $\overline{O_f^-(\xi)}$ has positive capacity. Thus $O_f^+(U) \cap \overline{O_f^-(\xi)} \neq \emptyset$. Since $O_f^+(U)$ is open we actually have $O_f^+(U) \cap O_f^-(\xi) \neq \emptyset$, which implies that $\xi \in O_f^+(U)$. We deduce that $\overline{\mathbb{C}}\backslash O_f^+(U) \subset E(f)$ and hence $\operatorname{card} \overline{\mathbb{C}}\backslash O_f^+(U) \leq 2$ whenever $\overline{\mathbb{C}}\backslash O_f^+(U)$ has capacity zero. This shows that the two definitions of $J(f)$ mentioned in the introduction agree for $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\deg(f) > K(f)$.

4. Proof of Theorem 2

Let $0 < \delta < 1/14$ and put $\lambda = 2e/\delta$. First we define f in

$$Z = \mathbb{C}\backslash(D(1,\delta) \cup D(-1,\delta)).$$

In order to do so we put $f(z) = \lambda(z^2-1)$ for $z \in \mathbb{C}\backslash(D(1,2\delta)\cup D(-1,2\delta))$, put $f(z) = \pm 2\lambda(z \mp 1)$ for $z \in \partial D(\pm 1,\delta)$, and define f by interpolation in the annuli $D(\pm 1, 2\delta)\backslash\overline{D}(\pm 1,\delta)$; that is, we put

$$\begin{aligned} f(z) &= \frac{|z\mp 1| - \delta}{\delta}\lambda(z^2-1) \pm \frac{2\delta - |z \mp 1|}{\delta} 2\lambda(z\mp 1) \\ &= \lambda\left((z\mp 1)^2\frac{|z\mp 1|}{\delta} \pm 2(z\mp 1) - (z\mp 1)^2\right) \end{aligned}$$

if $\delta \leq |z\mp 1| \leq 2\delta$. We now compute the dilatation of f in the annuli $D(\pm 1,2\delta)\backslash\overline{D}(\pm 1,\delta)$. For simplicity, we consider only $D(1,2\delta)\backslash\overline{D}(1,\delta)$, the argument for $D(-1,2\delta)\backslash\overline{D}(-1,\delta)$ being analogous. As

$$\left|\frac{\partial}{\partial z}|z-1|\right| = \left|\frac{\partial}{\partial z}\sqrt{(z-1)(\overline{z}-1)}\right| = \left|\frac{\overline{z}-1}{2|z-1|}\right| = \frac{1}{2}$$

and also

$$\left|\frac{\partial}{\partial \overline{z}}|z-1|\right| = \frac{1}{2},$$

we see that if $\delta < |z-1| < 2\delta$, then

$$\left|\frac{\partial f}{\partial \overline{z}}\right| = \frac{\lambda}{2\delta}|z-1|^2 \leq 2\lambda\delta$$

while

$$\begin{aligned} \left|\frac{\partial f}{\partial z}\right| &= \lambda\left|2(z-1)\frac{|z-1|}{\delta} + \frac{(z-1)^2}{\delta}\frac{\partial}{\partial z}|z-1| + 2 - 2(z-1)\right| \\ &\geq \lambda\left(2 - 2|z-1| - 2\frac{|z-1|^2}{\delta} - \frac{|z-1|^2}{2\delta}\right) \geq \lambda(2-14\delta). \end{aligned}$$

Thus

$$\left|\frac{\partial f}{\partial \overline{z}}(z)\Big/\frac{\partial f}{\partial z}(z)\right| \leq \frac{2\lambda\delta}{\lambda(2-14\delta)} = \frac{\delta}{1-7\delta}$$

for $\delta < |z-1| < 2\delta$. As mentioned, the same argument shows that the last inequality also holds for $\delta < |z+1| < 2\delta$. Choosing δ small we can thus achieve that f is K-quasiconformal in $D(\pm 1,2\delta)\backslash\overline{D}(\pm 1,\delta)$ and hence in the interior of Z.

Next we claim that

$$|f(z)| \geq 4 \quad \text{for } z \in Z. \tag{4.1}$$

In fact, if $z \in \mathbb{C}\backslash(D(1,2\delta)\cup D(-1,2\delta))$, then both terms $|z+1|$ and $|z-1|$ are greater than or equal to 2δ while at least one of them is greater than or equal to 1, so that

$$|f(z)| = \lambda|(z+1)(z-1)| \geq 2\lambda\delta = 4e \geq 4.$$

If $z \in D(1,2\delta)\backslash D(1,\delta)$, then

$$\begin{aligned}|f(z)| &\geq \lambda\left(2|z-1| - |z-1|^2 - \frac{|z-1|^3}{\delta}\right)\\ &= \lambda|z-1|\left(2 - |z-1| - \frac{|z-1|^2}{\delta}\right) \geq \lambda\delta(2-6\delta) = 2e(2-6\delta) \geq 4,\end{aligned}$$

and the same estimate holds for $z \in D(-1,2\delta)\backslash D(-1,\delta)$. Thus (4.1) holds.

If $|z| \geq 4$, then

$$|f(z)| = \lambda|z^2-1| \geq \lambda(|z|^2-1) \geq \lambda(|z|-1)(|z|+1) \geq 3\lambda|z| \geq 3|z|.$$

Hence

$$|f^k(z)| \geq 3^k|z| \quad \text{for } |z| \geq 4 \text{ and } k \in \mathbb{N}. \tag{4.2}$$

It follows from (4.1) and (4.2) that $f(Z) \subset Z$ and

$$f^k(z) \to \infty \quad \text{for } z \in Z \tag{4.3}$$

as $k \to \infty$.

So far we have defined f only in Z. We now extend f to the disks $D(\pm1,\delta)$. In order to do so, we put $t_0 = 4e$, $\alpha = \delta/4$ and define a sequence (t_n) by

$$t_n = t_0\alpha^n \exp\left(-\frac{K^n-1}{K-1}\right).$$

Note that $t_1 = t_0\alpha e^{-1} = \delta$ and

$$f(\partial D(\pm1,t_1)) = f(\partial D(\pm1,\delta)) = \partial D(0,4e) = \partial D(0,t_0).$$

More precisely,

$$f(\pm1 + t_1e^{i\varphi}) = \pm t_0e^{i\varphi}. \tag{4.4}$$

We also define sequences $(s_n)_{n\geq0}$ and $(r_n)_{n\geq1}$ by

$$s_n = t_n\exp(-K^n) \quad \text{and} \quad r_n = \frac{s_{n-1}}{s_0}$$

so that $r_1 = 1$. For later use we note that

$$\frac{t_{n+1}}{t_n} = \alpha\exp\left(-\frac{K^{n+1}-K^n}{K-1}\right) = \alpha\exp(-K^n) \tag{4.5}$$

and thus

$$t_{n+1} = \alpha t_n\exp(-K^n) = \alpha s_n \quad \text{and} \quad r_n = \frac{t_n}{\alpha s_0}. \tag{4.6}$$

For $n \in \mathbb{N}$ we put $\Sigma_n = \{-1,1\}^n$. We also put $\Sigma_0 = \{\emptyset\}$ and

$$\Sigma = \bigcup_{n=1}^{\infty}\Sigma_n \quad \text{and} \quad \Sigma' = \bigcup_{n=0}^{\infty}\Sigma_n = \Sigma\cup\{\emptyset\}.$$

The shift $\sigma\colon \Sigma \to \Sigma'$ is defined by

$$\sigma(u_1, \ldots, u_n) = (u_2, \cdots, u_n)$$

for $n \geq 2$. We also define $\tau\colon \Sigma \to \Sigma'$ by

$$\tau(u_1, \ldots, u_n) = u_1\sigma(u_1, \ldots, u_n) = (u_1u_2, u_1u_3, \ldots, u_1u_n)$$

for $n \geq 2$. For $n = 1$ we put $\sigma(1) = \tau(1) = \sigma(-1) = \tau(-1) = \emptyset$. Thus $\sigma(\Sigma_n) = \tau(\Sigma_n) = \Sigma_{n-1}$ for all $n \in \mathbb{N}$.

We define $a\colon \Sigma' \to \mathbb{C}$, writing a_u instead of $a(u)$, by $a_\emptyset = 0$ and

$$a_u = \sum_{j=1}^{n} u_j r_j \quad \text{for } u = (u_1, \ldots, u_n) \in \Sigma_n \text{ where } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, $u = (u_1, \ldots, u_n) \in \Sigma_n$ and $\varepsilon \in \{-1, 1\}$ we write

$$(u, \varepsilon) = (u_1, \ldots, u_n, \varepsilon) \in \Sigma_{n+1}$$

and put

$$X(u) = D(a_u, s_n) \backslash (D(a_{(u,1)}, t_{n+1}) \cup D(a_{(u,-1)}, t_{n+1}))$$

and

$$Y(u) = D(a_u, t_n) \backslash D(a_u, s_n),$$

see Figure 1.

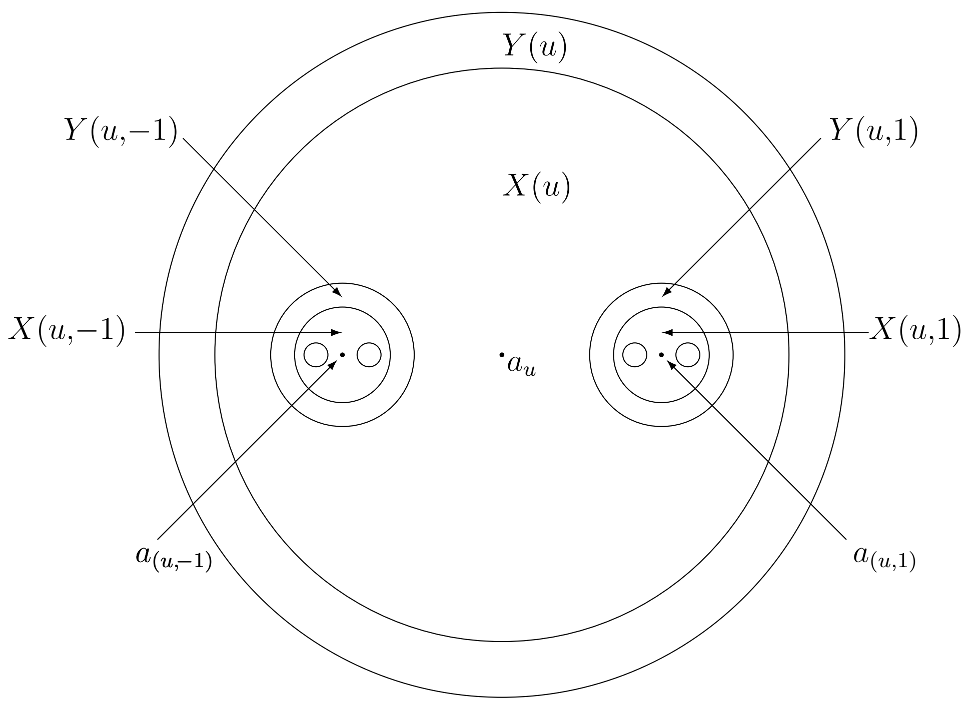

FIGURE 1. $X(u)$ and $Y(u)$, not to scale: the actual annuli $Y(u)$ are much wider and the $X(u, \pm 1)$ and $Y(u, \pm 1)$ are much smaller in comparison to $X(u)$ and $Y(u)$ than shown.

Let

$$X = \bigcup_{u\in\Sigma} X(u) \quad \text{and} \quad Y = \bigcup_{u\in\Sigma} Y(u).$$

We extend f to $X \cup Y$ in such a way that f maps $X(u)$ to $X(\tau(u))$ and $Y(u)$ to $Y(\tau(u))$ for all $u \in \Sigma$. Moreover, f is an affine function on each $X(u)$ and f is a suitably rescaled version of the function $z \mapsto z|z|^{1/K-1}$ on each $Y(u)$, implying that f is conformal on X and quasiconformal on Y with $K(f) = K$. More precisely, we put

$$f(z) = a_{\tau(u)} + u_1 \frac{s_{n-1}}{s_n}(z - a_u) \quad \text{for } z \in X(u) \tag{4.7}$$

and

$$f(z) = a_{\tau(u)} + u_1 \frac{s_{n-1}}{s_n^{1/K}}|z - a_u|^{1/K-1}(z - a_u) \quad \text{for } z \in Y(u). \tag{4.8}$$

Note that the expressions given by (4.7) and (4.8) agree for $z \in \partial X(u) \cap \partial Y(u)$, since for such z we have $|z - a_u| = s_n$ and hence

$$\frac{s_{n-1}}{s_n^{1/K}}|z - a_u|^{1/K-1} = \frac{s_{n-1}}{s_n^{1/K}} s_n^{1/K-1} = \frac{s_{n-1}}{s_n}.$$

The two expressions for f also agree for $z \in \partial X(u) \cap \partial Y(u, \varepsilon)$ where $\varepsilon \in \{1, -1\}$. In this case we have $|z - a_{(u,\varepsilon)}| = t_{n+1}$. Thus, by (4.5) and (4.6),

$$\begin{aligned} \frac{s_n}{s_{n+1}^{1/K}}|z - a_{(u,\varepsilon)}|^{1/K-1} &= \frac{s_n}{s_{n+1}^{1/K}} t_{n+1}^{1/K-1} = \frac{s_n}{t_{n+1}} \left(\frac{t_{n+1}}{s_{n+1}}\right)^{1/K} = \frac{1}{\alpha} \left(\exp(-K^{n+1})\right)^{1/K} \\ &= \frac{1}{\alpha} \exp(-K^n) = \frac{t_n}{t_{n+1}} = \frac{s_{n-1}}{s_n} = \frac{r_n}{r_{n+1}} \end{aligned}$$

so that

$$\begin{aligned} a_{\tau(u,\varepsilon)} + u_1 \frac{s_n}{s_{n+1}^{1/K}}|z - a_{(u,\varepsilon)}|^{1/K-1}(z - a_{(u,\varepsilon)}) &= u_1 a_{\sigma(u,\varepsilon)} + u_1 \frac{s_{n-1}}{s_n}(z - a_u - r_{n+1}\varepsilon) \\ &= u_1 a_{\sigma(u,\varepsilon)} + u_1 \frac{s_{n-1}}{s_n}(z - a_u) - u_1 r_n \varepsilon \\ &= u_1 (a_{\sigma(u,\varepsilon)} - r_n \varepsilon) + u_1 \frac{s_{n-1}}{s_n}(z - a_u) \\ &= a_{\tau(u)} + u_1 \frac{s_{n-1}}{s_n}(z - a_u). \end{aligned}$$

Hence the expressions given by (4.7) and (4.8) agree for $z \in \partial X(u) \cap \partial Y(u, \varepsilon)$.

We deduce that f is continuous and in fact quasiregular with $K(f) = K$ on $X \cup Y$. Moreover, we can deduce from (4.4) and (4.8) that f is continuous on

$$\partial D(1, t_1) \cup \partial D(-1, t_1) = \partial Y \cap \partial Z.$$

Thus f is continuous and quasiregular on $X \cup Y \cup Z$, which is the set where f has been defined so far.

In order to define f on $\mathbb{C} \backslash (X \cup Y \cup Z)$ we denote by Σ_∞ the set of all sequences $(u_k)_{k\in\mathbb{N}}$ with $u_k \in \{-1, 1\}$ for all $k \in \mathbb{N}$. The shift $\sigma \colon \Sigma_\infty \to \Sigma_\infty$ and the map

$\tau\colon \Sigma_\infty \to \Sigma_\infty$ are defined as before; that is,

$$\sigma((u_k)_{k\in\mathbb{N}}) = (u_{k+1})_{k\in\mathbb{N}} \quad \text{and} \quad \tau((u_k)_{k\in\mathbb{N}}) = u_1\sigma((u_k)_{k\in\mathbb{N}}).$$

For $u = (u_k)_{k\in\mathbb{N}} \in \Sigma_\infty$ we put $a_u = \sum_{k=1}^\infty u_k r_k$. Moreover, we put

$$C = \{a_u \colon u \in \Sigma_\infty\}.$$

Noting that $\operatorname{diam} X(u) = 2s_n$ for $u \in \Sigma_n$ and $s_n \to 0$ as $n \to \infty$, we easily see that $C = \mathbb{C}\backslash(X \cup Y \cup Z)$ and that f extends continuously and in fact quasiregularly to $\mathbb{C}$ by putting $f(a_u) = a_{\tau(u)}$ for $u \in \Sigma_\infty$. Moreover, the extended map satisfies $K(f) = K$.

We mention that the existence of a quasiregular extension of f from the domain $X \cup Y \cup Z = \mathbb{C}\backslash C$ to $\mathbb{C}$ also follows from a general removability result for quasiregular maps [1, Corollary 1.5], together with the assertion that $\dim C = 0$ proved below.

For $u \in \Sigma_n$ we have $f(X(u)) = X(\tau(u))$ and $f(Y(u)) = Y(\tau(u))$ so that

$$f^n(X(u)) = X(\emptyset) = D(0,s_0)\backslash(D(1,t_1) \cup D(-1,t_1))$$

and

$$f^n(Y(u)) = Y(\emptyset) = D(0,t_0)\backslash D(0,s_0).$$

Hence $f^k(z) \to \infty$ for $z \in X \cup Y$ by (4.3). On the other hand, $f(C) = C$. We deduce that $BO(f) = C$.

In order to estimate the Hausdorff measure of C we note that

$$C \subset \bigcup_{u\in\Sigma_n} D(a_u, s_n)$$

for all $n \in \mathbb{N}$. For the function h defined by (1.3) we thus have

$$\begin{aligned}\sum_{u\in\Sigma_n} h(\operatorname{diam} D(a_u,s_n)) &= 2^n h(2s_n) = 2^n h\left(\frac{2t_{n+1}}{\alpha}\right) = 2^n\left(\log\frac{\alpha}{2t_{n+1}}\right)^{-\frac{\log 2}{\log K}} \\ &= 2^n\left(-\log 2t_0 - n\log\alpha + \frac{K^{n+1}-1}{K-1}\right)^{-\frac{\log 2}{\log K}} \\ &= \left(K^{-n}\left(-\log 2t_0 - n\log\alpha\right) + \frac{K-K^{-n}}{K-1}\right)^{-\frac{\log 2}{\log K}}.\end{aligned}$$

We deduce that

$$H_h(C) \le \left(\frac{K}{K-1}\right)^{-\frac{\log 2}{\log K}} < \infty,$$

from which the conclusion follows since $C = BO(f)$. □

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