SOME EXAMPLES OF BAKER DOMAINS

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Abstract. We construct entire functions with hyperbolic and simply parabolic Baker domains on which the functions are not univalent. The Riemann maps from the unit disk to these Baker domains extend continuously to certain arcs on the unit circle. The results answer questions posed by Fagella and Henriksen, Baker and Domínguez, and others.

1. Introduction and main results

The *Fatou set* $\mathcal{F}(f)$ of an entire function f is the subset of the complex plane $\mathbb{C}$ where the iterates f^n of f form a normal family. Its complement $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$ is the *Julia set*. The connected components of $\mathcal{F}(f)$ are called *Fatou components*. As in the case of rational functions there exists a classification of periodic Fatou components (cf. [7, Section 4.2]), the new feature for transcendental functions being Baker domains. By definition, a periodic Fatou component U is called a *Baker domain* if $f^n|_U \to \infty$ as $n \to \infty$. The first example of a Baker domain was already given by Fatou [18, Exemple I] who proved that $f(z) = z + 1 + e^{-z}$ has a Baker domain U containing the right halfplane. Since then many further examples have been given; see [31] for a survey.

By a result of Baker [1], the domains today named after him are simply connected. A result of Cowen [14] then leads to a classification of Baker domains, which has turned out to be very useful. We introduce this classification following Fagella and Henriksen [17], but note that there are a number of equivalent ways to state it; see section 2 for a detailed discussion. For simplicity, and without loss of generality, we consider only the case of an invariant Baker domain; that is, we assume that $f(U) \subset U$. We define an equivalence relation on U by saying that $u, v \in U$ are equivalent if there exist $m, n \in \mathbb{N}$ such that $f^m(u) = f^n(v)$ and we denote by U/f the set of equivalence classes. The result of Fagella and Henriksen is the following [17, Proposition 1].

Theorem A. *Let U be an invariant Baker domain of the entire function f. Then U/f is a Riemann surface conformally equivalent to exactly one of the following cylinders:*

(a) $\{z \in \mathbb{C} : -s < \operatorname{Im} z < s\}/\mathbb{Z}$, *for some* $s > 0$;
(b) $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}/\mathbb{Z}$;
(c) $\mathbb{C}/\mathbb{Z}$.

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We call U *hyperbolic* in case (a), *simply parabolic* in case (b) and *doubly parabolic* in case (c). More generally, the above classification holds when U is a simply connected domain, $U \neq \mathbb{C}$, and $f : U \to U$ is holomorphic without fixed point in U.

While we defer a detailed discussion of this classification to section 2, we indicate where the names for the different types of Baker domains come from. Since U is simply connected, there exists a conformal map $\phi : \mathbb{D} \to U$, where $\mathbb{D}$ is the unit disk. Then $g = \phi^{-1} \circ f \circ \phi$ maps $\mathbb{D}$ to $\mathbb{D}$. (In fact, it can be shown that g is an inner function.) By the Denjoy-Wolff-Theorem, there exists $\xi \in \overline{\mathbb{D}}$ such that $g^n \to \xi$ as $n \to \infty$. As f has no fixed point in U and thus g has no fixed point in $\mathbb{D}$, we actually find that $\xi \in \partial\mathbb{D}$. Now suppose that g extends analytically to a neighborhood of ξ. Then $g(\xi) = \xi$ and U is hyperbolic if ξ is an attracting fixed point, U is simply parabolic if ξ is a parabolic point with one petal and U is doubly parabolic if ξ is a parabolic point with two petals.

We note that the Baker domain U of Fatou's example $f(z) = z + 1 + e^{-z}$ mentioned above is doubly parabolic. For example, this follows directly from Lemma 2.2 below. It is known (see the remark after Lemma 2.3 below) that if U is a doubly parabolic Baker domain, then $f|_U$ is not univalent. Equivalently, U contains a singularity of the inverse function of $f : U \to U$. (For Fatou's example it can be checked directly that $f : U \to U$ is not univalent, as U contains the critical points $2\pi ik$, with $k \in \mathbb{Z}$.) On the other hand, $f|_U$ may be univalent in a hyperbolic or simply parabolic Baker domain. The first examples of Baker domains where the function is univalent where given by Herman [21, p. 609] and Eremenko and Lyubich [16, Example 3]. We mention some further examples of Baker domains:

(1) simply parabolic Baker domains in which the function is univalent ([4, Theorem 3], [5, Section 5.3] and [17, Section 4, Example 2]);
(2) hyperbolic Baker domains in which the function is univalent ([5, Sections 5.1 and 5.2], [8, Theorem 1] and [17, Section 4, Example 1];
(3) hyperbolic Baker domains in which the function is not univalent ([30, Theorem 3], and [32, Theorems 2 and 3];

We note that there are no examples of simply parabolic Baker domains in which the function is not univalent. Therefore it was asked in [17, Section 4] and [38, p. 203] whether such domains actually exist. Our first result says that this is in fact the case.

Theorem 1.1. *There exists an entire function f with a simply parabolic Baker domain in which f is not univalent.*

It turns out that the function constructed also provides an answer to a question about the boundary of Baker domains which arises from the work of Baker and Weinreich [4], Baker and Domínguez [2], Bargmann [6] and Kisaka [22, 23].

For a conformal map $\phi : \mathbb{D} \to U$ let Ξ be the set of $\xi \in \partial\mathbb{D}$ such that ∞ is contained in the cluster set of ϕ at ξ and let Θ be the set of $\xi \in \partial\mathbb{D}$ such that $\lim_{r\to 1} \phi(r\xi) = \infty$. Clearly, $\overline{\Theta} \subset \Xi$. The sets Θ and Ξ depend on the choice of the conformal map ϕ, but we will only be concerned with the question whether Ξ is equal to $\partial\mathbb{D}$ or Θ is dense in $\partial\mathbb{D}$, and these statements are independent of ϕ.

Devaney and Goldberg [15] showed that if $\lambda \in \mathbb{C}\setminus\{0\}$ is such that $f(z) = \lambda e^z$ has an attracting fixed point and U denotes its attracting basin, then Θ is dense in $\partial\mathbb{D}$. Baker and Weinreich [4, Theorem 1] proved that if f is an arbitrary transcendental entire function and U is an unbounded invariant Fatou component of f which is not a Baker domain and which thus – by the classification of periodic Fatou components – is an attracting or parabolic basin or a Siegel disk, then $\Xi = \partial\mathbb{D}$. Baker and Domínguez [2, Theorem 1.1] showed, under the same hypothesis, that if $\Theta \neq \emptyset$, then $\overline{\Theta} = \partial\mathbb{D}$. Under additional hypotheses this had been proved before by Kisaka [22, 23].

As shown by Baker and Weinreich [4, Theorem 3], the above results need not hold for Baker domains: they gave an example of a Baker domain bounded by a Jordan curve on the sphere. Clearly, for this example the sets Ξ and Θ consist of only one point. On the other hand, Baker and Weinreich [4, Theorem 4] showed that if a Baker domain U is bounded by a Jordan curve on the sphere, then f is univalent in U.

If U is a Baker domain, then ∞ is accessible in U and thus we always have $\Theta \neq \emptyset$ for Baker domains. Baker and Domínguez [2, Theorem 1.2] showed that if U is a Baker domain where f is not univalent, then $\overline{\Theta}$ contains a perfect subset of $\partial\mathbb{D}$. Again, this had been proved before by Kisaka [22, 23] under additional hypotheses.

Baker and Domínguez [2, p. 440] asked whether even $\overline{\Theta} = \partial\mathbb{D}$ if U is a Baker domain where f is not univalent. It was shown by Bargmann [6, Theorem 3.1] that this is in fact the case for doubly parabolic Baker domains. With $\overline{\Theta}$ replaced by Ξ this result appears in [22]. The question whether these results also hold for hyperbolic and simply parabolic Baker domains was left open in these papers.

It turns out that the Baker domain constructed in Theorem 1.1 can be chosen such that

$$\Xi \neq \partial\mathbb{D}. \tag{1.1}$$

In particular, this implies that $\overline{\Theta} \neq \partial\mathbb{D}$. A modification of the method also yields an example of a hyperbolic Baker domain with this property. We thus have the following result.

Theorem 1.2. *There exists an entire function f_1 with a simply parabolic Baker domain U_1 such that $f_1|_{U_1}$ is not univalent and the set Ξ defined above satisfies* (1.1).

There also exists an entire function f_2 with a hyperbolic Baker domain U_2 satisfying (1.1) *such that $f_2|_{U_2}$ is not univalent.*

2. Classification of Baker domains

We describe the classification given by Cowen [14] following König [25] and Bargmann [6]; see also [31, Section 5] for a discussion of this classification. Let U be a domain and let $f : U \to U$ be holomorphic. We say that a subdomain V of U is *absorbing* for f, if V is simply connected, $f(V) \subset V$ and for any compact subset K of U there exists $n = n(K)$ such that $f^n(K) \subset V$. (Cowen used the term *fundamental* instead of absorbing.) Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be the right halfplane.

Definition 2.1. Let $f : U \to U$ be holomorphic. Then (V, φ, T, Ω) is called an *eventual conjugacy* of f in U, if the following statements hold:

(i) V is absorbing for f;
(ii) $\varphi : U \to \Omega \in \{\mathbb{H}, \mathbb{C}\}$ is holomorphic and φ is univalent in V;
(iii) T is a Möbius transformation mapping Ω onto itself and $\varphi(V)$ is absorbing for T;
(iv) $\varphi(f(z)) = T(\varphi(z))$ for $z \in U$.

König [25] used the term *conformal* conjugacy. With the terminology *eventual* conjugacy we have followed Bargmann [6].

If U is the basin of an attracting (but not superattracting) fixed point ξ, then an eventual conjugacy is given by the solution of the Schröder-Kœnigs functional equation $\varphi(f(z)) = \lambda\varphi(z)$, with $\lambda = f'(\xi) \neq 0$. Similarly, eventual conjugacies in parabolic domains are given by the solutions of Abel's functional equation $\varphi(f(z)) = \varphi(z) + 1$.

In what follows, we assume that f has no fixed points in U. Clearly, this is the case for Baker domains U. The result of Cowen can now be stated as follows.

Lemma 2.1. *Let $U \neq \mathbb{C}$ be a simply connected domain and $f : U \to U$ a holomorphic function without fixed point in U. Then f has an eventual conjugacy (V, φ, T, Ω). Moreover, T and Ω may be chosen as exactly one of the following possibilities:*

(a) *$\Omega = \mathbb{H}$ and $T(z) = \lambda z$, where $\lambda > 1$;*
(b) *$\Omega = \mathbb{H}$ and $T(z) = z + i$ or $T(z) = z - i$;*
(c) *$\Omega = \mathbb{C}$ and $T(z) = z + 1$.*

It turns out (cf. [17]) that the cases listed in Lemma 2.1 correspond precisely to the cases of Theorem A. Thus U is hyperbolic in case (a), simply parabolic in case (b) and doubly parabolic in case (c).

König [25, Theorem 3] has given the following useful characterization of the different cases.

Lemma 2.2. *Let $U \neq \mathbb{C}$ be an unbounded simply connected domain and $f : U \to U$ a holomorphic function such that $f^n|_U \to \infty$ as $n \to \infty$. For $w_0 \in U$ put*

$$w_n = f^n(w_0) \quad \text{and} \quad d_n = \operatorname{dist}(w_n, \partial U).$$

Then

(a) *U is hyperbolic if there exists $\beta > 0$ such that $|w_{n+1} - w_n|/d_n \geq \beta$ for all $w_0 \in U$ and all $n \geq 0$;*
(b) *U is simply parabolic if $\liminf_{n\to\infty} |w_{n+1} - w_n|/d_n > 0$ for all $w_0 \in U$, but*
$$\inf_{w_0 \in U} \limsup_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} = 0;$$
(c) *U is doubly parabolic if $\lim_{n\to\infty} |w_{n+1} - w_n|/d_n = 0$ for all $w_0 \in U$.*

Denote by $\rho_U(\cdot, \cdot)$ the hyperbolic metric in a hyperbolic domain U, using the normalization where the density $\lambda_{\mathbb{D}}$ in the unit disk is given by $\lambda_{\mathbb{D}}(z) = 2/(1-|z|^2)$. Considering the hyperbolic metric instead of the Euclidean metric in Lemma 2.2 leads to the following result; see [6, Lemma 2.6] or [38, Theorem 2.2.11].

Lemma 2.3. *Let $U \neq \mathbb{C}$ be a simply connected domain and $f : U \to U$ a holomorphic function without fixed point in U. For $z \in U$ put*

$$\ell(z) = \lim_{n\to\infty} \rho_U(f^{n+1}(z), f^n(z)).$$

Then

(a) *U is hyperbolic if $\inf_{z\in U} \ell(z) > 0$;*
(b) *U is simply parabolic if $\ell(z) > 0$ for all $z \in U$, but $\inf_{z\in U} \ell(z) = 0$;*
(c) *U is doubly parabolic if $\ell(z) = 0$ for all $z \in U$.*

Note that the sequence $(\rho_U(f^{n+1}(z), f^n(z)))_{n\in\mathbb{N}}$ is non-increasing by the Schwarz-Pick Lemma (Lemma 3.1 below). Thus the limit defining $\ell(z)$ exists.

Let now U be an invariant Baker domain of the entire function f. It is not difficult to see that if $f|_U$ is univalent, then $f(U) = U$. Also, the Schwarz-Pick Lemma says that if $f|_U$ is univalent, then

$$\rho_U(f^{n+1}(z), f^n(z)) = \rho_U(f(z), z)$$

for all $n \in \mathbb{N}$ and $z \in U$. It now follows from Lemma 2.3, as already mentioned in the introduction, that U cannot be doubly parabolic if $f|_U$ is univalent.

Lemma 2.4. *Let $f : U \to U$ be as in Lemma 2.1 and let U_0 be an absorbing domain for f. Then $f : U_0 \to U_0$ and $f : U \to U$ are of the same type according to the classification given in Theorem A or Lemma 2.1.*

Proof. Let (V, φ, T, Ω) be an eventual conjugacy of f in U. Since, by the definition of an absorbing domain, V and U_0 are simply connected, the components of $V \cap U_0$ are also simply connected. It was shown by Cowen [14, p. 79-80] (see also [6, Lemma 2.3]) that there exists a component W of $V \cap U_0$ which is absorbing for f in U. Moreover, $\varphi(W)$ is absorbing for T in Ω. Thus $(W, \varphi|_W, T, \Omega)$ is an eventual conjugacy of both $f : U \to U$ and $f : U_0 \to U_0$. The conclusion follows. □

We note that a somewhat different approach to classifying holomorphic self-maps of $\mathbb{H}$, based on the sequences $(\rho_{\mathbb{H}}(f^{n+1}(z), f^n(z)))_{n\in\mathbb{N}}$, was developed by Baker and Pommerenke [3, 28]; see also [12]. As shown by König [25, Lemma 3], this leads to the same classification as above.

We mention that Baker domains may also be defined for functions meromorphic in the plane. In general, Baker domains of meromorphic functions are multiply connected. König [25, Theorem 1] has shown that if U is a Baker domain of a meromorphic function with only finitely many poles, then an eventual conjugacy in U exists and the conclusion of Lemma 2.2 holds. On the other hand, eventual conjugacies need not exist for Baker domains of meromorphic functions with infinitely many poles [25, Theorem 2]. For further results on Baker domains of meromorphic functions we refer to [38]. In particular, we note that if U is a multiply connected Baker domain of a meromorphic function f such that there exists an eventual conjugacy in U, then U contains at least two singularities of f^{-1}; see [38, p. 200]. Thus $f|_U$ is not univalent in a multiply connected Baker domain U with an eventual conjugacy.

More generally, one may also consider functions meromorphic outside a small set. For such functions the classification of Baker domains is discussed in [37, Section 4].

The classification of Baker domains mentioned above appears in various other questions related to Baker domains. Besides the papers already cited we mention [10, 11, 13, 26].

3. Preliminary lemmas

The following result, already used in section 2, is known as the Schwarz-Pick Lemma; see, e.g., [34, p.12].

Lemma 3.1. *Let U and V be simply connected hyperbolic domains and $f : U \to V$ be holomorphic. Then $\rho_V(f(z_1), f(z_2)) \leq \rho_U(z_1, z_2)$ for all $z_1, z_2 \in U$. If there exists $z_1, z_2 \in U$ with $z_1 \neq z_2$ such that $\rho_V(f(z_1), f(z_2)) = \rho_U(z_1, z_2)$, then f is bijective.*

If $U \subset V$, then we may apply the Schwarz-Pick Lemma to $f(z) = z$ and obtain $\rho_V(z_1), z_2) \leq \rho_U(z_1, z_2)$ for all $z_1, z_2 \in U$.

The following result is the analogue of the Schwarz-Pick Lemma for quasiconformal mappings [27, Section II.3.3]. Note that a different normalization of the hyperbolic metric is used in [27].

Lemma 3.2. *Let U and V be simply connected hyperbolic domains and $f : U \to V$ be a K-quasiconformal mapping. Then*

$$\rho_V(f(z_1), f(z_2)) \leq M_K(\rho_U(z_1, z_2)) \tag{3.1}$$

for $z_1, z_2 \in U$, with

$$M_K(x) = 2 \operatorname{arctanh}\left(\varphi_K\left(\tanh \tfrac{1}{2}x\right)\right). \tag{3.2}$$

Here φ_K is the Hersch-Pfluger distortion function.

The function $\operatorname{arctanh} \circ \varphi_K \circ \tanh$ appearing on the right hand side of (3.2) has been studied in detail in a number of papers. For example, it was shown in [29, Theorem 1.6] that this function is strictly increasing and concave. Various estimates of this function in terms of elementary functions are known. We only mention [35, Theorem 11.2] where it was shown that the conclusion of Lemma 3.2 holds with (3.1) replaced by $\rho_V(f(z_1), f(z_2)) \leq K(\rho_U(z_1, z_2) + \log 4)$. We do not need any explicit estimate for M_K, but the fact that (3.1) holds for some non-decreasing function $M_K : [0, \infty) \to [0, \infty)$ suffices.

The following result [33, Lemma 1] is the fundamental lemma for quasiconformal surgery. Here $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

Lemma 3.3. *Let $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiregular mapping. Suppose that there are disjoint open subsets $E_1, \dots, E_m$ of $\overline{\mathbb{C}}$, quasiconformal mappings $\Phi_i : E_i \to E_i' \subset \overline{\mathbb{C}}$, for $i = 1, \dots, m$, and an integer $N \geq 0$ satisfying the following conditions:*

(i) *$g(E) \subset E$ where $E = E_1 \cup \dots \cup E_m$;*
(ii) *$\Phi \circ g \circ \Phi_i^{-1}$ is analytic in $E_i' = \Phi_i(E_i)$, where $\Phi : E \to \overline{\mathbb{C}}$ is defined by $\Phi|_{E_i} = \Phi_i$;*
(iii) *$g_{\overline{z}} = 0$ a.e. on $\overline{\mathbb{C}} \setminus g^{-N}(E)$.*

Then there exists a quasiconformal mapping $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *such that* $\psi \circ g \circ \psi^{-1}$ *is a rational function. Moreover,* $\psi \circ \Phi_i^{-1}$ *is conformal in* E_i' *and* $\psi_{\overline{z}} = 0$ *a.e. on* $\overline{\mathbb{C}} \setminus \bigcup_{n\geq 0} g^{-n}(E)$.

Shishikura stated the result in [33] for rational functions, but it holds for entire functions as well. A stronger result, stated for entire functions, can be found in [24, Theorem 3.1].

4. Proof of Theorem 1.1

Put $g(z) = e^{2\pi i\alpha} z e^z$, where $\alpha \in [0,1] \setminus \mathbb{Q}$ is chosen such that g has a Siegel disk S at 0, and put

$$h(z) = 2\pi i(\alpha + m) + z + e^z$$

where $m \in \mathbb{Z}$. Using $g(e^z) = \exp h(z)$ it can be shown that h has a Baker domain $V = \log S$ on which h is univalent. This example, with $m = 0$, is due to Herman [21, p. 609]; see also [4, 8, 9]. We shall assume, however, that $m \geq 3$. It is not difficult to see that V is simply parabolic.

There exists $r_0 \in (0,1)$ and a g-invariant domain $S_0 \subset S$ such that

$$D(0, r_0) \subset S_0 \subset D(0,1).$$

Here and in the following $D(a,r)$ denotes the open disk of radius r around a point a. With $x_0 = \log r_0$ we thus see that V contains $H_0 = \{z \in \mathbb{C} : \mathrm{Re} < x_0\}$. Moreover, if $z \in H_0$, then

$$\mathrm{Im}\, h^n(z) \geq 2\pi(\alpha + m) + \mathrm{Im}\, h^{n-1}(z) - 1 > \mathrm{Im}\, z + 2n\pi \quad \text{and} \quad \mathrm{Re}\, h^n(z) < 0 \tag{4.1}$$

for $n \in \mathbb{N}$.

For $x_1 < x_0 - \pi$ we define

$$S(x_1) = \{z \in \mathbb{C} : \mathrm{Re}\, z < x_1,\ |\mathrm{Im}\, z| < 2\pi\} \cup D(x_1, 2\pi)$$

and

$$T(x_1) = \{z \in \mathbb{C} : \mathrm{Re}\, z < x_1,\ |\mathrm{Im}\, z| < \pi\} \cup D(x_1, \pi)$$

We also put

$$k(z) = 2\pi i(\alpha + m) + z + \exp(e^{-z} - L),$$

for a large constant L to be determined later.

Now we define $F : \mathbb{C} \to \mathbb{C}$ as follows. For $z \in \mathbb{C} \setminus \underline{S(x_1)}$ we put $F(z) = h(z)$, for $z \in \overline{T(x_1)}$ we put $F(z) = k(z)$ and for $z \in S(x_1) \setminus \overline{T(x_1)}$ we define $F(z)$ by interpolation. Thus for $x \leq x_1$ and $\pi \leq y \leq 2\pi$ we put

$$F(x + iy) = \frac{y - \pi}{\pi} h(x + 2\pi i) + \frac{2\pi - y}{\pi} k(x + \pi i),$$

for $x \leq x_1$ and $-2\pi \leq y \leq -\pi$ we put

$$F(x + iy) = \frac{-y - \pi}{\pi} h(x - 2\pi i) + \frac{2\pi + y}{\pi} k(x - \pi i),$$

and for $\pi \leq r \leq 2\pi$ and $-\pi/2 \leq \varphi \leq \pi/2$ we put

$$F(x_1 + re^{i\varphi}) = \frac{r - \pi}{\pi} h(x_1 + 2\pi e^{i\varphi}) + \frac{2\pi - r}{\pi} k(x_1 + \pi e^{i\varphi}).$$

We claim that F is quasiregular. By definition, F is holomorphic in $T(x_1)$ and in $\mathbb{C} \setminus \overline{S(x_1)}$. So it suffices to consider the dilatation in $S(x_1) \setminus \overline{T(x_1)}$. We first consider the subregion $A = \{x + iy : x \leq x_1,\ \pi < y < 2\pi\}$ of $S(x_1) \setminus \overline{T(x_1)}$. For $z = x + iy \in A$ we have

$$F(z) = 2\pi i(\alpha + m) + z + P(z) \tag{4.2}$$

with

$$P(x + iy) = \frac{y - \pi}{\pi} e^x + \frac{2\pi - y}{\pi} \exp(-e^{-x} - L)). \tag{4.3}$$

It is easy to see that if $|x_1|$ is large enough, then $|P_x(z)| \leq 1/4$ and $|P_y(z)| \leq 1/4$ for $z \in A$. Thus $|P_z(z)| \leq 1/4$ and $|P_{\overline{z}}(z)| \leq 1/4$ and hence $|F_z(z)| \geq 3/4$ and $|F_{\overline{z}}(z)| \leq 1/4$ for $z \in A$. It follows that F is quasiregular in A. The argument for the domain $\overline{A} = \{x + iy : x \leq x_1,\ -2\pi < y < -\pi\}$ is analogous.

Now we consider the region $B = \{x_1 + re^{i\varphi} : \pi \leq r \leq 2\pi,\ -\pi/2 \leq \varphi \leq \pi/2\}$. For $z = x_1 + re^{i\varphi} \in B$ we have

$$F(z) = 2\pi i(\alpha + m) + z + Q(z) \tag{4.4}$$

with

$$\begin{aligned} Q(x_1 + re^{i\varphi}) = & \frac{r - \pi}{\pi} \exp\left(x_1 + 2\pi e^{i\varphi}\right) \\ & + \frac{2\pi - r}{\pi} \exp\left(-\exp\left(x_1 + \pi e^{i\varphi}\right) - L\right). \end{aligned} \tag{4.5}$$

The computation of the partial derivatives of Q is more cumbersome than for P, but again it follows $|Q_z(z)| \leq 1/4$ and $|Q_{\overline{z}}(z)| \leq 1/4$ for $z \in B$ if $|x_1|$ and $L = L(x_1)$ are large enough. As before this implies that F is quasiregular in B.

It follows from (4.2), (4.3), (4.4) and (4.5), together with the corresponding representation in $\overline{A}$, that

$$\operatorname{Im} F(z) > \operatorname{Im} z + 2\pi(\alpha + m) - 1 > 3\pi \quad \text{and} \quad \operatorname{Re} F(z) < x_0 \tag{4.6}$$

for $z \in \overline{S(x_1)} \setminus T(x_1)$, provided $|x_1|$ and L are large enough. Together with (4.1) this implies that if x_1 and L are suitably chosen, then every orbit passes through $\overline{S(x_1)} \setminus T(x_1)$, which is the set where F is not holomorphic, at most once.

It now follows from Lemma 3.3, applied with $g = F$, $N = 1$, $m = 1$,

$$E_1 = \bigcup_{n=1}^{\infty} F^n \left(S(x_1) \setminus \overline{T(x_1)}\right)$$

and $\Phi_1 = \mathrm{id}_{E_1}$, that there exists a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ such that $f = \psi \circ F \circ \psi^{-1}$ is an entire function.

It is easy to see that f is transcendental. For example, let $a \in \mathcal{J}(h)$ such that $h^{-1}(a)$ is infinite. The complete invariance of $\mathcal{J}(h)$ yields that $h^{-1}(a) \cap S(x_1) = \emptyset$ so that $h^{-1}(a) \subset F^{-1}(a) = \psi^{-1}(f^{-1}(\psi(a)))$. Therefore $f^{-1}(\psi(a))$ is infinite, which implies that f is transcendental.

In the sequel, we will apply the concepts of the Fatou-Julia theory also to the quasiregular function F. For example, we can define $\mathcal{J}(F)$ as the set where the iterates of F are not normal and find that $\mathcal{J}(f) = \psi(\mathcal{J}(F))$.

It follows from (4.1) and (4.6) that

$$\operatorname{Im} F^n(z) \to \infty \quad \text{for } z \in V \setminus \bigcup_{k=0}^{\infty} h^{-k}(T(x_1))$$

as $n \to \infty$. With $W_0 = \{z \in \mathbb{C} : \operatorname{Re} z < x_0,\ \operatorname{Im} z > 2\pi\}$ we have

$$V \setminus \bigcup_{k=0}^{\infty} h^{-k}(T(x_1)) \supset W_0$$

and thus find that F has a Baker domain W containing W_0. Using (4.6) we see that $\overline{S(x_1)} \setminus T(x_1) \subset W$.

We now show that $T(x_1) \cap \mathcal{J}(F) \neq \emptyset$. In order to do so we note that if $x \in \mathbb{R}$ is sufficiently large, then

$$\left|F\left(-x+i\frac{\pi}{2}\right)\right| = \left|k\left(-x+i\frac{\pi}{2}\right)\right| = \left|2\pi i(\alpha+m) - x + i\frac{\pi}{2} + \exp\left(ie^x - L\right)\right)\right| \leq 2x$$

while

$$\left|F'\left(-x+i\frac{\pi}{2}\right)\right| = \left|k'\left(-x+i\frac{\pi}{2}\right)\right| = \left|1 - ie^x \exp\left(ie^x - L\right)\right)\right| \geq e^{x-L} - 1 \geq e^{x/2}.$$

Given $a_1, a_2 \in \mathcal{J}(F)$ it now follows from Landau's theorem that if x is large enough, then there exists $j \in \{1, 2\}$ and $z \in D(-x + i\pi/2, 1)$ such that $F(z) = a_j$. By the complete invariance of $\mathcal{J}(F)$ we thus have $D(-x + i\pi/2, 1) \cap \mathcal{J}(F) \neq \emptyset$. In particular, $T(x_1) \cap \mathcal{J}(F) \neq \emptyset$.

Since F has the unbounded Fatou component W, a result of Baker [1] yields that F has no multiply connected Fatou components. Since $\overline{S(x_1)} \setminus T(x_1) \subset W \subset \mathcal{F}(F)$, this implies that $T(x_1)$ contains an unbounded component Γ of $\mathcal{J}(F)$.

For large $x \in \mathbb{R}$ we consider $w_1 = -x - 2\pi i$, $w_2 = F(w_1)$ and $w_3 = F(w_2)$. Obviously, $w_j \in W$ for $j = 1, 2, 3$. We note that w_1 is "below" the strip $T(x_1)$ while w_2 and w_3 are "above" this strip by (4.6). In fact, we have

$$w_2 = 2\pi i(\alpha + m - 1) - x + e^{w_1} \in W_0$$

and

$$w_3 = 2\pi i(\alpha + m) + w_2 + e^{w_2} = 2\pi i(2\alpha + 2m - 1) - x + e^{w_1} + e^{w_2} \in W_0.$$

We choose $\delta > 0$ such that $r_1 = 2\pi(\alpha + m) + \delta < r_2 = 2\pi(2\alpha + 2m - 3) - \delta$ and find that

$$w_2 \in D(w_3, r_1) \subset D(w_3, r_2)$$

for sufficiently large x. This implies that

$$\rho_{D(w_3,r_2)}(w_2, w_3) = \rho_{\mathbb{D}}((w_2 - w_3)/r_2, 0) \leq 2 \operatorname{arctanh}(r_1/r_2).$$

As $D(w_3, r_2) \subset W_0 \subset W$, the Schwarz-Pick Lemma now yields that

$$\rho_W(w_2, w_3) \leq \rho_{W_0}(w_2, w_3) \leq \rho_{D(w_3,r_2)}(w_2, w_3) \leq 2 \operatorname{arctanh}(r_1/r_2) \tag{4.7}$$

for large x.

Next we show that

$$\rho_W(w_1, w_2) \to \infty \quad \text{as } x \to \infty. \tag{4.8}$$

In order to do so, we note that the preimage of Γ under F contains an unbounded continuum Γ' which is contained in $\{z \in \mathbb{C} : \operatorname{Re} z < x_1,\ y_1 < \operatorname{Im} z < y_2\}$ for suitable y_1, y_2 satisfying $y_1 < y_2 < -2\pi$. Let now γ be a curve connecting w_1 and w_2 in W. It follows that there exists $x_2 \leq x_1$ such that if $t \leq x_2$, then there exists $z \in \gamma$, $\zeta \in \Gamma$ and $\zeta' \in \Gamma'$ such that $\operatorname{Re}\zeta = \operatorname{Re}\zeta' = \operatorname{Re} z = t$ and $y_1 < \operatorname{Im}\zeta' < \operatorname{Im} z < \operatorname{Im}\zeta < \pi$. This implies that the density λ_W of the hyperbolic metric in W satisfies

$$\lambda_W(z) \geq \frac{1}{2\operatorname{dist}(z, \partial W)} \geq \frac{1}{2\min\{|z-\zeta|, |z-\zeta'|\}} \geq \frac{1}{\pi - y_1}.$$

From this we can deduce that if $x < x_2$, then

$$\int_\gamma \lambda_W(z)|dz| \geq \frac{|x - x_2|}{\pi - y_1}.$$

As this holds for all curves γ connecting w_1 and w_2, we obtain

$$\rho_W(w_1, w_2) \geq \frac{|x - x_2|}{\pi - y_1},$$

from which (4.8) follows.

Put $U = \psi(W)$. Since $f = \psi \circ F \circ \psi^{-1}$ we find that U is a Baker domain of f. Let $v_j = \psi(w_j)$, for $j = 1, 2, 3$, and denote by K the dilatation of ψ. It follows from Lemma 3.2 and (4.7) that

$$\rho_U(v_2, v_3) \leq M_K(2 \operatorname{arctanh}(r_1/r_2)).$$

Suppose now that $f : U \to U$ is univalent. The Schwarz-Pick Lemma yields that $\rho_U(v_2, v_3) = \rho_U(f(v_1), f(v_2)) = \rho_U(v_1, v_2)$. Noting that ψ^{-1} is also K-quasiconformal, we deduce from Lemma 3.2 that

$$\rho_W(w_1, w_2) \leq M_K(\rho_U(v_1, v_2)) = M_K(\rho_U(v_2, v_3)).$$

Combining the last two estimates we obtain

$$\rho_W(w_1, w_2) \leq M_K(M_K(2 \operatorname{arctanh}(r_1/r_2))),$$

which contradicts (4.8). Thus $f : U \to U$ is not univalent.

It remains to show that U is a simply parabolic Baker domain. In order to do so we recall that V is a simply parabolic Baker domain of h. We also note that it follows from the construction of V that there exists an absorbing domain V_0 of h in V satisfying $V_0 \subset \{z \in \mathbb{C} : \operatorname{Re} z > 2\pi\}$. Clearly, V_0 is also an absorbing domain of F in W. Hence $U_0 = \psi(V_0)$ is an absorbing domain of f in U.

Since ψ is analytic in V_0, we find for $v \in V$ and $w = \psi(v) \in \psi(V) = U$ and large n that

$$\rho_{U_0}(f^{n+1}(w), f^n(w)) = \rho_{\psi(V_0)}(\psi(h^n(v)), \psi(h^{n+1}(v))) = \rho_{V_0}(h^{n+1}(v), h^n(v)).$$

Since V is a simply parabolic Baker domain of h, Lemma 2.4 now yields that U is simply parabolic. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

Let α, g, h, ψ and f be as in the proof of Theorem 1.1. Baker and Weinreich [4, Theorem 3] used results of Ghys [19] and Herman [20] to show that for suitably chosen α the boundary of the Siegel disk S of h is a Jordan curve. Actually, by a recent result of Zakeri [36], this is the case if α has bounded type. Thus in this case the Baker domain V of h is bounded by a Jordan curve on the sphere. Let γ be a Jordan arc in $\partial V \cap \{z : \operatorname{Im} z > 2\pi\}$. It follows from the construction that $\gamma \subset \partial W$ and that the points of γ are accessible from W. Thus ∂U contains the Jordan arc $\psi(\gamma)$ consisting of points accessible from U. This implies that $\Xi \neq \partial\mathbb{D}$. Thus $f_1 = f$ and $U_1 = U$ have the desired property.

The construction of f_2 and U_2 is similar. Here our starting point are the functions $g(z) = \frac{1}{2}z^2e^{2-z}$ and $h(z) = 2 - \log 2 + 2z - e^z$ considered in [8]. The function g has a superattracting basin B at the origin which is bounded by a Jordan curve and $V = \log B$ is a hyperbolic Baker domain of h where h is univalent. By a similar reasoning as in the proof of Theorem 1.1 we will now use quasiconformal surgery to construct an entire function f_2 with a Baker domain U_2 where f is not univalent.

Here we put, for large a positive integer M,

$$S(M) = \{z \in \mathbb{C} : |\operatorname{Re} z + 2\pi M| < 2\pi,\ \operatorname{Im} z < -2\pi\} \cup D(-2\pi M - 2\pi i, 2\pi)$$

and

$$T(M) = \{z \in \mathbb{C} : |\operatorname{Re} z + 2\pi M| < \pi,\ \operatorname{Im} z < -2\pi\} \cup D(-2\pi M - 2\pi i, \pi).$$

Next we put $k(z) = 2 - \log 2 + 2z + \exp(e^{-iz} - L)$ for a large constant L and define $F : \mathbb{C} \to \mathbb{C}$ by $F(z) = h(z)$ for $z \in \mathbb{C} \setminus S(M)$, by $F(z) = k(z)$ for $z \in \overline{T(M)}$ and by interpolation in $S(M) \setminus \overline{T(M)}$. Similarly as in the proof of Theorem 1.1 we see that F is quasiregular if M and $L = L(M)$ are large enough and that there exists a quasiconformal map ψ such that $f_2 = \psi \circ F \circ \psi^{-1}$ is a transcendental entire function. Noting that $V \cap \mathbb{H}$ is invariant under h and thus under F we see that F has a Baker domain W containing $V \cap \mathbb{H}$. Thus $U_2 = \psi(W)$ is a Baker domain of f_2, and the construction shows that ∂U_2 contains a Jordan arc consisting of points accessible from U_2.

To show that U_2 is hyperbolic we use again Lemma 2.4, noting that the domain $V_0 = \{z \in \mathbb{C} : \operatorname{Re} z < 3\pi M\}$ is absorbing for h and F in V and that ψ is analytic in V_0. Finally, the reasoning that f_2 is not univalent in U_2 is similar to that in the proof of Theorem 1.1.

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